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Inverse Scattering Problem based on the Method of Lines

조 선 대 학 교 대 학 원

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선 방법에 기초한 역 산란 문제

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대상물체에 전자기파 또는 음파를 투사하여 그 물체에서 산란된 파동으로부터 대상 물체의 모양을 추론하는 문제는 그 동안 많은 관심을 받아왔다.

본 논문에서는 2차원 헬름홀츠 방정식에 대한 형상 재구성 문제를 다루었다. 선 방법에 의한 반복법과 원거리 산란장 패턴을 이용하여 대상 물체의 모양을 역 추적하였다.

1. Introduction

The shape reconstruction problem is one of basic inverse problems in scattering theory. The inverse problem considered in this thesis is to reconstruct the shape of a scatterer from the scattered field and its far field pattern. It is known that the problem is nonlinear and severely ill-posed[1]. The method of lines is a widely applicable numerical method in which the given partial differential equation is replaced by a system of coupled ordinary differential equations obtained by discretizing all but one of independent variables[4]. Because of advances in the solution of ordinary differential equations, this method seems to be an attractive method for such approximations. Ma et al.[5] have developed an approximation method for solving scattering problems for cylinders using the Method of Lines. Instead of using the original Method of Lines in Cartesian coordinates, a modified Method of Lines in the cylindrical coordinates is used to solve the scattering from conduction cylinders. We followed the modified Method of Lines introduced by Ma et al.[5] to solve the scattering problem for the two-dimensional Helmholtz equation.

The method presented by M.A.Hooshyar[6] is a direct method using near field data. In this thesis, we consider the direct method using far field pattern and an iterative method based on the near field as well as the far field pattern. The synthetic near field and far field data were obtained from the integral representation of the Helmholtz equation to avoid an inverse crime[1]. Compared with those by M.A.Hooshyar our results can be extended to the inverse problem with higher approximation dimension as well as noise level up to 5%.

The forward problem for the scattering problem and the far field pattern by the modified Method of Lines are explained in Section 2 for the two-dimensional

Helmholtz equation. In Section 3, a direct method and an iterative method for solving the shape reconstruction problem are explained. Several simulations are presented in Section 4.

2. Forward Problem

2.1. Scattered Field

The direct obstacle scattering problem considered is to find the total field

$$u(x) = u^{inc}(x) + u^s(x) \tag{2.1}$$

such that u satisfies the Helmholtz equation

$$\triangle u(x) + k^2 u(x) = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D}, \tag{2.2}$$

the Dirichlet boundary condition

$$u(x) = 0 \quad \text{on} \quad \partial D, \tag{2.3}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \qquad r = |x|, \tag{2.4}$$

uniformly in all directions, where D is a sound-soft obstacle represented by a bounded simply connected domain with C^2 boundary ∂D in $\mathbb{R}^2, k > 0$ is the wave number such that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of D , $\triangle$ is the Laplace operator, u^{inc} is the incoming time-harmonic plane wave and u^s is the scattered field produced by the obstacle D due to the incident wave. The radiation condition (2.4) ensures uniqueness of the solution to the scattering problem and guarantees that the scattered wave is outgoing. By the smooth assumption on ∂D, there exists a unique solution $u \in C^2(\mathbb{R}^2/\bar{D}) \cap C(\mathbb{R}^2/D)$, where $\bar{D}$ is the closure of D[1].

The equations (2.1)-(2.4) can be represented in polar coordinates, i.e.,

$$u(r,\theta) = u^{inc}(r,\theta) + u^s(r,\theta),$$

$$u(r,\theta) = 0 \quad \text{on} \quad \partial D,$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0.$$

The following theorem is well-known[5, 6]. For the completeness of our presentation, we derive the Helmholtz equation in polar coordinates.

Theorem 2.1[5, 6]**.** *The Helmholtz equation (2.2) can be written in polar coordinates* (r,θ) *as*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\bar{D}. \tag{2.5}$$

Proof. Let $x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad \tan\theta = \frac{x_2}{x_1}$. Then

$$\begin{aligned}
r &= \sqrt{x_1^2+x_2^2}, \qquad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right),\\
\frac{\partial r}{\partial x_1} &= \frac{2x_1}{2\sqrt{x_1^2+x_2^2}} = \frac{x_1}{r}, \qquad \frac{\partial^2 r}{\partial x_1^2} = \frac{r - x_1\frac{\partial r}{\partial x_1}}{r^2} = \frac{r-\frac{x_1^2}{r}}{r^2} = \frac{x_2^2}{r^3},\\
\frac{\partial r}{\partial x_2} &= \frac{2x_2}{2\sqrt{x_1^2+x_2^2}} = \frac{x_2}{r}, \qquad \frac{\partial^2 r}{\partial x_2^2} = \frac{r - x_2\frac{\partial r}{\partial x_2}}{r^2} = \frac{r-\frac{x_2^2}{r}}{r^2} = \frac{x_1^2}{r^3},\\
\frac{\partial \theta}{\partial x_1} &= \frac{-\frac{x_2}{x_1^2}}{1+(\frac{x_2}{x_1})^2} = -\frac{x_2}{x_1^2+x_2^2} = -\frac{x_2}{r^2}, \qquad \frac{\partial^2\theta}{\partial x_1^2} = -x_2\frac{-2r\frac{\partial r}{\partial x_1}}{r^4} = \frac{2x_1x_2}{r^4},\\
\frac{\partial \theta}{\partial x_2} &= \frac{\frac{1}{x_1}}{1+(\frac{x_2}{x_1})^2} = \frac{x_1}{x_1^2+x_2^2} = \frac{x_1}{r^2}, \qquad \frac{\partial^2\theta}{\partial x_2^2} = x_1\frac{-2r\frac{\partial r}{\partial x_2}}{r^4} = -\frac{2x_1x_2}{r^4},
\end{aligned}$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_1}, \qquad \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_2},$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_1}\right)\\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x_1} + \frac{\partial u}{\partial r}\frac{\partial}{\partial x_1}\left(\frac{\partial r}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x_1} + \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x_1}\left(\frac{\partial \theta}{\partial x_1}\right)\\
&= \left\{\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x_1} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x_1}\right\}\frac{\partial r}{\partial x_1} + \frac{x_2^2}{r^3}\frac{\partial u}{\partial r}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \left\{ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x_1} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x_1} \right\} \frac{\partial \theta}{\partial x_1} + \frac{2x_1x_2}{r^4}\frac{\partial u}{\partial \theta} \\
= &\ \frac{x_1^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{x_2^2}{r^3}\frac{\partial u}{\partial r} + \frac{x_2^2}{r^4}\frac{\partial^2 u}{\partial \theta^2} \\
&\qquad -\frac{x_1x_2}{r^3}\left\{ \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right) \right\} + \frac{2x_1x_2}{r^4}\frac{\partial u}{\partial \theta},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x_2^2} &= \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_2}\right) \\
&= \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x_2} + \frac{\partial u}{\partial r}\frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x_2} + \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x_2}\left(\frac{\partial \theta}{\partial x_2}\right) \\
&= \left\{ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x_2} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x_2} \right\} \frac{\partial r}{\partial x_2} + \frac{x_1^2}{r^3}\frac{\partial u}{\partial r} \\
&\qquad + \left\{ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x_2} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x_2} \right\} \frac{\partial \theta}{\partial x_2} - \frac{2x_1x_2}{r^4}\frac{\partial u}{\partial \theta} \\
&= \frac{x_2^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{x_1^2}{r^3}\frac{\partial u}{\partial r} + \frac{x_1^2}{r^4}\frac{\partial^2 u}{\partial \theta^2} \\
&\qquad +\frac{x_1x_2}{r^3}\left\{ \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right) \right\} - \frac{2x_1x_2}{r^4}\frac{\partial u}{\partial \theta},
\end{aligned}
$$

$$
\begin{aligned}
\triangle u(x_1, x_2) &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \\
&= \left(\frac{x_1^2}{r^2} + \frac{x_2^2}{r^2}\right)\frac{\partial^2 u}{\partial r^2} + \left(\frac{x_2^2}{r^3} + \frac{x_1^2}{r^3}\right)\frac{\partial u}{\partial r} + \left(\frac{x_2^2}{r^4} + \frac{x_1^2}{r^4}\right)\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.
\end{aligned}
$$

This completes the proof. □

In order to simplify the problem and arrive at a set of ordinary differential equations, we apply the Method of Lines[2] to (2.5), and replace $\frac{\partial^2 u}{\partial \theta^2}$ by the central difference,

$$
\frac{\partial^2 u_n}{\partial \theta^2} = \frac{u(r, \theta_{n+1}) - 2u(r, \theta_n) + u(r, \theta_{n-1})}{(\Delta\theta)^2}, \quad u(r, \theta) = u(r, \theta + 2\pi).
$$

Then (2.5) is approximated by the following set of coupled ordinary differential equations:

$$\frac{\partial^2[u]}{\partial r^2} + \frac{1}{r}\frac{\partial[u]}{\partial r} + k^2[u] - \frac{[L]}{(r\Delta\theta)^2}[u] = 0, \tag{2.6}$$

where $[u] = [u_1 \quad u_2 \quad \cdots \quad u_N]^T$,

$$u_n = u(r, \theta_n)\ , \qquad \theta_n = (n-1)\Delta\theta, \qquad \Delta\theta = \frac{2\pi}{N}, \qquad \text{for} \qquad n = 1, 2, 3, \cdots, N,$$

and

$$[L] = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}_{N\times N}.$$

Here, A^T denotes the transpose of a vector or matrix A.

The set of all eigenvectors of a matrix A, each paired with its corresponding eigenvalue, is called the *eigensystem* of A. Then we can find an eigensystem $[P]$ of $[L]$ such that

$$[\Lambda] = [P]^{-1}[L][P] \qquad \text{with} \qquad [P]^T = [P]^{-1},$$

where $[\Lambda]$ is a diagonal matrix whose elements are the eigenvalues of $[L]$. The following theorem characterizes the eigensystem of $[L]$. The derivation of the eigensystem is based on the idea in [6].

Theorem 2.2. *For the $N \times N$ matrix $[L]$, the eigenvalues of $[L]$ are*

$$\lambda_m = 4\sin^2\frac{m\pi}{N}, \qquad m = 1, 2, \cdots, N,$$

and the eigensystem $[P]$ of $[L]$ becomes, for $n = 1, 2, \cdots, N$,

when N is even,

$$P_{nm} = \begin{cases} \sqrt{\frac{2}{N}} \cos \frac{2nm}{N}\pi \ , & m = 1, 2, \cdots, \frac{N}{2} - 1, \\ \frac{1}{\sqrt{N}} \cos n\pi \ , & m = \frac{N}{2}, \\ -\sqrt{\frac{2}{N}} \sin \frac{2nm}{N}\pi \ , & m = \frac{N}{2} + 1, \cdots, N - 1, \\ \frac{1}{\sqrt{N}} \ , & m = N, \end{cases}$$

when N is odd,

$$P_{nm} = \begin{cases} \sqrt{\frac{2}{N}} \cos \frac{2nm}{N}\pi \ , & m = 1, 2, \cdots, \frac{N-1}{2}, \\ -\sqrt{\frac{2}{N}} \sin \frac{2nm}{N}\pi \ , & m = \frac{N+1}{2}, \cdots, N - 1, \\ \frac{1}{\sqrt{N}} \ , & m = N, \end{cases}$$

where P_{nm} is an (n, m) element of $[P]$.

Proof. Let the column vector p with an element p_n be an eigenvector of $[L]$ corresponding the eigenvalue λ. It follows that

$$p_{n-1} - 2p_n + p_{n+1} = -\lambda p_n, \qquad 1 \le n \le N, \tag{2.7}$$

with $p_0 = p_N$ and $p_1 = p_{N+1}$. The periodicity condition $p_i = p_{N+i}$, $i = 0, 1, 2, \cdots$, is satisfied. In order to solve (2.7), we assume that the components of the eigenvector p take the form[6]

$$p_n = a\alpha^{-n} + b\alpha^n, \tag{2.8}$$

where a and b are the constants. From the periodicity condition, we have

$$p_n = p_{N+n} = a\alpha^{-(N+n)} + b\alpha^{N+n}.$$

Multiply α^N and α^{-N} to the above equation,

$$\begin{aligned} \alpha^N p_n = \alpha^N p_{N+n} &= a\alpha^N \alpha^{-N-n} + b\alpha^N \alpha^{N+n} \\ &= a\alpha^{-n} + b\alpha^{2N+n}, \\ \alpha^{-N} p_n = \alpha^{-N} p_{N+n} &= a\alpha^{-N} \alpha^{-N-n} + b\alpha^{-N} \alpha^{N+n} \\ &= a\alpha^{-2N-n} + b\alpha^n. \end{aligned}$$

From the above two equations,

$$\begin{aligned}\left(\alpha^N + \alpha^{-N}\right) p_n &= \left(a\alpha^{-n} + b\alpha^n\right) + \left(a\alpha^{-(2N+n)} + b\alpha^{(2N+n)}\right)\\ &= p_n + p_{2N+n} = 2p_n,\end{aligned}$$

$$\alpha^N + \alpha^{-N} = 2, \qquad \alpha^N = 1 = e^{i2\pi}.$$

Substitution of (2.8) into (2.7) leads to

$$a\alpha^{-n+1} + b\alpha^{n-1} - 2p_n + a\alpha^{-n-1} + b\alpha^{n+1} = -\lambda p_n,$$

$$\alpha\left(a\alpha^{-n} + b\alpha^n\right) + \alpha^{-1}\left(a\alpha^{-n} + b\alpha^n\right) - 2p_n = -\lambda p_n,$$

$$\alpha p_n + \alpha^{-1} p_n - 2p_n = -\lambda p_n,$$

$$\lambda = 2 - \alpha - \alpha^{-1}.$$

To generalize, let

$$[P] = [P_1 \; P_2 \; \cdots \; P_m \; \cdots \; P_N], \qquad P_m = [P_{1m} \; P_{2m} \; \cdots \; P_{nm} \; \cdots \; P_{Nm}]^T,$$

where the column vector P_m with an element P_{nm} is the eigenvector corresponding to the eigenvalue λ_m. Then (2.8) becomes,

$$P_{nm} = a_m \alpha_m^{-n} + b_m \alpha_m^n,$$

and we have

$$\alpha_m^N = 1 = e^{i2m\pi}, \qquad m = 1, 2, \cdots, N.$$

Thus the eigenvalues of $[L]$ are

$$\begin{aligned}\lambda_m &= 2 - \alpha_m^{-1} - \alpha_m = 2 - e^{i\left(-\frac{2m\pi}{N}\right)} - e^{i\left(\frac{2m\pi}{N}\right)}\\ &= 2 - \left\{\cos\left(-\frac{2m\pi}{N}\right) + i\sin\left(-\frac{2m\pi}{N}\right)\right\} - \left\{\cos\frac{2m\pi}{N} + i\sin\frac{2m\pi}{N}\right\}\\ &= 2\left(1 - \cos\frac{2m\pi}{N}\right) = 4\sin^2\frac{m\pi}{N},\end{aligned}$$

and the components P_{nm} of $[P]$ become,

$$P_{nm} = a_m e^{-i(\frac{2mn\pi}{N})} + b_m e^{i(\frac{2mn\pi}{N})}, \qquad n,m = 1,2,3,\cdots,N,$$

where a_m and b_m are constants. Since the number of repeated eigenvalues for even N and odd N are different, we choose the constants as for even N,

$$\begin{cases} a_m = \frac{1}{\sqrt{2N}},\ b_m = a_m\ , & m = 1,2,\cdots,\frac{N}{2}-1, \\ a_m = -\frac{i}{\sqrt{2N}},\ b_m = -a_m\ , & m = \frac{N}{2}+1,\cdots,N-1, \\ a_m = \frac{1}{2\sqrt{N}},\ b_m = a_m\ , & m = \frac{N}{2}, N, \end{cases} \tag{2.9}$$

and for odd N,

$$\begin{cases} a_m = \frac{1}{\sqrt{2N}},\ b_m = a_m\ , & m = 1,2,\cdots,\frac{N-1}{2}, \\ a_m = -\frac{i}{\sqrt{2N}},\ b_m = -a_m\ , & m = \frac{N+1}{2},\cdots,N-1, \\ a_m = \frac{1}{2\sqrt{N}},\ b_m = a_m\ , & m = N. \end{cases} \tag{2.10}$$

□

Lemma 2.3. *Let the constants a_m and b_m be (2.9) or (2.10), then $[P]$ is an orthogonal matrix.*

Proof. There is a proof in [6] for odd N, we prove in the case of even N.

$$\begin{aligned} ([P][P]^T)_{nm} &= \sum_{k=1}^{N} P_{nk}P_{mk} \\ &= \sum_{k=1}^{\frac{N}{2}-1} P_{nk}P_{mk} + P_{n\frac{N}{2}}P_{m\frac{N}{2}} + \sum_{k=\frac{N}{2}+1}^{N-1} P_{nk}P_{mk} + P_{nN}P_{mN} \\ &= \frac{2}{N}\sum_{k=1}^{\frac{N}{2}-1} \cos\frac{2nk}{N}\pi\cos\frac{2mk}{N}\pi + \frac{1}{N}\cos n\pi\cos m\pi \\ &\qquad + \frac{2}{N}\sum_{k=\frac{N}{2}+1}^{N-1} \sin\frac{2nk}{N}\pi\sin\frac{2mk}{N}\pi + \frac{1}{N} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1}\left\{\cos\frac{2(n+m)k}{N}\pi + \cos\frac{2(n-m)k}{N}\pi\right\} + \frac{\cos n\pi\cos m\pi}{N} \\
&\qquad -\frac{1}{N}\sum_{k=\frac{N}{2}+1}^{N-1}\left\{\cos\frac{2(n+m)k}{N}\pi - \cos\frac{2(n-m)k}{N}\pi\right\} + \frac{1}{N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2(n-m)k}{N}\pi - \frac{1}{N}\cos{(n-m)\pi} + \frac{1}{N}\cos n\pi\cos m\pi \\
&\qquad +\frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1}\cos\frac{2(n+m)k}{N}\pi - \frac{1}{N}\sum_{k=\frac{N}{2}+1}^{N-1}\cos\frac{2(n+m)k}{N}\pi \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2(n-m)k}{N}\pi \\
&\qquad +\frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1}\left\{\cos\frac{2(n+m)k}{N}\pi - \cos\frac{2(n+m)(N-k)}{N}\pi\right\} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2(n-m)k}{N}\pi = \frac{1}{N}Re\left\{\sum_{k=0}^{N-1} e^{i\frac{2(n-m)k}{N}\pi}\right\} = \delta_{nm}
\end{aligned}
$$

where δ_{nm} is the Kronecker delta. Thus, $[P]^{-1} = [P]^T$, that is, $[P]$ is orthogonal.

□

Let

$$[f] = [P]^T[u]. \tag{2.11}$$

Then (2.6) becomes

$$r^2\frac{\partial^2 f_n}{\partial r^2} + r\frac{\partial f_n}{\partial r} + \left\{(kr)^2 - \left(\frac{\sqrt{\lambda_n}}{\Delta\theta}\right)^2\right\} f_n(r) = 0, \qquad 1 \le n \le N, \tag{2.12}$$

where f_n is the n-th element of $[f]$.

Note that Bessel's equation with order α is

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + (x^2 - \alpha^2)y = 0.$$

By substituting $x = kr$, the Bessel equation can be represented by

$$r^2 \frac{\partial^2 y}{\partial r^2} + r\frac{\partial y}{\partial r} + \left\{(kr)^2 - \alpha^2\right\} y = 0,$$

since

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{k}\frac{\partial y}{\partial r}, \quad \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial y}{\partial r}\right) = \frac{1}{k^2}\frac{\partial^2 y}{\partial r^2}.$$

Therefore, (2.12) is just the Bessel equation with order $v_n = \left(\sqrt{\lambda_n}/\Delta\theta\right)$, and its solution is the Bessel function

$$f_n(r) = A_n H_{v_n}^{(1)}(kr), \qquad 1 \le n \le N. \tag{2.13}$$

Because the scattered fields are outside the scatterer, the first kind of Hankel function is chosen. By substituting (2.13) into (2.11), we obtain

$$u_n^s(r) = \sum_{m=1}^{N} P_{nm} A_m H_{v_m}^{(1)}(kr), \qquad 1 \le n \le N. \tag{2.14}$$

The coefficients A_m can be determined by the boundary condition in (2.3), i.e.,

$$u^{inc} + u^s = 0 \qquad \text{on} \qquad \partial D,$$

where $u^{inc} = e^{ikr\cos(\theta-\alpha)}$ is the incident field and α is the direction of incoming wave. Hence, for each (r_n, θ_n) on the boundary

$$-e^{ikr_n\cos(\theta_n-\alpha)} = \sum_{m=1}^{N} P_{nm} A_m H_{v_m}^{(1)}(kr_n), \qquad 1 \le n \le N. \tag{2.15}$$

2.2. Far-field pattern

Now we consider the asymptotic behavior of the scattered field. It is known that the scattered field u^s and the Hankel function have the following behaviors[1]:

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right\}, \qquad r = |x| \to \infty, \quad \hat{x} = \frac{x}{|x|},$$

$$H_v^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{v\pi}{2} - \frac{\pi}{4})} \left\{ 1 + O\left(\frac{1}{r}\right) \right\}, \qquad r \to \infty. \tag{2.16}$$

Substituting (2.16) into (2.14)

$$\begin{aligned} u_n^s(r) &\approx \sqrt{\frac{2}{\pi kr}} e^{ikr} \sum_{m=1}^{N} P_{nm} A_m e^{-i(\frac{vm}{2}\pi + \frac{\pi}{4})} \\ &\approx \frac{e^{ikr}}{\sqrt{r}} u_{\infty_n}(\hat{x}), \qquad 1 \leq n \leq N, \end{aligned}$$

we have

$$u_{\infty_n}(\hat{x}) \approx \sqrt{\frac{2}{\pi k}} \sum_{m=1}^{N} P_{nm} A_m e^{-i(\frac{vm}{2}\pi + \frac{\pi}{4})}, \qquad 1 \leq n \leq N. \tag{2.17}$$

3. Inverse Problem

3.1. Optimization

Assume that ∂D is identified by a real column distance vector

$$r = [\ r_1 \quad r_2 \quad \cdots \quad r_N\ ]^T \in \mathbb{R}^N, \tag{3.1}$$

$$\partial D := \{\ (r_n, \theta_n) \mid \theta_n = (n-1)\frac{2\pi}{N},\ n = 1, 2, \cdots, N\ \}.$$

Here, $\mathbb{R}^N$ is the N-dimensional real vector space.

We consider the following cost functional:

$$\chi^2(r) = \sum_{i=1}^{N} |X_i|^2 = \sum_{i=1}^{N} \bar{X}_i X_i, \qquad X_i = X(r_i), \tag{3.2}$$

where X is a function of r. Consider finding a minimum of (3.2) by using Newton's method to search for zero of the gradient of the function χ^2 in (3.2). Let D be the second derivative matrix(Hessian matrix) of χ^2 with respect to r. Near the current point r^{cur}, we have the second order approximation[7]

$$\chi^2(r) \approx \chi^2(r^{cur}) + (r - r^{cur}) \cdot \nabla\chi^2(r^{cur}) + \frac{1}{2}(r - r^{cur}) \cdot D \cdot (r - r^{cur}). \tag{3.3}$$

By taking the derivative with respect to r for the above equation, we have

$$\nabla\chi^2(r) = \nabla\chi^2(r^{cur}) + D \cdot (r - r^{cur}).$$

In Newton's method, we set $\nabla\chi^2(r) = 0$ to determine the next iteration point:

$$r - r^{cur} = -D^{-1} \cdot \nabla\chi^2(r^{cur}), \tag{3.4}$$

when D is nonsingular. Let the increment vector $\delta = r - r^{cur}$ be

$$\delta = [\ \delta_1 \quad \delta_2 \quad \cdots \quad \delta_N\ ]^T \in \mathbb{R}^N.$$

Then (3.4) can be written as the following system of equations

$$\delta = -\left[\frac{\partial}{\partial r}\left(\frac{\partial \chi^2}{\partial r}\right)\right]^{-1}\left[\frac{\partial \chi^2}{\partial r}\right].$$

To treat the possible singularities of D, we take the following complex version of Levenberg type regularization formula

$$\delta = -\left[\frac{\partial}{\partial r}\left(\frac{\partial \chi^2}{\partial r}\right) + \lambda I\right]^{-1}\left[\frac{\partial \chi^2}{\partial r}\right], \tag{3.5}$$

where $\lambda > 0$ is the regularization parameter and I is the $N \times N$ identity matrix. The regularization parameter λ in (3.5) can be adjusted automatically during iterations based on the following selection criterion:

$$\chi^2(r+\delta) < \chi^2(r),$$

where χ^2 is the cost defined by (3.2). For a small value of λ, the formula is similar to the Newton type method, and for a large value of λ, it is close to the steepest descent method.

We now present an algorithm for solving (3.5)[7].

Algorithm 3.1

Step 1. Choose the initial guesses for the distance vector r and the regularization parameter $\lambda > 0$. Perform Step 2 - Step 4 until the cost (3.2) reaches the minimum.

Step 2. Compute the cost $\chi^2(r)$ by (3.2).

Step 3. Solve the linear system (3.5) for δ, and compute $\chi^2(r+\delta)$ by (3.2).

Step 4. If $\chi^2(r+\delta) \geq \chi^2(r)$, update λ by 10λ and go back to step 3.
If $\chi^2(r+\delta) < \chi^2(r)$, update λ by $\frac{\lambda}{10}$, update r by $r+\delta$, and go back to step 3.

3.2. Direct Method

To obtain $\{A_n\}$ in (2.14), we choose an observation distance R and measure the scattered field $y_m = u_m^s(R)$. Then, from (2.14), we have

$$A_n = \frac{1}{H_{v_n}^{(1)}(kR)} \sum_{m=1}^{N} P_{nm}^T y_m, \qquad 1 \le n \le N. \tag{3.6}$$

From (2.15), let

$$f(r_n) = e^{ikr_n \cos(\theta_n - \alpha)} + \sum_{m=1}^{N} P_{nm} A_m H_{v_m}^{(1)}(kr_n), \qquad 1 \le n \le N. \tag{3.7}$$

Then the inverse problem is to find a r_n, $1 \le n \le N$, such that $f(r_n) = 0$.

We consider the following cost functional:

$$\chi^2(r) = \sum_{n=1}^{N} |f_n|^2 = \sum_{n=1}^{N} \bar{f}_n f_n, \qquad f_n = f(r_n). \tag{3.8}$$

The gradient of χ^2 with respect to the distance vector r in (3.1) becomes

$$\begin{aligned} \frac{\partial \chi^2}{\partial r_j} &= \sum_{n=1}^{N} \left\{ \frac{\partial f_n}{\partial r_j} \bar{f}_n + f_n \frac{\partial \bar{f}_n}{\partial r_j} \right\} = 2Re \left\{ \sum_{n=1}^{N} \frac{\partial f_n}{\partial r_j} \bar{f}_n \right\} \\ &= 2Re \left\{ \frac{\partial f_j}{\partial r_j} \bar{f}_j \right\}, \qquad 1 \le j \le N, \end{aligned} \tag{3.9}$$

since $\frac{\partial f_n}{\partial r_j} = 0$ for $n \ne j$.

The following property of the Hankel function is well-known[3],

$$\frac{\partial}{\partial x} H_v^{(1)}(x) = \frac{1}{2} \left(H_{v-1}^{(1)}(x) - H_{v+1}^{(1)}(x) \right).$$

For each j, $j = 1, 2, \cdots, N$, we have

$$\frac{\partial f_j}{\partial r_j} = ik \cos(\theta_j - \alpha) e^{ikr_j \cos(\theta_j - \alpha)} + \frac{1}{2} k \sum_{m=1}^{N} P_{jm} A_m (H_{v_m-1}^{(1)}(kr_j) - H_{v_m+1}^{(1)}(kr_j)).$$

To obtain the Hessian matrix D in (3.3), we take an additional partial derivative for $\frac{\partial \chi^2}{\partial r_i}$ with respect to r_j, $i, j = 1, 2, \cdots, N$.

$$\begin{aligned}
\frac{\partial}{\partial r_j}\left(\frac{\partial \chi^2}{\partial r_i}\right) &= \sum_{n=1}^{N}\left\{\frac{\partial}{\partial r_j}\left(\frac{\partial f_n}{\partial r_i}\right)\bar{f}_n + \frac{\partial f_n}{\partial r_i}\frac{\partial \bar{f}_n}{\partial r_j} + \frac{\partial}{\partial r_j}\left(\frac{\partial \bar{f}_n}{\partial r_i}\right)f_n + \frac{\partial \bar{f}_n}{\partial r_i}\frac{\partial f_n}{\partial r_j}\right\} \\
&= 2Re\left[\sum_{n=1}^{N}\left\{\frac{\partial}{\partial r_j}\left(\frac{\partial f_n}{\partial r_i}\right)\bar{f}_n + \frac{\partial f_n}{\partial r_i}\frac{\partial \bar{f}_n}{\partial r_j}\right\}\right] \\
&= \begin{cases} 2Re\left\{\frac{\partial}{\partial r_j}\left(\frac{\partial f_j}{\partial r_j}\right)\bar{f}_j + \frac{\partial f_j}{\partial r_j}\frac{\partial \bar{f}_j}{\partial r_j}\right\}, & i = j, \\ 0, & i \neq j, \end{cases}
\end{aligned} \tag{3.10}$$

since $\frac{\partial f_n}{\partial r_j} = 0$ for $n \neq j$. Here,

$$\begin{aligned}
\frac{\partial}{\partial r_j}\left(\frac{\partial f_j}{\partial r_j}\right) &= \{ik\cos(\theta_j - \alpha)\}^2 e^{ikr_j\cos(\theta_j-\alpha)} \\
&\quad + \frac{1}{4}k^2\sum_{m=1}^{N}P_{jm}A_m\{H^{(1)}_{v_m-2}(kr_j) - 2H^{(1)}_{v_m}(kr_j) + H^{(1)}_{v_m+2}(kr_j)\}.
\end{aligned}$$

Let $k > 0$ be a wave number. Let N be the number of observation angles. Then a minimum of χ^2 in (3.8) can be found from (3.6),(3.7),(3.9),(3.10) and Algorithm 3.1 in Section 3.1.

For the far field pattern applications, $\{A_n\}$ in (2.17) can be found using the true far field pattern y_{∞_m}, i.e.,

$$A_n = \sqrt{\frac{\pi k}{2}}e^{i(\frac{v_m}{2}\pi + \frac{\pi}{4})}\sum_{m=1}^{N}P^T_{nm}y_{\infty_m}, \qquad 1 \leq n \leq N. \tag{3.11}$$

3.3. Iterative Method

Let the scattered field u^s in (2.14) be

$$u^s = [\ u^s(r_1) \quad u^s(r_2) \quad \cdots \quad u^s(r_N)\ ]^T, \tag{3.12}$$

and the true scattered field measurements be

$$Y = [\ y_1 \quad y_2 \quad \cdots \quad y_N\ ]^T \in \mathbb{C}^N, \tag{3.13}$$

where y_i is the true scattered field complex value measured at the location (R, θ_i).

We consider the following cost functional:

$$\chi^2(r) = \sum_{n=1}^{N} |y_n - u^s(r_n)|^2. \tag{3.14}$$

The gradient of χ^2 with respect to the distance vector r in (3.1) becomes

$$\begin{aligned} \frac{\partial \chi^2}{\partial r_j} &= \sum_{n=1}^{N} \left\{ -\frac{\partial u_n^s}{\partial r_j} \overline{(y_n - u_n^s)} - (y_n - u_n^s) \frac{\partial \bar{u}_n^s}{\partial r_j} \right\} \\ &= -2Re \left\{ \sum_{n=1}^{N} \frac{\partial u_n^s}{\partial r_j} \overline{(y_n - u_n^s)} \right\}, \qquad 1 \le j \le N, \end{aligned} \tag{3.15}$$

From (2.14), the derivative of u^s with respect to the distance vector r in (3.1) at ∂D,

$$\left[\frac{\partial u^s}{\partial r} \right]_{(n,j)} := \frac{\partial u_n^s}{\partial r_j} = \sum_{m=1}^{N} P_{nm} H_{v_m}^{(1)}(kR) \frac{\partial A_m}{\partial r_j},$$

where the derivative $\frac{\partial A}{\partial r}$ of $A = [A_1 \ A_2 \ \cdots \ A_N]^T \in \mathbb{C}^N$ is obtained from the following theorem.

Theorem 3.2. *The derivative of A with respect to the distance vector r in (3.1) at ∂D has the following relation.*

$$\begin{aligned} &\sum_{m=1}^{N} P_{nm} H_{v_m}^{(1)}(kr_n) \frac{\partial A_m}{\partial r_j} \\ &= \begin{cases} -ik\cos(\theta_n - \alpha) e^{ikr_n \cos(\theta_n - \alpha)} & \\ \qquad -\frac{1}{2}k \sum_{m=1}^{N} P_{nm} \left(H_{v_m-1}^{(1)}(kr_n) - H_{v_m+1}^{(1)}(kr_n) \right) A_m, & j = n, \\ \qquad 0, & j \neq n. \end{cases} \end{aligned}$$

Proof. In (2.15), let

$$E = [\ E_1\ E_2\ \cdots\ E_n\ \cdots\ E_N\ ]^T, \qquad S = [\ S_1\ S_2\ \cdots\ S_n\ \cdots\ S_N\ ]^T,$$

where

$$E_n = -e^{ikr_n\cos(\theta_n-\alpha)}, \qquad S_n = \sum_{m=1}^{N} P_{nm} H^{(1)}_{v_m}(kr_n) A_m.$$

Then, the derivative of E with respect to the distance vector r becomes

$$\frac{\partial E}{\partial r} = \left[\ \frac{\partial E}{\partial r_1}\ \ \frac{\partial E}{\partial r_2}\ \ \cdots\ \ \frac{\partial E}{\partial r_j}\ \ \cdots\ \ \frac{\partial E}{\partial r_N}\ \right],$$

$$\frac{\partial E}{\partial r_j} = \left[\ \frac{\partial E_1}{\partial r_j}\ \ \frac{\partial E_2}{\partial r_j}\ \ \cdots\ \ \frac{\partial E_n}{\partial r_j}\ \ \cdots\ \ \frac{\partial E_N}{\partial r_j}\ \right]^T,$$

where $\frac{\partial E_n}{\partial r_j}$ is,

$$\frac{\partial E_n}{\partial r_j} = \begin{cases} -ik\cos(\theta_n-\alpha)e^{ikr_n\cos(\theta_n-\alpha)}, & j = n. \\ 0, & j \neq n, \end{cases}$$

The derivative of S with respect to the distance vector r becomes

$$\frac{\partial S}{\partial r} = \left[\ \frac{\partial S}{\partial r_1}\ \ \frac{\partial S}{\partial r_2}\ \ \cdots\ \ \frac{\partial S}{\partial r_j}\ \ \cdots\ \ \frac{\partial S}{\partial r_N}\ \right],$$

$$\frac{\partial S}{\partial r_j} = \left[\ \frac{\partial S_1}{\partial r_j}\ \ \frac{\partial S_2}{\partial r_j}\ \ \cdots\ \ \frac{\partial S_n}{\partial r_j}\ \ \cdots\ \ \frac{\partial S_N}{\partial r_j}\ \right]^T,$$

where $\frac{\partial S_n}{\partial r_j}$ is,

$$\frac{\partial S_n}{\partial r_j} = \begin{cases} \sum_{m=1}^{N} P_{nm}\left(\frac{\partial H^{(1)}_{v_m}(kr_n)}{\partial r_j}A_m + H^{(1)}_{v_m}(kr_n)\frac{\partial A_m}{\partial r_j}\right), & j = n, \\ \\ \sum_{m=1}^{N} P_{nm} H_{v_m}(kr_n)\frac{\partial A_m}{\partial r_j}, & j \neq n. \end{cases}$$

Because of $\frac{\partial S}{\partial r} = \frac{\partial E}{\partial r}$, we obtain

$$\begin{cases} -\sum_{m=1}^{N} P_{nm} H^{(1)}_{v_m}(kr_n)\frac{\partial A_m}{\partial r_j} = & \\ ik\cos(\theta_n-\alpha)e^{ikr_n\cos(\theta_n-\alpha)} + \sum_{m=1}^{N} P_{nm}\frac{\partial H^{(1)}_{v_m}(kr_n)}{\partial r_j}A_m, & j = n, \\ \\ \sum_{m=1}^{N} P_{nm} H^{(1)}_{v_m}(kr_n)\frac{\partial A_m}{\partial r_j} \ = \ 0, & j \neq n. \end{cases}$$

From the following relation[3]:

$$\frac{\partial}{\partial x} H_v^{(1)}(x) = \frac{1}{2}\left(H_{v-1}^{(1)}(x) - H_{v+1}^{(1)}(x)\right),$$

we have,

$$\sum_{m=1}^{N} P_{nm} H_{v_m}^{(1)}(kr_n) \frac{\partial A_m}{\partial r_j}$$

$$= \begin{cases} -ik\cos(\theta_n - \alpha) e^{ikr_n \cos(\theta_n - \alpha)} & \\ \qquad -\frac{1}{2}k \sum_{m=1}^{N} P_{nm} \left(H_{v_m-1}^{(1)}(kr_n) - H_{v_m+1}^{(1)}(kr_n)\right) A_m, & j = n, \\ \qquad 0, & j \neq n. \end{cases}$$

□

To obtain the Hessian matrix D in (3.3), we take an additional partial derivative for $\frac{\partial \chi^2}{\partial r_i}$ with respect to r_j, $i, j = 1, 2, 3, \cdots, N$.

$$\begin{aligned} \frac{\partial}{\partial r_j}\left(\frac{\partial \chi^2}{\partial r_i}\right) &= -\sum_{n=1}^{N}\left\{\frac{\partial}{\partial r_j}\left(\frac{\partial u_n^s}{\partial r_i}\right)\overline{(y_n - u_n^s)} - \frac{\partial u_n^s}{\partial r_i}\frac{\partial \bar{u}_n^s}{\partial r_j}\right. \\ &\qquad \left. + \frac{\partial}{\partial r_j}\left(\frac{\partial \bar{u}_n^s}{\partial r_i}\right)(y_n - u_n^s) - \frac{\partial \bar{u}_n^s}{\partial r_i}\frac{\partial u_n^s}{\partial r_j}\right\}. \end{aligned} \tag{3.16}$$

Note that the components $\frac{\partial}{\partial r_j}\left(\frac{\partial \chi^2}{\partial r_i}\right)$ of the Hessian matrix depend both on the first derivatives and on the second derivatives of u^s with respect to r. We will ignore the second derivative of u^s. Because the second derivative term can be small enough to be negligible when compared to the term involving the first derivative[7]. So, (3.16) becomes, for $i, j = 1, 2, \cdots, N$

$$\begin{aligned} \frac{\partial}{\partial r_j}\left(\frac{\partial \chi^2}{\partial r_i}\right) &\approx \sum_{n=1}^{N}\left\{\frac{\partial u_n^s}{\partial r_i}\frac{\partial \bar{u}_n^s}{\partial r_j} + \frac{\partial \bar{u}_n^s}{\partial r_i}\frac{\partial u_n^s}{\partial r_j}\right\} \\ &= 2Re\sum_{n=1}^{N}\left\{\frac{\partial u_n^s}{\partial r_i}\frac{\partial \bar{u}_n^s}{\partial r_j}\right\}. \end{aligned} \tag{3.17}$$

Let $k > 0$ be the wave number. Let N be the number of observation angles. Let Y be the target scattered field measurements as in (3.13). Then a minimum of χ^2 in (3.14) can be found from (3.12),(3.15),(3.17) and Algorithm 3.1 in Section 3.1.

For the far field pattern applications, χ^2 in (3.14) becomes

$$\chi^2(r) = \sum_{n=1}^{N} |y_{\infty_n} - u_{\infty_n}|^2, \tag{3.18}$$

where y_∞ is the true measured far field pattern and u_∞ is the estimated far field pattern. The derivative of u_∞ with respect to the distance vector r in (3.1) at ∂D,

$$\left[\frac{\partial u_\infty}{\partial r}\right]_{(n,j)} := \frac{\partial u_{\infty_n}}{\partial r_j} = \sqrt{\frac{2}{\pi k}} e^{-i(\frac{v_m}{2}\pi + \frac{\pi}{4})} \sum_{m=1}^{N} P_{nm} \frac{\partial A_m}{\partial r_j},$$

where $\frac{\partial A_m}{\partial r_j}$ can be obtained from the Theorem 3.2

4. Examples

We consider the following circle with center $(1, 1)$ and radius 6. The boundary of the circle can be represented in polar form as

$$\partial D \ : \ r = \cos\theta + \sin\theta + \sqrt{35 + 2\cos\theta\sin\theta}, \qquad 0 \le \theta \le 2\pi.$$

The wave number $k = 1$ was chosen for the numerical experiments. The starting regularization parameter $\lambda > 0$ in (3.5) was chosen as 0.001.

In the direct method, the stopping criterion for global iteration, we used the following sequential difference:

$$\chi^2(r^{(n)}) - \chi^2(r^{(n+1)}) < \text{tolerance} \tag{4.1}$$

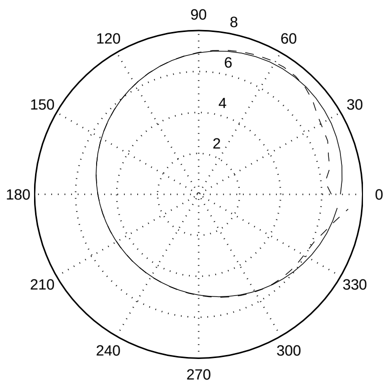
with tolerance 10^{-2}, where $r^{(n)}$ is the n-th iteration distance vector and χ^2 is defined in (3.8).

The number of observation angles $N = 64$, the incident angle $\alpha = 0$, the observation distance $R = 15$, and the initial guess for $r = 5$ were chosen. Table 1 shows the convergence of χ^2 for the scattered field by the direct method.

Table 1: Convergence of χ^2 (Noise 0%, $\alpha = 0$, scattered field)

Iter	χ^2	$\chi^2 Seq$	Iter	χ^2	$\chi^2 Seq$
0	64.8948	64.8948	5	0.1826	0.4104
1	19.7524	45.1424	6	0.1014	0.0812
2	6.4980	13.2544	7	0.0939	0.0075
3	2.0473	4.4507	8	0.0937	0.0002
4	0.5930	1.4543			

Figures 1-4 show the reconstruction results for the scattered field using the direct method with incident angle $\alpha = 0,\ \pi,\ \pi/2,\ \pi/3$, respectively.



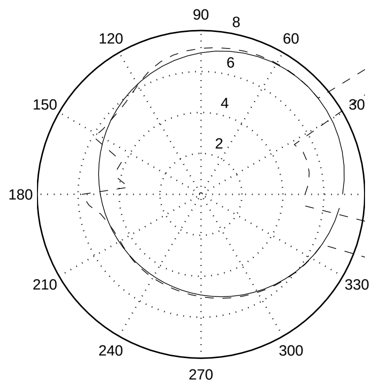
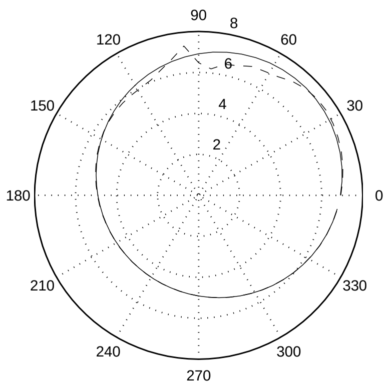


Figure 1. Shape ($\alpha = 0$)

Figure 2. Shape ($\alpha = \pi$)

(scattered field, exact(solid line), reconstruction(dashed line))



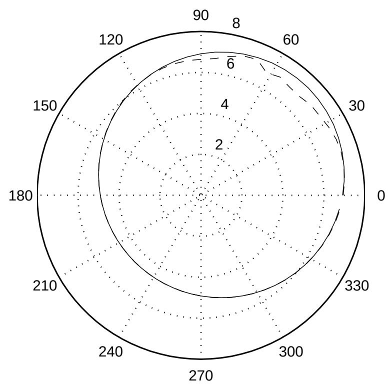


Figure 3. Shape ($\alpha = \frac{\pi}{2}$)

Figure 4. Shape ($\alpha = \frac{\pi}{3}$)

(scattered field, exact(solid line), reconstruction(dashed line))

For the far field pattern applications, the number of observation angles $N = 64$, the incident angle $\alpha = 0$, and the initial guess for $r = 5$ were chosen.

Table 2 shows the convergence of χ^2 for the far field pattern by the direct method.

Table 2: Convergence of χ^2 (Noise 0%, $\alpha = 0$, far field pattern)

Iter	χ^2	$\chi^2 Seq$	Iter	χ^2	$\chi^2 Seq$
0	64.7878	64.7878	5	0.3238	0.4271
1	19.6410	45.1468	6	0.2236	0.1001
2	6.5424	13.0986	7	0.2087	0.0149
3	2.1900	4.3525	8	0.2078	0.0009
4	0.7509	1.439			

Figures 5-8 show the reconstruction results for the far field pattern using the direct method and incident angle $\alpha = 0,\ \pi,\ \pi/2,\ \pi/3$, respectively.

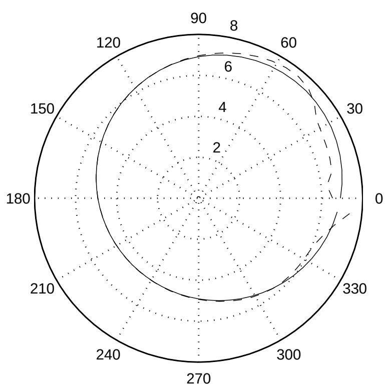


Figure 5. Shape ($\alpha = 0$)

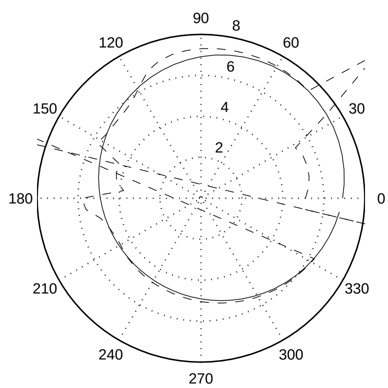
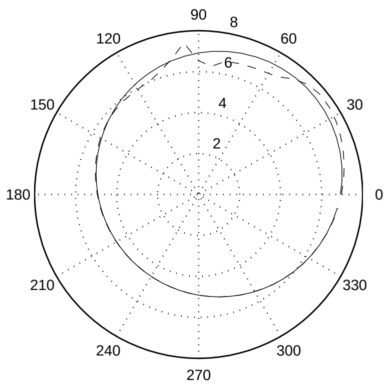


Figure 6. Shape ($\alpha = \pi$)

(far field pattern, exact(solid line), reconstruction(dashed line))

The random noise was added as 5% to the exact data. We observed that if the number of observation angles N is greater than or equal to 64, χ^2 in (3.8)



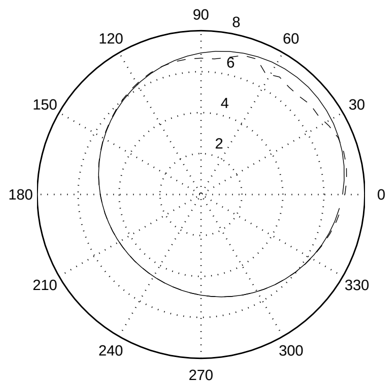


Figure 7. Shape ($\alpha = \frac{\pi}{2}$) Figure 8. Shape ($\alpha = \frac{\pi}{3}$)

(far field pattern, exact(solid line), reconstruction(dashed line))

diverged. The number of observation angles $N = 32$, the incident angle $\alpha = 0$, the observation distance $R = 15$, and the initial guess for $r = 5$ were chosen. Tables 3-4 show the convergence of χ^2 for the scattered field and the far field pattern, respectively.

Table 3: Convergence of χ^2 (Noise 5%, $\alpha = 0$, scattered field)

Iter	χ^2	$\chi^2 Seq$	Iter	χ^2	$\chi^2 Seq$
0	44.3170	44.3170	4	7.4208	0.6650
1	19.3376	24.9794	5	7.1784	0.2424
2	10.4686	8.8690	6	7.1008	0.0776
3	8.0857	2.3829	7	7.0829	0.0179

Figures 9-10 show the reconstruction results using the direct method for the scattered field and the far field pattern, respectively.

Table 4: Convergence of χ^2 (Noise 5%, $\alpha = 0$, far field pattern)

Iter	χ^2	$\chi^2 Seq$	Iter	χ^2	$\chi^2 Seq$
0	47.3650	47.3650	4	7.8922	0.9541
1	22.2199	25.1452	5	7.6466	0.2456
2	12.5555	9.6644	6	7.5521	0.0945
3	8.8463	3.7092	7	7.5411	0.0110

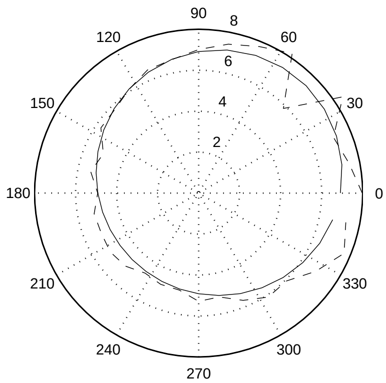


Figure 9. Shape(scattered field)

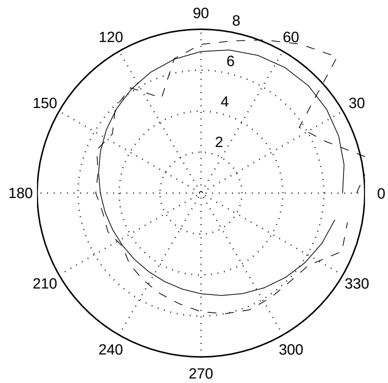


Figure 10. Shape(far field pattern)

(Noise 5%, $\alpha = 0$, exact(solid line), reconstruction(dashed line))

In the iterative method, the following relative residual stopping criterion for global iteration was used,

$$Res(r) = \frac{\left[\sum_{i=1}^{N} |y_i - u^s(r_i)|^2\right]^{\frac{1}{2}}}{\left[\sum_{i=1}^{N} |y_i|^2\right]^{\frac{1}{2}}}.$$

The global iteration was terminated when

$$Res(r^{(n)}) - Res(r^{(n+1)}) < \text{tolerance} \tag{4.2}$$

with tolerance 10^{-2}, where $r^{(n)}$ is the distance vector from the n-th iteration.

The number of observation angles $N = 64$, the incident angle $\alpha = 0$, the observation distance $R = 15$, and the initial guess for $r = 5.5$ were chosen. Table 5 shows the convergence of residual for the scattered field by the iterative method.

Table 5: Convergence of residual (Noise 0%, $\alpha = 0$, scattered field)

Iter	χ^2	Res	$ResSeq$	Iter	χ^2	Res	$ResSeq$
0	8.0647	0.6531	0.6531	5	0.4227	0.1495	0.0117
1	2.9656	0.3960	0.2570	6	0.0390	0.0454	0.1041
2	1.3599	0.2682	0.1278	7	0.0171	0.0301	0.0153
3	1.1006	0.2413	0.0269	8	0.0089	0.0217	0.0084
4	0.4914	0.1612	0.0800				

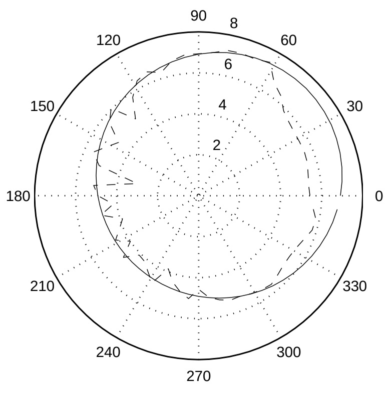


Figure 11. Shape ($\alpha = 0$)

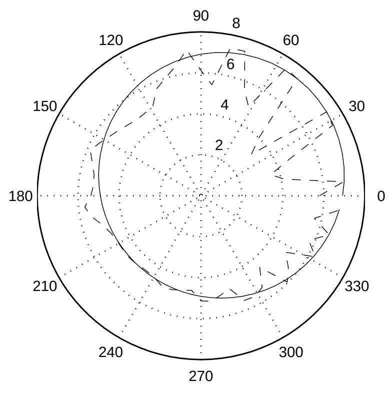
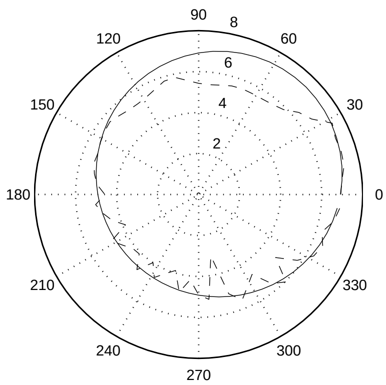


Figure 12. Shape ($\alpha = \pi$)

(scattered field, exact(solid line), reconstruction(dashed line))



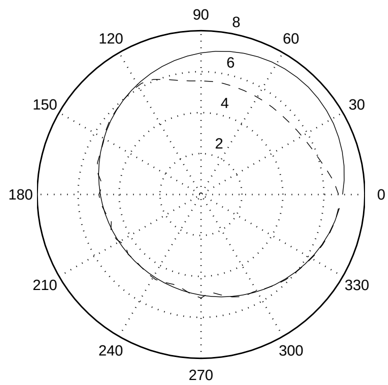


Figure 13. Shape ($\alpha = \frac{\pi}{2}$) Figure 14. Shape ($\alpha = \frac{\pi}{3}$)

(scattered field, exact(solid line), reconstruction(dashed line))

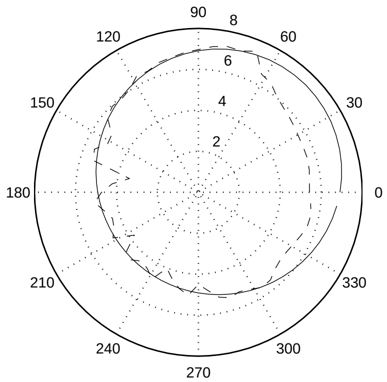
Figures 11-14 show the reconstruction results for the scattered field using the iterative method and incident angle $\alpha = 0,\ \pi,\ \pi/2,\ \pi/3$, respectively.

For the far field pattern applications, the number of observation angles $N = 64$, the incident angle $\alpha = 0$, the initial guess for $r = 5.5$ were chosen. Table 6 shows the convergence of residual for the far field pattern by the iterative method.

Table 6: Convergence of residual (Noise 0%, $\alpha = 0$, far field pattern)

Iter	χ^2	Res	$ResSeq$	Iter	χ^2	Res	$ResSeq$
0	118.1380	0.6546	0.6546	4	0.9590	0.0590	0.0813
1	62.4856	0.4761	0.1785	5	0.3910	0.0377	0.0213
2	14.2059	0.2270	0.2491	6	0.3876	0.0375	0.0002
3	5.4254	0.1403	0.0867				

Figures 15-18 show the reconstruction results for the far field pattern using the iterative method and incident angle $\alpha = 0,\ \pi,\ \pi/2,\ \pi/3$, respectively.



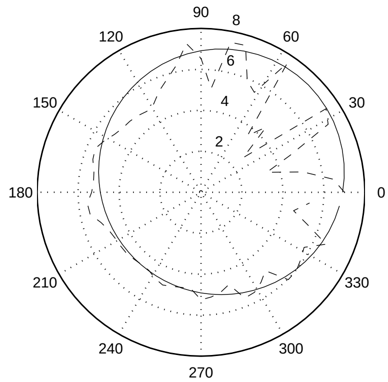
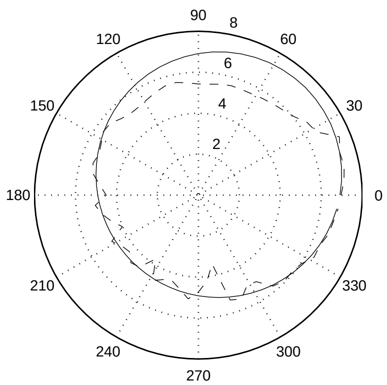


Figure 15. Shape ($\alpha = 0$) Figure 16. Shape ($\alpha = \pi$)

(far field pattern, exact(solid line), reconstruction(dashed line))



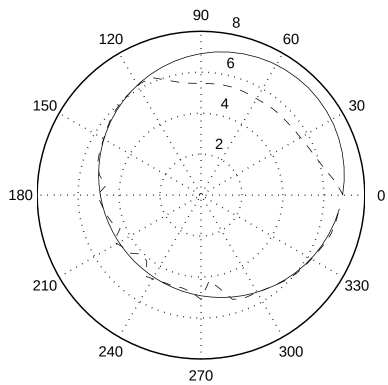


Figure 17. Shape ($\alpha = \frac{\pi}{2}$) Figure 18. Shape ($\alpha = \frac{\pi}{3}$)

(far field pattern, exact(solid line), reconstruction(dashed line))

The random noise was added as 5% to the exact data. The number of observation angles $N = 64$, the incident angle $\alpha = 0$, the observation distance

$R = 15$, and the initial guess for $r = 5.5$ were chosen. Tables 7-8 show the convergence of χ^2 for the scattered field and the far field pattern, respectively.

Table 7: Convergence of residual (Noise 5%, $\alpha = 0$, scattered field)

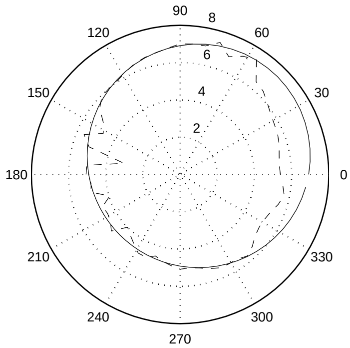
Iter	χ^2	Res	$ResSeq$	Iter	χ^2	Res	$ResSeq$
0	8.0309	0.6511	0.6511	4	0.0372	0.0443	0.0673
1	2.9059	0.3916	0.2594	5	0.0279	0.0384	0.0059
2	0.8093	0.2067	0.1850	6	0.0258	0.0369	0.0015
3	0.2360	0.1116	0.0951	7	0.0256	0.0367	0.0002

Table 8: Convergence of residual (Noise 5%, $\alpha = 0$, far field pattern)

Iter	χ^2	Res	$ResSeq$	Iter	χ^2	Res	$ResSeq$
0	119.9382	0.6573	0.6573	4	0.9180	0.0575	0.0408
1	57.4011	0.4547	0.2026	5	0.5841	0.0459	0.0116
2	11.2586	0.2014	0.2534	6	0.5627	0.0450	0.0008
3	2.6849	0.0984	0.1030				

Figures 19-20 show the reconstruction results using the iterative method for the scattered field and the far field pattern, respectively. Comparing with Figures 11 and 15, one can observe that the reconstruction accuracy is similar to those of without noise.

From Tables 1-8, one can see that the sequence convergence for stopping criterion in (4.1) and (4.2) has achieved in 6-8 iterations under the tolerance



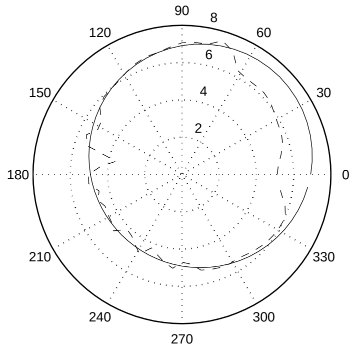


Figure 19. Shape(scattered field) Figure 20. Shape(far field pattern)

(Noise 5%, $\alpha = 0$, exact(solid line), reconstruction(dashed line))

10^{-2}. From Figures 1-20, we observed that the reconstruction errors were related to the incident angles.

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