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博士學位論文

A STUDY ON THE NONEXISTENCE OF CONFORMAL DEFORMATIONS ON SPACE-TIMES WITH PRESCRIBED SCALAR CURVATURES

朝鮮大學校　大學院

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- 주어진 스칼라 曲率을 갖는 時空間 위에서 等角 變形의 非存在性에 관한 硏究 -

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주어진 스칼라 曲率을 갖는 時空間 위에서 等角 變形의 非存在性에 관한 硏究

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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다.

연구방법으로는 일반적으로 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

N이 compact Riemannian manifold일때, 휜다양체 $M=(0,\infty)\times N$위에 적당한 함수가 주어지면 그 함수를 scalar curvature로 갖는 등각 변형이 존재하는지를 연구하였다. 이는 적당한 편미분 방정식의 해가 존재하는지에 따라 결정된다.

제 1장은 최근 연구 동향을 소개하였다.

제 2장은 휜다양체의 정의 및 여러 가지 성질, 그리고 연구해야 할 편미분방정식을 유도하였다.

제 3장에서는 엽다양체의 scalar curvature이 음의 상수일때, 주어진 함수가 적당한 조건을 만족하면 등각변형에 의한 metric의 스칼라 곡률이 존재하지 않음을 보였다.

1. Introduction

One of the basic problems in the differential geometry is to study the set of curvature funtions over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwise conformal to some metric with constant scalar curvature or not.

In a recent study [14, 15, 16], Leung has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. In this paper, we study the existence and nonexistence of Lorentzian warped metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner [10, 11, 12], if N is a compact Riemannian n-manifold without boundary $n \geq 3$, then N belongs to one of the following three categories:

(A) A smooth function on N is the scalar curvature of some Reimannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Reimannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Reimannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [10, 11, 12], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvature of complete Riemannian metrics on an open manifold [3, 6, 7, 13, 14]. Results of Gromov and Lawson [9] show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded [9], [13,p.322].

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature [7]. It follows from the results of Aviles and McOwen [3] that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In [14,15], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [8], the authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifolds, results in [8], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [8] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold, we consider the nonexistence of conformal deformations on a warped product manifold $M = (a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. That is, it is shown that if the fiber manifold N belongs to class (A) then M does not admit a Lorentzian metric with some positive scalar curvature near the end outside a compact set.

2. Preliminaries on a warped product manifold

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\Im(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(D1) $\nabla_V W$ is $\Im$-linear in V.

(D2) $\nabla_V W$ is R-linear in W.

(D3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \Im(M)$.

(D4) $[V, W] = \nabla_V W - \nabla_W V$.

(D5) $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$ for all $X, V, W \in \mathfrak{X}(M)$.

If ∇ satisfies axioms (D1) $\sim$ (D3), then $\nabla_V W$ is called the covariant derivative of W with respect to V for the connection ∇. If ∇ satisfies axioms (D4) $\sim$ (D5), then ∇ is called the Levi - Civita connection of M, which is characterized by the Koszul formula ([17]).

As indicated above, (D4) is the condition that ∇ is torsion free, and (D5) is the condition that the connection ∇ is compatible with metric g.

For semi-Riemannian manifolds, the connection coefficients are given by

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{a=1}^{n} g^{ak}\left(\frac{\partial g_{ia}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{a}} + \frac{\partial g_{aj}}{\partial x^{i}}\right) \tag{2.1}$$

where (g^{ij}) represents the (2,0) tensor defined by

$$\sum_{a=1}^{n} g^{ia} g_{aj} = \delta_{j}^{i} \quad for \quad 1 \le i, j \le n.$$

Definition 2.2. The curvature tensor of the connention ∇ is a linear transformation valued tensor R in $\mathrm{Hom}(\mathfrak{X}(M), \mathfrak{X}(M))$ defined by :

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathfrak{X}(M)$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well-known that $R(X, Y)Z$ at p depends only upon the values of X, Y, and Z at p ([17]).

If $\omega \in T_p^*(M)$ is a cotangent vector at p and $x, y, z \in T_p(M)$ are tangent vectors at P, then one defines

$$R(\omega, x, y, z) = (\omega, R(X, Y)Z) = \omega(R(X, Y)Z)$$

for X, Y, and Z smooth vector fields extending x, y, and z, respectively.

The curvature tensor R is a (1,3) tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

Notice that

$$R(X, Y)Z = -R(Y, X)Z, \quad R(\omega, X, Y, Z) = -R(\omega, Y, X, Z)$$

and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \Sigma \frac{X^i \partial}{\partial x^i}$, $Y = \Sigma \frac{Y^i \partial}{\partial x^i}$, $Z = \Sigma \frac{Z^i \partial}{\partial x^i}$ and $\omega = \Sigma \omega_i dx^i$ then

$$R(X, Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} Z^j X^k Y^m \frac{\partial}{\partial x^i}$$

and

$$R(\omega, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \omega_i Z^j X^k Y^m.$$

Consequently, one has $R(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}) = R^i_{jkm}$.

The local representations g^{ij} and g_{ij} may be used to raise and lower indices.

For example, if the upper index of the curvature tensor is lowered, one obtains the components of the Riemann-Christoffel tensor which is also known as the covariant curvature tensor

$$R_{ijkm} = \sum_{a=1}^{n} g_{ai} R^a_{jkm}.$$

Alternatively, one may define the Riemann-Christoffel tensor $\widetilde{R}$ as the (0, 4) tensor such that

$$\widetilde{R}(W, Z, X, Y) = g(W, R(X, Y)Z).$$

Some standard curvature identities satisfied by the components of this tensor are

$$R_{ijkm} = R_{kmij} = -R_{jikm} = -R_{ijmk}$$

and

$$R_{ijkm} + R_{ikmj} + R_{imjk} = 0.$$

The trace of the curvature tensor is the *Ricci curvature*, a symmetric (0, 2) tensor. For each $p \in M$, the Ricci curvature may be interpreted as a symmetric bilinear map $\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R}$. To evaluate $Ric(v, w)$, let $e_1, e_2, \cdots, e_n$ be an orthonormal basis for T_pM. Then

$$Ric(v, w) = \sum_{i=1}^{n} g(e_i, e_i) g(R(e_i, w)v, e_i).$$

or equivalently,

$$Ric(v, w) = \sum_{i=1}^{n} g(e_i, e_i) \widetilde{R}(e_i, v, e_i, w).$$

One may express v and w in the natural basis as $v = \sum v^i \frac{\partial}{\partial x^i}$ and $w = \sum w^i \frac{\partial}{\partial x^i}$ and then write

$$Ric(v, w) = \sum_{i,j=1}^{n} R_{ij} v^i w^j$$

where

$$R_{ij} = \sum_{a=1}^{n} R^a_{iaj}. \tag{2.2}$$

Definition 2.3. A semi-Riemannian metric g for a manifold M is a smooth symmetric tensosr field of type (0,2) on M which assign to each point $p \in M$ a nondegenerate inner product $g|_p : T_p(M) \times T_p(M) \to \mathbb{R}$ of signature $(-, \cdots, -, +, \cdots, +)$.

Here nondegenerate means that for each nontrivial vector $v \in T_p(M)$ there is some $w \in T_p(M)$ such that $g_p(v, w) \neq 0$. If g has components g_{ij} in local coordinates, then the nondegeneracy assumption is equivalent to the condition that the determinant of the matrix (g_{ij}) is nonzero.

In local coordinates $(U, (x^1, x^2, \cdots, x^n))$ on M, the metric g is represented by

$$g|U = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j$$

with

$g_{ij} = g_{ji}$ and $\quad det(g_{ij}) \neq 0.$

If g has s negative eigenvalues and $r = n - s$ positive eigenvalues, then the signature of g will be denoted by (s, r). For each fixed $p \in M$, there exist local coordinates $(U, (x^1, x^2, \cdots, x^n))$ such that $g_p = g|T_p(M)$ can be represented as the diagonal matrix diag $\{-, \cdots, -, +, \cdots, +\}$. For each semi-Riemannian manifold (M, g) there is an associated semi-Riemannian manifold $(M, -g)$ obtained by replacing g with $-g$. Aside from some minor changes in sign, there is no essential difference between (M, g) and $(M, -g)$. Thus, results for spaces of signature (s, r) may always be translated into corresponding results for spaces of signature (r, s) by appropriate sign changes and inequality reversals.

Definition 2.4. The trace of the Ricci curvature is the scalar curvature τ. That is, $S = \sum_{ij=1}^{n} R_{ij} g^{ij}$.

Thus if $e_1, e_2, \cdots, e_n$ is an orthonormal basis of T_pM, one has

$$\tau = R = \sum_{i=1}^{n} g(e_i, e_i) Ric(e_i, e_i).$$

The gradient and Hessian are defined for semi-Riemannian manifolds just as for Riemannian manifolds. If $f : M \to \mathbb{R}$ is a smooth function, then df is a (0, 1) tensor field (i.e., one-form) on M, and grad f is the (1,0) tensor field (i.e., vector field) which corresponds to df. Thus,

$$Y(f) = df(Y) = g(\text{grad } f, Y)$$

for an arbitrary vector field Y. In local coordinates $(U, (x^1, x^2, \cdots, x^n))$, the vector field grad f is represented by

$$\text{grad } f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

The Hessian H^f is defined to be the second covariant differential of f:

$$H^f = \nabla(\nabla f).$$

For a given $f \in \Im(M)$, the Hessian H^f is a symmetric (0,2) tensor field which is related to the gradient of f through the formula

$$H^f(X, Y) = g(\nabla_x(\text{grad } f), Y)$$

for arbitrary vector fields X and Y.

Definition 2.5. (Laplacian) The Laplacian of f is defined to be the divergence of the gradient of f. That is, $\triangle f = \mathrm{div}$ (grad f). The Laplacian in a local chart can be written as follows:

$$\begin{aligned}\Delta\varphi &= \nabla_i(g^{ij}\nabla_j\varphi) \\ &= \partial_i(g^{ij}\partial_j\varphi) + g^{kj}\partial_j\varphi\Gamma^i_{ik} \\ &= |g|^{-\frac{1}{2}}\partial_i[g^{ij}\sqrt{|g|}\partial_j\varphi],\end{aligned}$$

because $\Gamma^i_{ik} = \partial_k \log\sqrt{|g|}$ ([2,17]).

Definition 2.6. A tangent vector $v \in T_p(M)$ is classified as timelike, nonspacelike, null or spacelike if $g(v,v)$ is negative, nonpositive, zero, or positive, respectively:

(A) $g(v,v) < 0$ (timelike).

(B) $g(v,v) \leq 0$ (nonspacelike or causal).

(C) $g(v,v) = 0$ (null or lightlike).

(D) $g(v,v) > 0$ (spacelike).

The set of all null vectors in $T_p(M)$ is called the nullcone at $p \in M$ null vectors are also said to be lightlike.

A Lorentzian manifold is a semi-Riemannian manifold (M, g) of signature $(1, n-1)$ [$i.e., (-, +, \cdots, +)$]. At each point $p \in M$ the induced metric on the tangent space is Minkowskian. Each point of a Lorentzian manifold has timelike, null, and spacelike tangent vectors. A smooth curve is said to be timelike, null, or spacelike if its tangent vectors are always timelike, null, or spacelike, respectively.

A timelik curve in a Lorentzian manifold corresponds to the path of an observer moving at less than the speed of light. Null curves correspond to moving at the speed of light, and spacelike curves correspond to the geometric equivalent of moving faster than light, spacelike curves are of clear geometric interest.

Definition 2.7. A Lorentzian manifold (M, g) is a connected smooth manifold of dimension ≥ 2 with a countable basis together with a smooth Lorentzian metric g of signature $(-, +, +, +, \cdots, +)$ ([4]).

Definition 2.8. A vector field X on M is timelike if $g(X(p), X(p)) < 0$ at all points of $p \in M$. A Lorentzian manifold with a given timelike vector field X is said to be timeoriented by X. A space-time is a time oriented Lorentzian manifold.

Theorem 2.9. ([2]) *Let (M, g) be a semi-Riemannian manifold with scalar curvature R. Let $g' = e^f g$ be a conformal metric with $f \in C^\infty(M)$ and R' be a scalar curvature of g'. Then R' is given by*

$$R' = e^{-f}\ [R + (n-1)\Delta f - \frac{(n-1)(n-2)}{4}|\nabla f|^2].$$

Proof. By equation (2.1), if Γ'^{l}_{ij} and Γ^{l}_{ij} denote the Christoffel symbols relating to g' and g respectively:

$$\begin{aligned}\Gamma'^{l}_{ij} - \Gamma^{l}_{ij} &= \frac{1}{2}\left[g_{kj}\partial_i f + g_{ki}\partial_j f - g_{ij}\partial_k f\right] g^{kl} \\ &= \frac{1}{2}\left[\delta^l_j \partial_i f + \delta^l_i \partial_j f - g_{ij}\nabla^l f\right].\end{aligned}$$

According to equation (2.2),

$$\begin{aligned}R'_{ij} = R'^{k}_{ikj} = R_{ij} &- \frac{n-2}{2}\nabla_{ij} f + \frac{n-2}{4}\nabla_i f \nabla_j f \\ &- \frac{1}{2}\left(-\Delta f + \frac{n-2}{2}|\nabla f|^2\right) g_{ij}\end{aligned}$$

so

$$R' = e^{-f}\left[R + (n-1)\nabla^v_v f - \frac{(n-1)(n-2)}{4}\nabla^v f \nabla_v f\right].$$

■

If we consider the conformal deformation in the form $g' = \varphi^{\frac{4}{n-2}}$ (with $\varphi \in C^\infty, \varphi > 0$), the scalar curvature R' satisfies the equation:

$$4((n-1)/(n-2))\Delta\varphi + R\varphi + R'\varphi^{(n+2)/(n-2)} = 0, \tag{2.3}$$

where $\quad \Delta\varphi = \nabla^v \nabla_v \varphi$.

So Yamabe problem is equivalent to solve the equation (2.3) with constant R', and the solution φ must be smooth and strictly positive ([1, 19]).

We briefly recall some results on warped product manifolds. Complete details may be found in ([3]), or ([17]). On a semi-Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.10. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)\sigma^*(g_F)$$

where g_B and g_F are metric tensors of B and F, respectively. In other words, if v is tangent to M at (p, q), then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f(p)g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base manifold of M and F the fiber manifold ([17]).

Remark 2.11. Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows ;

(1) For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B\times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(p)=p\times F}$ is positive homothetic onto F with scale factor $\frac{1}{\sqrt{f(p)}}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts play a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p, q)

by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F. Similarly, if Y is a vector field on F, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

Lemma 2.12. *If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B.*

Proof. We must show that grad $(h \circ \pi)$ is horizonal and π-related to grad h on B. If v is vertical tangent vector to M, then

$$(\text{grad }(h \circ \pi), v) = v(h \circ \pi) = d\pi(v)h = 0, \quad \textit{since} \quad d\pi = 0.$$

Thus grad $(h \circ \pi)$ is horizonal. If x is horizonal,

$$\begin{aligned} g(d\pi(\text{grad }(h \circ \pi)), d\pi(x)) &= g(\text{grad }(h \circ \pi), x) \\ &= x(h \circ \pi) \\ &= d\pi(x)h \\ &= g(\text{grad } h, d\pi(x)). \end{aligned}$$

Hence at each point, $d\pi(\text{grad }(h \circ \pi)) = \text{grad } h$. ∎

We denote the metric g by $<,>$. In view of Remark 2.11 (1) and Lemma 2.12, we may also denote the metric g_B by $<,>$. The metric g_F will be denoted by $(,)$.

In view of Lemma 2.12, we simplify the notations by writing h for $h \circ \pi$ and grad (h) for grad $(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a $(1, s)$-tensor, and if $v_1, v_2, \cdots, v_s \in T_{(p,q)}(M)$ then $\overline{A}(v_1, \cdots, v_s) = A(d\pi(v_1), \cdots, d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in ([17]).

In order to induce the d'Alembertian for $M = B \times_f F$, we will consider the general warped product $(B \times_f F, g)$ where $g = \pi^*(g_B) + (f \circ \pi)\sigma^*(g_F)$, (F, g_F) is Riemannian and (B, g_B) is equipped with a metric of signature $(-, +, \cdots, +)$. Let ∇^1 denote the Levi-Civita connection for (B, g_B) and ∇^2 denote the Levi-Civita connection for (F, g_F). Recall that the connection ∇ for $(B \times_f F, g)$ is related to the metric g by the formula

$$2g\ (\nabla_X Y, Z) = Xg\ (Y, Z) + Yg\ (X, Z) - Zg\ (X, Y)$$
$$+ g\ ([X, Y], Z) - g\ ([X, Z], Y) - g\ ([Y, Z], X).$$

Using this formula and setting $\phi = \ln f$, we obtain the following formula for ∇ for vector fields $X = (X_1, 0) + (0, X_2)$ and $Y = (Y_1, 0) + (0, Y_2)$:

$$\nabla_X Y = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2 \tag{2.4}$$

$$+\frac{1}{2}\{X_1(\phi)Y_2 + Y_1(\phi)X_2 - g\ (X_2, Y_2)\ \text{grad}\ \phi\}.$$

Here grad Φ denotes the gradient of the function Φ on (B, g_B) and we are identifying the vector $(\nabla^1_{X_1} Y_1|_p) \in T_p(B)$ with the vector $(\nabla^1_{X_1} Y_1|_p, 0_q) \in T_{(p,q)}(B \times F)$, and so on.

Now, we will calculate the d'Alembertian for Lorentzian warped products using the method of separation of variables. From now on, we refer the results in [5]. Recall that if is a semi-Riemannian manifold and $\Phi : M \to \mathbb{R}$ is a smooth function, then the symmetric (0,2) Hessian tensor Hess(Φ) associated to Φ is given by

$$\text{Hess}(\Phi)(x, y) = g\ (\nabla_x\ \text{grad}\ \Phi, y) \tag{2.5}$$

for any tangent vectors $x, y \in T_pM$. The d'Alembertian operator $\Box$: $C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ may then be defined by setting

$$\Box\Phi = tr \circ \text{Hess}(\Phi) \tag{2.6}$$

or in local coordinates $(U, x^1, \cdots, x^n)$:

$$\Box\phi = |g|^{-1/2} \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \frac{\partial \phi}{\partial x^j} \right). \tag{2.7}$$

It may be verified that for $\phi_1, \phi_2 \in C^\infty(M, \mathbb{R})$

$$\Box(\phi_1 \cdot \phi_2) = \phi_1 \Box\phi_2 + 2g\ (\text{grad}\ \phi_1, \text{grad}\ \phi_2) + \phi_2 \Box\phi_1. \tag{2.8}$$

We now restrict our attention to Lorentzian warped products $M = (B \times F,\, g)$. Recall that $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ denote the projection maps given by $\pi(p,q) = p$ and $\sigma(p,q) = q$, respectively. We will use the isomorphism $T_{(p,q)}(B \times F) \cong T_pB \oplus T_qF$ to decompose vector fields X on M as $X = (X_1, X_2)$. Also since we wish to use the method of separation of variables, we will fix smooth functions $\phi_1 : B \to \mathbb{R}$ and $\phi_2 : F \to \mathbb{R}$ and set

$$\Phi = (\phi_1 \circ \pi)(\phi_2 \circ \sigma),$$

i.e., $\Phi(p,q) = \phi_1(p)\phi_2(q)$ for all $(p,q) \in M$. Letting grad Φ, $\text{grad}_B\ \phi_1$ and $\text{grad}_F\ \phi_2$ denote the gradient vector fields on (M,g), (B,g_B) and (F,g_F) respectively, it follows from the warped product formula that

$$\text{grad}\ (\phi_1 \circ \pi)(p,q) = (\text{grad}_B\ \phi_1(p), 0_q) \tag{2.9}$$

and

$$\text{grad}\ (\phi_2 \circ \sigma)(p,q) = (0_p, \frac{1}{f(p)}\ \text{grad}_F\ \phi_2(q)) \tag{2.10}$$

where 0_p and 0_q denote the zero tangent vectors of T_pB and T_qF respectively. We will let ∇, ∇^1 and ∇^2 denote the Levi-Civita connections of (M,g), (B,g_B) and (F,g_F) respectively. As an aid to calculating $\Box\Phi$, we consider a (1,1) tensor field

$$H_\Phi : T(M) \times T(M)$$

given by:

$$H_\Phi(\xi) = \nabla_\xi \text{ grad } \Phi \tag{2.11}$$

from which it follows using (2.5) that

$$\text{Hess } (\Phi)\ (\xi_1, \xi_2) = g\ (H_\Phi(\xi_1), \xi_2) \tag{2.12}$$

for all $\xi_1, \xi_2 \in T_{(p,q)}M$. The following proposition may now be established using (2.4).

Proposition 2.13. *Let $\xi = (v, w) \in T_{(p,q)}(M)$. Then,*

$$H_{\phi_1 \circ \pi}(\xi) = \left(H_{\phi_1}(v), \frac{1}{2f(p)} (\text{grad}_B\ \phi_1)|_p(f)w \right)$$

and

(2.13)

$$H_{\phi_2 \circ \eta}(\xi) = \left(-\frac{w(\phi_2)}{2f(p)} \text{ grad } f(p), \frac{1}{f(p)} H_{\phi_2}(w) - \frac{v(f)}{2(f(p))^2} \text{ grad}_F\ \phi_2(q) \right).$$

Proof. First

$$\begin{aligned} H_{\phi_1 \circ \pi}(\xi) &= \nabla^1_v \text{ grad } \phi_1 + \frac{1}{2} (\text{grad } \phi_1)(\ln f) w \\ &= H_{\phi_1}(v) + \frac{1}{2f} \text{ grad } \phi_1(f) w \\ &= \left(H_{\phi_1}(v), \frac{1}{2f} \text{ grad } \phi_1(f) w \right), \end{aligned}$$

where we have used (2.4) to calculate the covariant derivative and (2.9) to decompose grad ϕ_1 into its components on B and F. Now

$$
\begin{aligned}
H_{\phi_2\circ\eta}(\xi) &= \nabla_\xi \frac{1}{f}\mathrm{grad}_F\ \phi_2 = \ \xi\left(\frac{1}{f}\right)\ \mathrm{grad}_F\ \phi_2 + \frac{1}{f}\nabla_\xi \frac{1}{f}\ \mathrm{grad}_F\ \phi_2 \\
&= v\left(\frac{1}{f}\right)\mathrm{grad}_F\ \phi_2 + \frac{1}{f}\{\nabla^2_w\ \mathrm{grad}_F\ \phi_2 \\
&\quad + \frac{1}{2}[v(\ln f)\mathrm{grad}_F\ \phi_2 - g(w, \mathrm{grad}_F\ \phi_2)\mathrm{grad}(\ln f)]\} \\
&= \ -\frac{v(f)}{f^2}\ \mathrm{grad}_F\ \phi_2 + \frac{1}{f}H_{\phi^2}(w) + \frac{v(f)}{2f^2}\ \mathrm{grad}_F\ \phi_2 \\
&\quad - h(w, \mathrm{grad}_F\ \phi_2)\frac{1}{2f}\ \mathrm{grad}\ f \\
&= \ -\frac{v(f)}{f^2}\ \mathrm{grad}_F\ \phi_2 + \frac{1}{f}H_{\phi^2}(w) - \frac{w(\phi^2)}{2f}\ \mathrm{grad}\ f \\
&= \left(-\frac{w(\phi^2)}{2f}\ \mathrm{grad}\ f, \frac{1}{f}H_{\phi^2}(w) - \frac{v(f)}{f^2}\ \mathrm{grad}_F\ \phi_2\right).
\end{aligned}
$$

■

With proposition 2.14 in hand, we are ready to calculate $\Box(\phi_1\circ\pi)$ and $\Box(\phi_2\circ\sigma)$. We will let $\Box^B$ denote the d'Alembetian of (B, g_B) and Δ^F denote the Laplace operator on (F, g_F) which is defined just as in (2.6). Also let $m = \dim\ B$ and $n = \dim\ F$ below.

Proposition 2.14. *If $\phi_1 : B \to \mathbb{R}$ is a smooth function, then*

$$\Box(\phi_1\circ\pi)(p,q) = \Box^M\phi_1(p) + \frac{\dim\ F}{2f(p)}\ \mathrm{grad}_B\ \phi_1|_p(f). \tag{2.14}$$

Proof. Let $\{e_1, e_2, \cdots, e_m\}$ be a Lorentzian orthonormal basis for $T_p(B)$ with e_1 timelike and let $\{e_{m+1}, \cdots, e_{m+n}\}$ be chosen in $T_q(F)$. If $\overline{e}_j = (0_p, e_j)$ for $m+1 \leq j \leq m+n$, then $\{\overline{e}_{m+1}, \cdots, \overline{e}_{m+n}\}$ are g-orthonormal. Also set $\overline{e}_j = (e_j, 0_b)$ for $1 \leq j \leq m$. Then,

$$
\begin{aligned}
\Box(\phi_1 \circ \pi) &= -\text{ Hess } (\phi_1 \circ \pi)(\overline{e}_1, \overline{e}_1) + \sum_{j=2}^{m+n} \text{Hess } (\phi_1 \circ \pi)(\overline{e}_j, \overline{e}_j) \\
&= -\text{ Hess } (\phi_1)(e_1, e_1) + \sum_{j=2}^{m} \text{Hess } (\phi_1)(e_j, e_j) \\
&\quad - \frac{1}{2f(p)}(\text{grad}_B\ \phi_1)|_p(f) f(p) h(\sigma_* \overline{e}_1, \sigma_* \overline{e}_1) \\
&\quad + \frac{1}{2f(p)}(\text{grad}_B\ \phi_1)|_p(f) \sum_{j=2}^{m+n} f(p) h(\sigma_* \overline{e}_j, \sigma_* \overline{e}_j) \\
&= \Box^M \phi_1(p) + \frac{\dim F}{2f(p)}\ \text{grad}_B\ \phi_1|_p(f).
\end{aligned}
$$

■

Proposition 2.15. *If $\phi_2 : F \to \mathbb{R}$ is a smooth function, then*

$$
\Box(\phi_2 \circ \eta)(p, b) = \frac{1}{f(p)} \Delta^F \phi_2(b). \tag{2.15}
$$

Proof. Let $\{\overline{e_1}, \cdots, \overline{e_{n+m}}\}$ be as in the proof of Proposition 2.14. Setting

$v_j = (f(p))^{1/2} e_j$, we have using (2.13) that

$$\begin{aligned}\Box(\phi_2 \circ \eta) &= \frac{1}{f} \sum_{m+1}^{m+n} f g_F(H_{\phi_2}(e_j), e_j) \\ &= \frac{1}{f} \sum_{m+1}^{m+n} \text{Hess } (\phi_2)(v_j, v_j) \\ &= \frac{1}{f} \Delta^H \phi_2,\end{aligned}$$

since $g(\text{grad}_H\ \phi_2, \overline{e}_j) = 0$ if $1 \leq j \leq m$ and $g(\text{grad}_M\ f, \overline{e}_j) = 0$ if $m + 1 \leq j \leq m + n$. ■

Combining these preliminary propositions with (2.8), we obtain the following result.

Proposition 2.16. *Let $\Phi : (B \times_f F, g) \to \mathbb{R}$ be a smooth function of the form $\Phi = (\phi_1 \circ \pi)(\phi_2 \circ \sigma)$, where $\phi_1 : B \to \mathbb{R}$ and $\phi_2 : F \to \mathbb{R}$ are smooth. Then,*

$$\begin{aligned}\Box\Phi(p, q) = &\left\{ \Box^M \phi_1(p) + \frac{\dim F}{2f(p)} (\text{grad}_M\ \phi_1)|_p(f) \right\} \phi_2(q) \\ &+ \frac{\phi_1(p)}{f(p)} \Delta^H \phi_2(q).\end{aligned} \tag{2.16}$$

Proof. This is immediate since $g(\text{grad } (\phi_1 \circ \pi), \text{grad } (\phi_2 \circ \eta)) = 0$, using formulas (2.9) and (2.10). ■

3. Main results

In this section, we let (N, g_0) be a compact Riemannian n-dimensional manifold with $n \geq 3$ and without boundary.

We consider the $(n+1)$-dimensional Riemannian warped manifold $M = [a, \infty) \times_f N$ with the metric $g = -dt^2 + f(t)^2 g_0$, where f is a positive function on $[a, \infty)$. Let $u(t,x)$ be a positive smooth function on M and let g have a scalar curvature equal to $r(t,x)$. If the conformal metric $g_c = u(t,x)^{\frac{4}{n-1}} g$ has a scalar curvature $R(t,x)$, which is an arbitrary smooth function in $C^\infty(M)$, by equation (2.3) then $u(t,x)$ satisfies equation

$$\frac{4n}{n-1} \Box_g u(t,x) - r(t,x)u(t,x) + R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0 \tag{3.1}$$

where $\Box_g$ is the d'Alembertian for a Lorentzian warped manifold $M = [a, \infty) \times_f N$.

Proposition 3.1. *Let $M = (a, \infty) \times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2 g_0$. Then the d'Alembertian $\Box_g$ is given by*

$$\Box_g = -\frac{\partial^2}{\partial t^2} - \frac{nf'(t)}{f(t)} \frac{\partial}{\partial t} + \frac{1}{f(t)^2} \Delta_x,$$

where Δ_x is the Laplacian on fiber manifold N.

Proof. In the case that dim $B = 1$, and (B, g_B) is given by an interval $((a,\infty), -dt^2)$, we have $\Box^B \phi_1(t) = -\phi_1^{''}(t)$ and $\mathrm{grad}_B\ \phi_1(t) = -\phi_1^{'}(t)(\frac{\partial}{\partial t})$. Thus in this case, using $f(t)^2$ instead of $f(p)$ in (2.16), formula (2.16) simplifies to

$$\Box\Phi(t,b) = -\left(\phi_1^{''}(t) + \frac{\dim H}{f(t)}\phi_1^{'}(t)f'(t)\right)\phi_2(b) + \frac{\phi_1(t)}{f(t)^2}\Delta_F^H\phi_2(b).$$

■

By Proposition 3.1 equation (3.1) is changed into the following equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial u(t,x)}{\partial t} - \frac{1}{f(t)^2}\Delta_x u(t,x) + \frac{n-1}{4n}r(t,x)u(t,x) - \frac{n-1}{4n}R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0. \tag{3.2}$$

We may assume that in Lorentzian warped manifold $M = [a,\infty)\times_f N$ admits a negative constant scalar curvature $r(t,x) = -c > 0$, where $c > 0$, and the warping function $f(t)$ with $f(t) \to \infty$ as $t \to \infty$ and $\left|\frac{f'(t)}{f(t)}\right| \leq constant.$

If $u(t,x) = u(t)$ is a positive function with only variable t, then equation (3.2) becomes

$$u^{''}(t) + \frac{nf^{'}(t)}{f(t)}u^{'}(t) - c_n u(t) - H(t,x)u(t)^{\frac{n+3}{n-1}} = 0, \tag{3.3}$$

where $c_n = \dfrac{n-1}{4n}c$ and $H(t,x) = \dfrac{n-1}{4n}R(t,x)$. In order to prove the following theorem, we develop the idea used in proof of Theorem 4.9 in [16].

Theorem 3.2. *Let $u(t)$ be a positive solution of equation (3.3) and let $H(t,x) = H(t)$ be a smooth function with only variable t such that $H(t) \geq c_1$, where c_1 is a positive constant. Assume that there exist positive constants t_0 and C_0 such that $\left|\dfrac{f'(t)}{f(t)}\right| \leq C_0$ for all $t > t_0$. Then $u(t)$ is bounded from above.*

Proof. From equation (3.3) we have

$$\frac{(f^n u')'}{f^n} = c_n u + H(t)u^{\frac{n+3}{n-1}}. \tag{3.4}$$

Let $\chi \in C_0^\infty((a,\infty))$ be a cut-off function. Multiplying both sides of equation (3.4) by $\chi^{n+1}u$ and then using integration by parts, we obtain

$$\begin{aligned} -\int_a^\infty (f^n u')\left(\frac{\chi^{n+1}u}{f^n}\right)' dt &= c_n \int_a^\infty \chi^{n+1}u^2 dt + \int_a^\infty H(t)\chi^{n+1}u^{\frac{2n+2}{n-1}} dt \\ &\geq c_1 \int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}} dt. \end{aligned}$$

From the left side of the above equation, we have

$$-(f^n u')\left(\frac{\chi^{n+1}u}{f^n}\right)' = -(n+1)\chi^n u\chi' u' - \chi^{n+1}|u'|^2 + n\chi^{n+1}uu'\frac{f'}{f}.$$

Applying Cauchy's inequality, we get

$$\begin{aligned}-(n+1)\chi^n u\chi' u' &= -2((n+1)\chi^{\frac{n+1}{2}-1}u\chi')\left(\frac{1}{2}\chi^{\frac{n+1}{2}}u'\right)\\ &\leq (n+1)^2\chi^{n-1}u^2|\chi'|^2+\frac{1}{4}\chi^{n+1}|u'|^2\end{aligned}$$

and

$$\begin{aligned}n\chi^{n+1}uu'\frac{f'}{f} &= 2\left(n\chi^{\frac{n+1}{2}}u\frac{f'}{f}\right)\left(\frac{1}{2}\chi^{\frac{n+1}{2}}u'\right)\\ &\leq n^2\chi^{n+1}\left(\frac{f'}{f}\right)^2u^2+\frac{1}{4}\chi^{n+1}|u'|^2.\end{aligned}$$

Together with the above equations, we obtain

$$\begin{aligned}&\int_a^\infty\left(\frac{f'}{f}\right)^2\chi^{n+1}u^2dt+\int_a^\infty\chi^{n-1}u^2|\chi'|^2dt\\ &\qquad\geq c_1\int_a^\infty\chi^{n+1}u^{\frac{2n+2}{n-1}}dt+\frac{1}{2}\int_a^\infty\chi^{n+1}|u'|^2dt.\end{aligned}$$

Applying Young's inequality and using the bound $\left|\frac{f'}{f}\right|\leq C_0$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^\infty\chi^{n+1}|u'|^2dt+c_1\int_a^\infty\chi^{n+1}u^{\frac{2n+2}{n-1}}dt\\ &\qquad\leq C'\int_a^\infty(|\chi'|^{n+1}+\chi^{n+1})dt,\end{aligned}\tag{3.5}$$

where C' is a positive constant. Let $\chi\equiv 0$ on $(a,r]\cup[r+3,\infty)$ with $r>t_0$ and $\chi\equiv 1$ on $[r+1,r+2]$, $\chi\geq 0$ on $[a,\infty)$ and $|\chi'|\leq\frac{1}{2}$. From equation (3.4) we have

$$\int_{r+1}^{r+2}|u'|^2dt+\int_{r+1}^{r+2}u^{\frac{2n+2}{n-1}}dt\leq C''$$

for all $r > t_0$, where $C^{''}$ is a constant independent on r. Therefore u is bounded from above. ■

Theorem 3.3. *Let (M,g) be a Lorentzian manifold with scalar curvature equal to $-c$ $(c > 0)$. Assume that there exist positive constants t_0 and C_0 such that $\left|\frac{f^{'}(t)}{f(t)}\right| \leq C_0$ for all $t > t_0$. If $H(t,x) = H(t)$ is a scalar curvature satisfying $H(t) \geq c_1$, where c_1 is a positive constant, then equation (3.3) has no positive solution.*

Proof. If $u = u(t)$ is a positive solution of equation (3.3), then by Theorem 3.2 $u(t)$ is bounded from above on (a,∞). Then, by Omori-Yau maximum principle ([18]), there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} u(t_k) = \sup_{t\in[a,\infty)} u(t)$, $|u'(t_k)| \leq \frac{1}{k}$ and $u^{''}(t_k) \leq \frac{1}{k}$. Since $\sup_{t\in[a,\infty)} u(t) = c_2 > 0$, there exist a number $\epsilon > 0$ and K such that

$$(c_n u(t_k) + H(t_k)u(t_k)^{\frac{n+3}{n-1}}) > \epsilon$$

for all $k > K$, which is a contradiction to the fact that

$$u^{''}(t_k) + \frac{nf^{'}(t_k)}{f(t_k)}u^{'}(t_k) \leq \frac{1+nC_0}{k}$$

for all $k > K$. Therefore equation (3.3) has no positive solution. ■

The following corollary is derived easily from the previous theorem 3.3.

Corollary 3.4. *Let* $(M, g) = ((a, \infty) \times_f N, g)$ *be a Lorentzian manifold with scalar curvature equal to* $h(t) \leq 0$. *Assume that there exist positive constants* t_0 *and* C_0 *such that* $\left|\frac{f'(t)}{f(t)}\right| \leq C_0$ *for all* $t > t_0$. *If* $H(t) = C$, *where* C *is a positive constant, then the following equation*

$$u''(t) + \frac{nf'(t)}{f(t)}u'(t) = \frac{4n}{n-1}h(t)u(t) + Cu(t)^{\frac{n+3}{n-1}}$$

also has no positive solution.

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