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碩士學位論文

CONVERGENCE OF FOURIER SERIES

朝 鮮 大 學 校 大 學 院

數　學　科

池　希　貞

CONVERGENCE OF FOURIER SERIES

- 푸리에 급수의 수렴성-

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指導敎授　　洪 成 金

이 論文을 理學碩士學位申請 論文으로 提出함.

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적분 가능한 주기함수를 우리는 다음과 같이 푸리에 급수로 표현 할 수 있다.

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, D(x-y)\, dy.$$

그런데 푸리에 급수 $S_N f$ 는 함수 f 에 점별 수렴하지 않음이 잘 알려져 있다 ([6] 참조). 우리는 이 논문에서 푸리에 급수를 합하는 방법을 다르게 하여 푸리에 부분 합을 평균하여 세사로 합의 의미로 수렴 성을 조사하고 아벨 합의미로의 수렴 성을 증명하였다. 구체적으로 우리는 세사로 합과 아벨 합을 연속함수 f 와 Good kernel 을 합성 곱하여 연산자로 정의하고 이 연산자들이 연속함수 f 에 수렴함을 증명하였다. 더 나아가서 급수의 수렴성과 세사로 합과 아벨합의 관계를 조사하였다.

감사의 글

본 논문이 완성되기까지 많은 지도를 해주신 홍성금 교수님께 감사드립니다. 그리고, 바쁘신 가운데서도 세심하고 자상하게 심사해주신 정윤태 교수님, 김남권 교수님께 깊은 감사를 드립니다.

또한 대학원 생활을 하는 동안 편히 할 수 있도록 뒤에서 지켜봐주시고, 도와주신 임향주 선생님과, 김홍주 선생님께 마음 속 깊은 감사를 드립니다.

오늘의 제가 있기까지 늘 곁에서 자식들을 위해 기도하신 사랑하는 우리 엄마, 힘든 상황에서도 삶을 긍정적으로 보라보시는 아빠, 두분을 너무나 사랑합니다. 더불어 때로는 친구처럼 더 많은 것을 함께 해 온 동생 길환이와 바른길로 갈 수 있도록 불을 비춰주는 친구 진아에게 고마움을 전합니다.

2008년 2월

지 희 정 올림

1. Introduction

If f is an integrable function given on an interval $[a,b]$ of length L (that is, $b-a=L$), then the n^{th} Fourier coefficient of f is defined by

$$\widehat{f}(n)=\frac{1}{L}\int_a^b f(x)\,e^{-2\pi inx/L}\,dx,\quad n\in\mathbb{Z}.$$

The Fourier series of f is given formally by

$$\sum_{n=-\infty}^{\infty}\widehat{f}(n)\,e^{2\pi inx/L}.$$

We shall sometimes write a_n for the Fourier coefficients of f, and use the notation

$$f(x)\sim\sum_{n=-\infty}^{\infty}a_n\,e^{2\pi inx/L}$$

to denote that the series on the right-hand side is the Fourier series of f.
Moreover, if we take $a=-\pi$ and $b=\pi$, then the n^{th} Fourier coefficient of f is

$$\widehat{f}(n)=a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,e^{-in\theta}\;d\theta,\quad n\in\mathbb{Z},$$

and the Fourier series of f is

$$f(x)\sim\sum_{n=-\infty}^{\infty}a_n\,e^{in\theta}.$$

We express the partial sums of the Fourier series of f as follow:

$$\begin{aligned}S_N(f)(x)&=\sum_{n=-N}^{N}\widehat{f}(n)e^{inx}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)(\sum_{n=-N}^{N}e^{in(x-y)})dy\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,D_N(x-y)\,dy,\end{aligned}$$

where $D_N(x)=\sum_{n=-N}^{N}e^{inx}$. Now we define the convolution $f*D_N$ on $[-\pi,\pi]$ by

$$(f*D_N)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(x-y)\,f(y)\,dy.$$

In general, given a family of functions $\{K_n\}$, we consider the limiting properties as n tends to infinity of the convolutions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x-y)\, f(y)\, dy.$$

We find that if the family $\{K_n\}$ satisfies the three properties of "good kernels", then the convolution tend to $f(x)$ as $n \to \infty$ when f is continuous (For the main properties of convolutions and good kernels, see [6]). However, the Dirichlet kernels D_N do not belong to the category of good kernels (see Chapter 4).

In this thesis we prove other methods of summing the Fourier series of a function. The first method, which involves average of partial sums in the sense of Cesàro, lead to convolutions with good kernels. Second, we may also sum the Fourier series in the sense of Abel and again encounter a family of good kernels.

More precisely, we shall show

Theorem 1. (**Bernstein's Theorem**) *If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series of f converges absolutely.*

Theorem 2. *If f is integrable on the circle, then the Fourier series of f is Cesàro summable to f at every point of continuity of f. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesàro summable to f.*

Theorem 3. *The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f.*

Theorem 4. *If a series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ, then it is Abel summable to σ.*

Remark 1. (i) Theorem 1 is called Bernstein's theorem. We reprove it in Chapter 3.

(ii) The converse of Theorem 4 is not true (see Chapter 6).

In Chapter we study some examples Fourier series of a function and some condition for series convergence.

In Chapter 3 we consider the Fourier coefficient and estimate its decays and prove Theorem 1.

In Chapter 4 we study the characteristic properties of convolution, good kernel. Also by using convolutions, we show how these kernels can be used to recover a given function.

In Chapter 5 we prove our mains results Theorems 2 and 3.

In Chapter 6 we show that Abel summability is stronger than the standard or Cesàro methods of summation.

In Appendix we study infinite-dimensional vector spaces and pre-Hilbert space to understand the mean square convergence of Fourier series.

Throughout this paper the different constants will be denoted by the same letter C. Each Chapter is based on [1] through [8] .

2. Fourier series

In this Chapter we consider some examples of Fourier series of a function. The following Lemma 1 shows that the given integrals are independent of the chosen interval.

Lemma 1. *Suppose f is 2π-periodic and integrable on finite interval. If $a, b \in \mathbb{R}$, then*

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx \tag{2.1}$$

and

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx. \tag{2.2}$$

Proof. We first show (2.1). By change of valuables x with $u+2\pi$, we have

$$\int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_a^b f(u+2\pi)\,du.$$

Since $f(u+2\pi) = f(u)$, the above integral is

$$\int_a^b f(u+2\pi)\,du = \int_a^b f(u)\,du.$$

Similarly, if we change of variables with x with $u-2\pi$, we get

$$\int_{a-2\pi}^{b-2\pi} f(x)\,dx = \int_a^b f(u-2\pi)\,du = \int_a^b f(u)\,du.$$

We now turn to the proof of (2.2). By change of valuables $x+a$ with u, we have

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi+a}^{\pi+a} f(u)\,du.$$

Clearly, changing of variables with x with $u-a$, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(u-a)\,d(u-a) = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

We complete the proof. □

Before proceeding some examples and properties of Fourier series of a function, we first show the following lemma.

Lemma 2. *Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.*

(a) *The summation by parts formula is*

$$\sum_{n=M}^{N} a_n b_n = a_{N+1}B_N - a_M B_{M-1} - \sum_{n=M}^{N}(a_{n+1}-a_n)B_n.$$

(b) *If the partial sums of the series $\sum b_n$ are bounded, and a_n is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.*

Proof. (a). We write

$$\begin{aligned}
\sum_{n=1}^{N} a_n b_n &= \sum_{n=1}^{N} a_n(B_n - B_{n-1}) \\
&= \sum_{n=1}^{N} a_n B_n - \sum_{n=1}^{N} a_{n+1}B_n + a_{N+1}B_N \\
&= a_{N+1}B_N - \sum_{n=1}^{N}(a_{n+1}-a_n)B_n.
\end{aligned}$$

Hence if $M < N$, then we have

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=1}^{N} a_n b_n - \sum_{n=1}^{M-1} a_n b_n \\
&= \{B_N a_{N+1} - \sum_{n=1}^{N} B_n(a_{n+1}-a_n)\} - \{B_{M-1}a_M - \sum_{n=1}^{M-1} B_n(a_{n+1}-a_n)\} \\
&= a_{N+1}B_N - a_M B_{M-1} - \sum_{n=M}^{N}(a_{n+1}-a_n)B_n.
\end{aligned}$$

Next we prove (b). We note that $B_k = \sum_{n=1}^{k} b_n$. Since B_n is bounded their exists a positive real number L such that

$$|B_n| \le L \quad \text{for all} \quad n.$$

In order to show the convergence of $\sum a_n b_n$, it suffices to show that for any $\epsilon > 0$

$$\left|\sum_{n=M}^{N} a_n b_n\right| < \epsilon\ .$$

From the assumption for a sequence a_n, for give $\epsilon > 0$ there exists a positive integer K such that

$$|a_M - a_N| < \epsilon/3L,$$

and

$$|a_M| < \epsilon/3L$$

for all $M, N \geq K$.
Hence

$$\begin{aligned} |\sum_{n=M}^{N} a_n b_n| &\leq |a_{N+1}||B_N| + |a_M||B_M| + \sum_{n=M}^{N-1} |a_{n+1} - a_n||B_N| \\ &\leq \frac{2\epsilon}{3} + L \sum_{n=M}^{N-1} |a_{n+1} - a_n|. \end{aligned}$$

Since a_n decreases monotonically, we have

$$\begin{aligned} \sum_{n=M}^{N-1} |a_{n+1} - a_n| &= |a_{M+1} - a_M| + |a_{M+1} - a_{M+1}| + \cdots + |a_N - a_{N-1}| \\ &= (a_M - a_{M+1}) + (a_{M+1} - a_{M+2}) + \cdots + (a_{N-1} - a_N) \\ &= (a_M - a_N). \end{aligned}$$

Therefore, we get the desired estimate

$$|\sum_{n=M}^{N} a_n b_n| \leq \frac{2\epsilon}{3} + L\frac{\epsilon}{3L} = \epsilon.$$

□

Example 1. We consider the sawtooth function

$$f(x) = \begin{cases} \frac{\pi}{2} - \frac{x}{2} & \text{if} \quad -\pi < x < 0, \\ \\ \frac{\pi}{2} - \frac{x}{2} & \text{if} \quad 0 < x < \pi, \end{cases}$$

where $f(0) = 0$.
The Fourier coefficients are obtained by the integration by parts as follows :
First, if $n \neq 0$, then

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{0} (-\frac{\pi}{2} - \frac{x}{2}) e^{-inx} \, dx \ + \ \frac{1}{2\pi} \int_{0}^{\pi} (\frac{\pi}{2} - \frac{x}{2}) e^{-inx} \, dx \\ &= \frac{1}{2in}. \end{aligned}$$

If $n = 0$, we clearly have $\widehat{f}(n) = 0$. Thus, the Fourier series of f is given by

$$f(x) \sim \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}.$$

If we take $a_n = 1/n$ and $b_n = e^{inx}$ in Lemma 2, we see that this series $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ converges.

Example 2. Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b] \subset [-\pi,\pi]$ where

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if} \quad x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

First, if $n \neq 0$, then

$$\widehat{f}(n) \;=\; \frac{1}{2\pi}\int_a^b 1\, e^{inx}\, dx = \frac{1}{2\pi in}(-e^{-inb} + e^{-ina}).$$

As for $n = 0$ it is obvious that

$$\widehat{f}(0) \;=\; \frac{1}{2\pi}\int_a^b 1\, dx = \frac{b-a}{2\pi}.$$

Thus, the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0}\frac{e^{-ina} - e^{-inb}}{2\pi in}\, e^{inx}.$$

If $a \neq -\pi$ or $b \neq -\pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x. Because

$$\sum_{n\neq 0}\left|\frac{e^{-ina} - e^{-inb}}{2\pi in}\, e^{inx}\right| \;\geq\; \sum_{n\neq 0}\frac{|sinn\theta_0|}{\pi n} \;\geq\; \sum_{n\neq 0}\frac{c}{\pi n},$$

where $\theta_0 = (b-a)/2$ and $c > 0$.

However, Fourier series converges at every point since in Lemma 2 we can consider a_n as $1/n$ and b_n as $(e^{-ina} - e^{-inb})\, e^{inx}$.

3. Fourier coefficients

In this Chapter we state the uniqueness of the Fourier series in [6] and estimate the decay estimates of the Fourier coefficients.

Theorem 5. *Suppose that f is an integrable function on the circle with $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.*

Corollary 1. *If f is continuous on the circle and $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$.*

Corollary 2. *Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$. Then the Fourier series converges uniformly to f, that is,*

$$lim_{N\to\infty}\, S_N(f)(\theta) \ = f(\theta) \ \textit{uniformly} \ \ \textit{in} \ \ \theta.$$

We consider the decay estimates of the Fourier coefficients.

Lemma 3. *Suppose f is a periodic function of period 2π which belongs to the class C^k. Then there exists a constant C such $|\widehat{f}(n)| \leq C/|n|^k$.*

Proof. If we integrate by parts k times, then we have

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta = \cdots \\ &= \frac{1}{2\pi(in)^k}\int_0^{2\pi} f^{(k)}(\theta)e^{-in\theta}\,d\theta. \end{aligned}$$

Thus,

$$|\widehat{f}(n)| \ \leq \ \frac{1}{2\pi|n|^k}\int_0^{2\pi} |f^{(k)}(\theta)|d\theta \ \leq \ C'\frac{1}{|n|^k},$$

where C' is a bound for $f^{(k)}$. $\square$

Definition 1. f is said to be a Hölder condition of order α $(0 < \alpha \leq 1)$ if

$$\sup_\theta |f(\theta + t) - f(\theta)| \leq A|t|^\alpha \ \ \text{for} \ \ \text{all} \ \ t.$$

Lemma 4. *Let f be 2π-periodic and Riemann integrable on $[-\pi, \pi]$.*

(a) *We have*

$$\widehat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + \pi/n)e^{-inx}dx$$

and so

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x) - f(x + \pi/n)]e^{-inx}dx.$$

(b) *Now assume that f satisfies a Hölder condition of order α, namely*

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some $0 < \alpha \leq 1$, and all x, h. Then there exists a constant $C > 0$ such that

$$\widehat{f}(n) \leq C/|n|^\alpha.$$

(c) *The above result (b) cannot be improved.*

Proof. We first prove (a). We recall that the Fourier coefficient is

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

From Lemma 1, we also have

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-in(x+\frac{\pi}{n})}dx \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-inx}dx.\end{aligned}$$

Using the above

$$\begin{aligned}2\widehat{f}(n) &= \widehat{f}(n) + \widehat{f}(n) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-inx}. \qquad (3.1)\end{aligned}$$

If we divide by 2 in both side (3.1), we obtain the desired estimates.

Next we prove (b). From (a) it follows that

$$|\widehat{f}(n)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\pi/n)|\,dx.$$

If we use Definition 1, then we have

$$|f(x) - f(x+\pi/n)| \leq A(\pi/n)^\alpha \text{ for all } x.$$

Hence, we obtain the desired estimate.

Lastly, we prove (c). We consider the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x}, \quad 0 < \alpha < 1.$$

We want to show that

$$|f(x+h) - f(x)| \leq C|h|^\alpha,$$

and $\widehat{f}(N) = 1/N^\alpha$ whenever $N = 2^k$.

Since $f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x}$, we can rewrite

$$f(x+h) - f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k(x+h)} - \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x}.$$

If we split $\sum_{k=0}^{\infty}$ as $\sum_{2^k \le 1/|h|} + \sum_{2^k > 1/|h|}$, we have

$$\begin{aligned}
|f(x+h) - f(x)| &= \Big(\sum_{2^k \le 1/|h|} 2^{-k\alpha} e^{i2^k(x+h)} - \sum_{2^k \le 1/|h|} 2^{-k\alpha} e^{i2^k(x)} \Big) \\
&\quad + \Big(\sum_{2^k > 1/|h|} 2^{-k\alpha} e^{i2^k(x+h)} - \sum_{2^k > 1/|h|} 2^{-k\alpha} e^{i2^k(x)} \Big) \\
&:= I + II.
\end{aligned}$$

Applying the mean value theorem we obtain

$$I \le C_1 \sum_{2^k \le 1/|h|} 2^{k(1-\alpha)} |h|.$$

From the fact $|e^{i2^k(x+h)} - e^{i2^k(x)}| \le 2$, the estimate II is obtained as

$$II \le 2 \sum_{2^k > 1/|h|} 2^{-k\alpha}.$$

If we combine I and II together, we have

$$\begin{aligned}
|f(x+h) - f(x)| &\le C_1 \frac{|h|}{|h|^{1-\alpha}} + 2|h|^{\alpha} \\
&\le C|h|^{\alpha}.
\end{aligned}$$

□

Now, we shall show that the Fourier series of f converges absolutely if f satisfies a Hölder condition of order $\alpha > 1/2$.

Theorem 6. (**Parseval's identity**) *Let f be an integrable function on the circle with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Then we have Parseval's identity*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta.$$

Proof. See the details in [6]. □

We first consider the Hölder condition of order $\alpha = 1$.

Proposition 1. *Let f be a 2π-periodic function which satisfies a Hölder condition of order $\alpha = 1$ with constant K; that is,*

$$|f(x) - f(y)| \le K|x - y| \quad \textit{for all } x,\, y.$$

(a) *For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |g_h(x)|^2 dx = \sum_{n=-\infty}^{\infty} 4|sin\ nh|^2 |\hat{f}(n)|^2,$$

and
$$\sum_{n=-\infty}^{\infty} |sin\ nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2.$$

(b) *Let p be a positive integer. If we choose $h = \pi/2^{p+1}$, we have*
$$\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) *The Fourier series of f converges absolutely and uniformly.*

Proof. (a). Since $g_h(x) \sim \sum_{n=-\infty}^{\infty} \widehat{g_h}(n) e^{in\theta}$, we can express the Fourier coefficient of g_h as
$$\widehat{g_h}(n) = e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n) = 2i\, sinnh\, \widehat{f}(n).$$
If we use Parseval's identity, then we have
$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2 dx &= \sum_{n=-\infty}^{\infty} |\widehat{g_h}(x)|^2 \\ &= \sum_{n=-\infty}^{\infty} 4|sin\ nh|^2 |\hat{f}(n)|^2. \end{aligned}$$
Moreover,
$$\begin{aligned} \sum_{n=-\infty}^{\infty} |sin\ nh|^2 |\hat{f}(n)|^2 &\leq \frac{1}{8\pi}\int_0^{2\pi} |f(x+h) - f(x-h)|^2 dx \\ &\leq \frac{1}{8\pi}\int_0^{2\pi} \big(K \cdot 2h\big)^2 dx \qquad (3.2) \\ &= K^2 h^2. \end{aligned}$$

(b). From given $2^{p-1} < |n| \leq 2^p$, $h = \pi/2^{p+1}$, we see that $\dfrac{\pi}{4} < |n|h \leq \dfrac{\pi}{2}$. So it is clear that $|sin\ nh|^2 \geq 1/2$, and thus $\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq 2\,K^2h^2$. Plugging in $h = \pi/2^{p+1}$, we conclude that
$$\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c). For the proof, if we apply (b) and Cauchy Schwartz's inequality, then
$$\begin{aligned} \sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)| &\leq \Big(\sum_{2^{p-1}<|n|\leq 2^p} |\widehat{f}(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{2^{p-1}<|n|\leq 2^p} 1^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\frac{K^2\pi^2}{2^{2p+1}}\Big)^{\frac{1}{2}} \cdot 2^{\frac{p}{2}} = \frac{K\pi}{2^{\frac{p+1}{2}}}. \end{aligned}$$

Furthermore, the Fourier coefficient of f is bounded by

$$\sum_{n=-\infty}^{\infty}|\widehat{f}(n)| \le \sum_{p=1}^{\infty}\sum_{2^{p-1}<|n|\le 2^p}|\widehat{f}(n)| \quad \le \quad \frac{K\pi}{\sqrt{2}}\sum_{p=1}^{\infty}\frac{1}{2^{\frac{p}{2}}} = \frac{K\pi}{(2-\sqrt{2})}.$$

$\square$

In fact, if we modify the argument slightly used in Proposition 1, we can reprove Bernstein's theorem, that is Theorem 1.

Proof of Theorem 1. If we replace $(2h)$ by $(2h)^\alpha$ in (3.2) and use (b), we obtain

$$\sum_{2^{p-1}<|n|\le 2^p}|\widehat{f}(n)|^2 \le \Big(\frac{K\pi^\alpha}{2^{(2p+2)\alpha-1}}\Big)^2.$$

Repeating the same procedures as those in (c), we get

$$\sum_{2^{p-1}<|n|\le 2^p}|\widehat{f}(n)| \quad \le \quad \frac{K\pi^\alpha}{2^{(\alpha-\frac{1}{2})(p+1)}}.$$

If we assume $\alpha > 1/2$, the Fourier coefficient of f converges absolutely, that is

$$\sum_{n=-\infty}^{\infty}|\widehat{f}(n)| \le \sum_{p=1}^{\infty}\sum_{2^{p-1}<|n|\le 2^p}|\widehat{f}(n)| \quad \le \quad K\pi^\alpha\sum_{p=1}^{\infty}\frac{1}{2^{(\alpha-\frac{1}{2})(p+1)}}.$$

We complete the proof. $\square$

4. Convolutions and Good kernels

In this Chapter we study the characteristic properties of convolution, good kernel. Also by using convolutions, we show how these kernels can be used to recover a given function.

Definition 2. A family of kernels $\{K_n(x)\}_1^\infty$ on the circle is said to be a family of good kernels if it satisfies the following properties :

(i) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\ dx = 1.$$

(ii) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|\ dx \leq M.$$

(iii) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\ dx \to 0, \quad \text{as} \quad n \to \infty.$$

Theorem 7. *Let $\{K_n(x)\}_1^\infty$ be a family of good kernels, and f an integrable function on the circle. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, the above limit is uniform.

Proof. See pp.49–50 in [6] for the details. □

We now revisit the partial sum of the Fourier series of f

$$S_N(f)(x) = (f * D_N)(x),$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ is the Dirichlet kernel. It is natural now consider whether D_N is a good kernel. Unfortunately, this is not the case. The following proposition 2 tells us that D_N is not a good kernel.

Lemma 5. *The Dirichlet kernel D_N is*

$$D_N(x) = \frac{\sin(N + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})}.$$

Proof. We decompose

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \sum_{n=0}^{N} e^{inx} + \sum_{n=-N}^{-1} e^{inx}.$$

By change of valuables e^{ix} with ω, we have

$$\begin{aligned}\sum_{n=0}^{N}\omega^n+\sum_{n=-N}^{-1}\omega^{-n} &= \frac{1-\omega^{N+1}}{1-\omega}+\frac{\omega^{-N}-1}{1-\omega}\\ &= \frac{\omega^{-N-1/2}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}}\\ &= \frac{sin((N+1/2)x)}{sin(x/2)}.\end{aligned}$$

$\square$

Proposition 2. *The Dirichlet kernel D_N is not a good kernel.*

Proof. It suffices to show that D_N does not hold the second property such that

$$\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta \geq C\log N, \quad \text{as} \quad N\to\infty.$$

First, we define

$$L_N=\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(\theta)|\,d\theta.$$

From Lemma 5, we express D_N as

$$D_N(\theta)=\frac{sin((N+1/2)\theta)}{sin(\theta/2)}.$$

In the denominator since the period of $sin(\theta/2)$ is 4π, it is obvious that $|\theta|>|sin(\theta/2)|$, and so

$$|D_N(\theta)|\geq\frac{|sin((N+1/2)\theta)|}{|\theta|}.$$

By changing of variable $(N+1/2)\theta$ by ϑ, we have

$$\begin{aligned}L_N &\geq \frac{1}{\pi}\int_0^{(2N+1)\pi}\frac{|sin\vartheta|}{|\vartheta|}\,d\vartheta\\ &= \frac{1}{\pi}\Big(\int_0^{\pi}+\int_{\pi}^{N\pi}+\int_{(N+1)\pi}^{(2N+1)\pi}\Big)\frac{|sin\vartheta|}{|\vartheta|}\,d\vartheta\\ &\geq \frac{1}{\pi}\int_{\pi}^{N\pi}\frac{|sin\vartheta|}{|\vartheta|}\,d\vartheta.\end{aligned}$$

Now since we split the integral $\int_{\pi}^{N\pi}$ as $\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}$, we conclude that

$$\begin{aligned}\int_{\pi}^{N\pi}\frac{|sin\vartheta|}{|\vartheta|}\,d\vartheta &= \sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{|sin\vartheta|}{|\vartheta|}\,d\vartheta \\ &\geq C\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{1}{(k+1)\pi}\,d\vartheta \\ &= C\sum_{k=1}^{N-1}\frac{1}{k+1}.\end{aligned}$$

From the fact

$$\sum_{k=1}^{N}\frac{1}{k}\geq\sum_{k=1}^{N-1}\int_{k}^{k+1}\frac{1}{\vartheta}\,d\vartheta=\int_{1}^{N}\frac{1}{\vartheta}\,d\vartheta=C\,logN,$$

we have that

$$L_N\geq C\,logN.$$

□

Remark (i) We note that the formula for D_N as a sum of exponentials immediately gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(x)dx=1.$$

(ii) The fact that the mean value of D_N is 1 is a result of cancellations.

We now turn to the standard positive result. Although $S_N(f)(x)=(f*D_N)(x)$ does not converge to f pointwisely, we instead have the following theorem.

Theorem 8. *If f is an integrable function on the circle, then $\|S_N-f\|_\infty\to 0$ as $N\to\infty$.*

Proof. See [4]. □

5. Proofs of Theorems 2, 3

The Dirichlet kernels fail to belong to the family of good kernels, since one has

$$\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta \geq C \log N, \quad \text{as} \quad N \to \infty.$$

However, their averages are very well behaved functions, in the sense that they do form a family of good kernels.

Definition 3. We define N^{th} Fejér kernel given by

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

From $S_n(f) = f * D_n$, we find that

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where $F_N(x)$ is the N^{th} Fejér kernel.

Lemma 6. *The Fejér kernel is given by*

$$F_N(x) = \frac{sin^2(Nx/2)}{N\ sin^2(x/2)}.$$

Proof. From Definition 3 we write

$$N\, F_N(x) = D_0(x) + \cdots + D_{N-1}(x).$$

If we denote $\omega = e^{ix}$, then $D_l(x) = \sum_{n=0}^{l} \omega^n + \sum_{n=-l}^{-1} \omega^n$ and their sums are

$$\frac{1-\omega^{l+1}}{1-\omega}, \quad \text{and} \quad \frac{\omega^{-l}-1}{1-\omega}, \tag{5.1}$$

respectively. If you plug l into $0, 1, \cdots, N-1$ in (5.1), this leads

$$\begin{aligned}
\sum_{l=0}^{N-1} D_l(x) &= \frac{1-\omega}{1-\omega} + \frac{\omega^{-1}-\omega^2}{1-\omega} + \cdots + \frac{\omega^{-N+1}-\omega^N}{1-\omega} \\
&= \frac{(1 + 1/\omega + \cdots + 1/\omega^{N-1})}{1-\omega} - \frac{(\omega + \omega^2 + \cdots + \omega^N)}{1-\omega} \\
&= \frac{\omega(\omega^{-N}-1) - \omega(1-\omega^N)}{(1-\omega)^2} \\
&= \frac{(\omega^{-N/2}-\omega^{N/2})^2}{(\omega^{-1/2}-\omega^{1/2})^2} = \frac{sin^2(Nx/2)}{sin^2(x/2)}.
\end{aligned}$$

If we divide the above estimate by N, this completes the proof. $\square$

We let the partial sum s_n by $s_n = \sum_{k=0}^{n} c_k$, and say that the series converges to s if

$$lim_{n\to\infty} s_n = s.$$

Definition 4. We define the average of the first N partial sums by

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

The quantity σ_N is called the N^{th} Cesàro mean of the sequence $\{s_n\}$ or the N^{th} Cesàro sum of the series $\sum_{k=0}^{\infty} c_k$. If σ_N converges to a limit σ as N tends to infinity, we say that the series $\sum_{k=0}^{\infty} c_n$ is Cesàro summable to the same limit s.

Example 3. Consider the series

$$1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k.$$

Its partial sums form the sequence $\{1, 0, 1, 0, ...\}$ which has no limit. However, the series Cesàro summable to $1/2$.

We proceed to the proof of Theorem 2.

Proof of Theorem 2. In view of Theorem 7 we first show that the Fejér kernel is a good kernel. For $N \geq 1$, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1$ since $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$ and Definition 3. Clearly,

$$F_N(x) = \frac{sin^2(Nx/2)}{N\ sin^2(x/2)}$$

from Lemma 6. However, $sin^2(x/2) \geq C(\delta) > 0$, if $\delta \leq |x| \leq \pi$, hence $F_N(x) \leq \frac{1}{C(\delta)\,N}$, from which it follow that

$$\int_{\delta \leq |x| \leq \pi} \big|F_N(x)\big| dx \ \leq \ \frac{1}{C(\delta)\,N} \ \to 0 \ \text{ as } \ N \to \infty.$$

We now consider the N^{th} Cesàro mean of the Fourier series

$$\sigma_N(x) = \frac{S_0(x) + \cdots + S_{N-1}(x)}{N}.$$

Since $N\,F_N(x) = D_0(x) + \cdots + D_{N-1}(x)$, we have $\sigma_N(x) = f * F_N(x)$. Hence if we apply Theorem 7, we have the desired results. □

Corollary 3. *If f is integrable on the circle and $\widehat{f}(n) = 0$ for all n, then $f = 0$ at all point of continuity of f.*

Proof. Since all the partial sums are 0, hence all the Cesàro means are 0. □

Corollary 4. *Continuous function on the circle can be uniformly approximated by trigonometric polynomials.*

Proof. Since the Cesro means are trigonometric polynomials, it follows from Theorem 2. □

Remark. Corollary 4 is the periodic analogue of Weierstrass approximation theorem (see [6], pp.159-160).

Another method of summation is Abel summation.

Definition 5. A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said be Abel summable to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$lim_{r\to 1} A(r) = s.$$

The quantities $A(r)$ are called the Abel means of the series.

In the context of Fourier series, we define the Abel means of the function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

We note that since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \leq r < 1$. In fact,

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} (\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) (\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)})\, d\varphi, \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series. These Abel means can be written as convolutions

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ is the Poisson Kernel given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{5.2}$$

We turn to the proof of Theorem 3.

Proof of Theorem 3. Let $0 \leq r < 1$. By change of valuables $re^{i\theta}$ with ω, we have

$$
\begin{aligned}
P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} &= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n. \\
&= \frac{1}{1-\omega} + \frac{\overline{\omega}}{1-\overline{\omega}} \\
&= \frac{1-|\omega|^2}{|1-\omega|^2} = \frac{1-r^2}{1-2r\,cos\theta + r^2}.
\end{aligned}
$$

To show that P_r is a good kernel, we first note that

$$1 - 2r\,cos\theta + r^2 = (1-2r)^2 + 2r(1-cos\theta).$$

Hence if $1/2 \leq r \leq 1$ and $\delta \leq |\theta| \leq \pi$, then

$$1 - 2r\,cos\theta + r^2 \geq c_\delta > 0.$$

Thus $P_r(\theta) \leq (1-r^2)/c_\delta$ when $\delta \leq |\theta| \leq \pi$, and

$$\int_{\delta \leq |x| \leq \pi} |P_r(\theta)|\, d\theta \leq (1-r^2)/c_\delta \to 0 \ \text{ as } \ r \to 1.$$

Clearly $P_r(\theta) \geq 0$, and integrating the expression (5.2) term by term yield

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) d\theta = 1.$$

Therefore, we conclude that P_r is a good kernel. Combining this lemma with Theorem 7 we complete the proof. $\square$

6. Cesàro and Abel summability

In this Chapter we show that Abel summability is stronger than the standard or Cesàro methods of summation.

Lemma 7. *If a series of complex numbers $\sum_{n=0}^{\infty} c_n$ converges to s, then $\sum_{n=0}^{\infty} c_n$ is Cesàro summable to s.*

Proof. Let
$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$
We want to show that σ_n converges to s as n goes to infinity. Now we consider
$$\begin{aligned} \sigma_n - s &= \frac{s_1 + s_2 + \cdots + s_n}{n} - \frac{ns}{n} \\ &= \frac{(s_1 - s) + (s_2 - s) + \cdots + (s_n - s)}{n}. \end{aligned}$$
Since s_n converges to s by assumption, for give $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n \geq N$. By using this and triangle inequality, we have
$$\begin{aligned} |\sigma_n - s| &\leq \frac{|s_1 - s| + |s_2 - s| + \cdots + |s_N - s|}{n} \\ &\quad + \frac{|s_{N+1} - s| + |s_2 - s| + \cdots + |s_n - s|}{n} \\ &\leq \frac{|s_1 - s| + |s_2 - s| + \cdots + |s_N - s|}{n} + \frac{\epsilon(n-N)}{n}. \end{aligned}$$
Therefore, since $\lim sup_{n\to\infty}|\sigma_n - s| \leq \epsilon$ and ϵ is arbitrary, we obtain the desired result. □

Lemma 8. *If the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit s, then the series is Abel summable to s,*

Proof. Let $S_N = c_1 + c_2 + \cdots + c_N$. Since $s = s(1-r)\sum_{n=1}^{\infty} r^n$ for $|r| < 1$,
$$\begin{aligned} \sum_{n=1}^{N} c_n r^n - s &= \sum_{n=1}^{N} (S_n - S_{n-1}) r^n - s \\ &= \sum_{n=1}^{N} S_n r^n - r \sum_{n=1}^{N} S_{n-1}\, r^{n-1} - S_N\, r^{N+1} + S_N\, r^{N+1} - s \\ &= \sum_{n=1}^{N} S_n r^n (1-r) + S_N\, r^{N+1} - s. \end{aligned}$$

If we let N go to infinity, we have

$$\sum_{n=1}^{\infty} c_n r^n - s = \sum_{n=1}^{\infty} (S_n - s) r^n (1-r) + \lim_{N\to\infty} S_N \, r^{N+1}.$$

Since $\lim_{N\to\infty} S_N \, r^{N+1} = 0$ and $\lim_{r\to 1} \sum_{n=1}^{\infty} (S_n - s) r^n (1-r) = 0$, we obtain

$$\lim_{r\to 1} \sum_{n=1}^{\infty} c_n r^n = s.$$

□

The following example shows that there exists a series which is Abel summable does not converge.

Example 4. ([6]) Take $c_n = (-1)^n$. Then $\lim_{r\to 1} \sum_{n=1}^{\infty} (-1)^n r^n = -\frac{1}{2}$, but $\sum_{n=1}^{\infty} (-1)^n$ does not converge.

We prove Theorem 4. This tells us that Cesro methods of summation implies Abel summation.

Proof of Theorem 4. Let $S_n = \sum_{k=1}^{n} c_k$, $\sigma_n = \frac{S_1 + S_2 + \cdots + S_{n-1}}{n}$ and so $n\sigma_n = S_1 + S_2 + \cdots + S_{n-1}$. In view of Lemma 8 it suffices to show that $\sigma = 0$. Now

$$\begin{aligned} \sum_{n=1}^{N} c_n r^n &= \sum_{n=1}^{N} (S_n - S_{n-1}) r^n \\ &= (1-r) \sum_{n=1}^{N} S_n \, r^n + S_N \, r^{N+1}. \end{aligned} \tag{6.1}$$

Since $S_n = n\sigma_n - (n-1)\sigma_{n-1}$, we rewrite

$$\begin{aligned} \sum_{n=1}^{N} S_n r^n &= \sigma_1 r^1 + (2\sigma_2 - \sigma_1) r^2 + \ \cdots \ + (N\sigma_N - (N-1)\sigma_{N-1}) r^N \\ &= (\sigma_1 r^1 + \ \cdots \ + N\sigma_N r^N) - r(\sigma_1 r^1 + \cdots \ + (N-1)\sigma_{N-1} \, r^{N-1}) \\ &= (1-r) \sum_{n=1}^{N} n\sigma_n r^n + N\sigma_N r^{N+1}. \end{aligned}$$

Thus if we plug the above in (6.1), we have

$$\begin{aligned} \sum_{n=1}^{N} c_n r^n &= (1-r)[(1-r) \sum_{n=1}^{N} n\sigma_n r^n + N\sigma_N \, r^{N+1}] + S_N \, r^{N+1} \\ &= (1-r)^2 \sum_{n=1}^{N} n\sigma_n r^n + (1-r) N\sigma_n \, r^{N+1} + S_N r^{N+1}. \end{aligned}$$

If we let N go to ∞, then $\lim_{N\to\infty} \sigma_n N r^{N+1} = 0$ and $\lim_{N\to\infty} S_N r^{N+1} = 0$. Hence it follows that

$$\sum_{n=1}^{\infty} c_n r^n \quad = \quad (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n \, r^n.$$

Finally, if we let r go to 1, we obtain the desires estimate. $\square$

The next example explains that a series that is Abel summable is not Cesàro summable.

Example 5. ([6]) If we consider the series

$$1 - 2 + 3 - 4 + \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1),$$

then one can show that it is Abel summable to 1/4 since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2}.$$

On the other hand, to show this series is not Cesàro summable we put $N = 2k$ for $k \in \mathbb{Z}$. Then

$$\lim_{k\to\infty} \frac{S_1 + S_2 + \cdots + S_{2k}}{2k} = \lim_{k\to\infty} \frac{-k}{2k} = -\frac{1}{2},$$

If $N = 2k+1$ for $k \in \mathbb{Z}$, then

$$\lim_{k\to\infty} \frac{S_1 + S_2 + \cdots + S_{2k} + S_{2k+1}}{2k} = \lim_{k\to\infty} \frac{k+1}{2k+1} = \frac{1}{2}.$$

Remark 2. The results above can be summarized by the following implications about series:

$$\text{Convergent} \quad \Rightarrow \text{Cesàro summable} \Rightarrow \text{Abel summable},$$

and the fact that none of the arrows can be reversed.

The next Theorem 9 shows with the above arrows can be reversed with some additional condition.

Theorem 9. (a) *If $\sum_{n=0}^{\infty} c_n$ is Cesàro summable to σ and $c_n = o(1/n)$ (that is, $nc_n \to 0$), then $\sum_{n=0}^{\infty} c_n$ converge to σ.*

(b) *The above statement holds if we replace Cesàro summable by Abel summable.*

Proof. (*a*) Let $\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}$ and $S_n = \sum_{k=0}^{n} c_k$. Then

$$\begin{aligned} S_n - \sigma_n &= (c_1 + \cdots + c_n) - \frac{c_1 + (c_1 + c_2) + \cdots + (c_1 + \cdots + c_n)}{n} \\ &= (c_1 - \frac{nc_1}{n}) + (c_2 - \frac{(n-1)c_2}{n}) + \cdots + (c_n - \frac{c_n}{n}) \\ &= \frac{(n-1)c_n + \cdots + c_2}{n}. \end{aligned}$$

Next, we estimate the size of $S_n - \sigma_n$. For this, we consider for any $\epsilon > 0$ there exists K such that $|nc_n| < \epsilon$ and

$$|S_n - \sigma_n| \leq \frac{c_1}{n} + \cdots + \frac{Kc_K}{n} + \frac{(n-K)\epsilon}{n}$$

for all $n \geq K$.
Lastly, we have

$$\limsup_{n\to\infty} |S_n - \sigma_n| \leq \epsilon.$$

Since ϵ is arbitrary, we complete the proof.

(b) It suffices to show that the difference between $\sum_{n=1}^{N} c_n$ and $\sum_{n=1}^{N} c_n(1-1/N)^n$ where $r = 1 - 1/N$. Then we have

$$\begin{aligned} & \sum_{n=1}^{N} c_n(1-(1-1/N)^n) \\ = & \sum_{n=1}^{N} c_n\Big(\frac{n}{N} - \frac{n(n-1)}{2}(\frac{1}{N})^2 + \cdots + (-1)^n(\frac{1}{N})^n\Big), \end{aligned}$$

If we repeat the same argument used in (a), we finish the proof. $\square$

7. Appendix : Mean square Convergence of Fourier Series

In this Chapter we study infinite-dimensional vector spaces and pre-Hilbert space to understand the following mean square convergence of Fourier series:
Theorem 10. *Suppose f is integrable on the circle. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(\theta)-S_N(f)(\theta)\right|^2 d\theta \to 0 \quad as \quad N\to\infty.$$

We now review the definitions of a vector space over $\mathbb{R}$ or $\mathbb{C}$, an inner product, and its associated norm.

Definition 6. A vector space V over the real numbers $\mathbb{R}$ is a set whose elements may be "added" together, and "multiplied" by scalars. More precisely, we may associate to any pair $X, Y \in V$ an element in V called their sum and denoted by $X + Y$.

Definition 7. An inner product on a vector space V over $\mathbb{R}$ associates to any pair X, Y of elements in V a real number which we denote by (X, Y). In particular, the inner product must be by symmetric $(X, Y) = (Y, X)$ and linear in both variables; that is,

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z)$$

whenever $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in V$. Also, we require that the inner product be positive-definite, that is, $(X, X) \geq 0$ for all X in V.

Definition 8. An inner product $(\cdot, \cdot)$ we may define the norm of X by

$$||X|| = (X, Y)^{1/2}.$$

If in addition $||X|| = 0$ implies $X = 0$, we say that the inner product is strictly positive-definite.

Definition 9. For vector spaces over the complex numbers, the inner product of two elements is a complex number. Moreover, these inner products are called Hermitian (instead of symmetric) since they must satisfy $(X, Y) = \overline{(Y, X)}$. Hence the inner product is linear in the first variable, but conjugate-linear in the second:

$$(\alpha X + \beta Y, Z) = \alpha(X, Z) + \beta(Y, Z) \text{ and}$$

$$(X, \alpha Y + \beta Z) = \overline{\alpha}(X, Y) + \overline{\beta}(X, Z).$$

Also, we must gave $(X, X) \geq 0$, and the norm of X is defined by $||X|| = (X, Y)^{1/2}$as before. Again, the inner product is strictly positive-definite if $||X|| = 0$ implies $X = 0$.

Definition 10. Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with inner product $(\cdot, \cdot)$ and associated norm $||\cdot||$. Two elements X and Y are orthogonal if $(X, Y) = 0$, and we write $X \perp Y$.

Three important results can be derived from this notion of orthogonality :

Lemma 9. *We have*
(i) The Pythagorean theorem: if X and Y are orthogonal, then

$$||X + Y||^2 = ||X||^2 + ||Y||^2.$$

(ii) The Cauchy-Schwarz inequality: for any $X, Y \in V$ we have

$$|(X, Y)| \leq ||X|| \, ||Y||.$$

(iii) The triangle inequality: for any $X, Y \in V$ we have

$$||X + Y|| \leq ||X|| + ||Y||.$$

Proof. See [6]. □

We study infinite-dimensional vector spaces.

Definition 11. The vector space $l^2(\mathbb{Z})$ over $\mathbb{C}$ is the set of all (two-sided) infinite sequence of complex numbers

$$(\ldots, a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, \ldots)$$

such that

$$\sum_{n \in \mathbb{Z}} |a_n|^2 \; < \; \infty.$$

Addition is defined componentwise, and so is scalar multiplication. The inner product between the two vectors $A = (\ldots, a_{-1}, a_0, a_1, \ldots)$ and $B = (\ldots, b_{-1}, b_0, b_1, \ldots)$ is defined by the absolutely convergent series

$$(A, B) = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}.$$

The norm of A is then given by

$$||A|| = (A, A)^{1/2} = \big(\sum_{n \in \mathbb{Z}} |a_n|^2 \big)^{1/2}.$$

Lemma 10. *$l^2(\mathbb{Z})$ is a vector space.*

Proof. See pp.73–74 in [6]. □

We now consider Pre-Hilbert space.
In the three example $\mathbb{R}^d, \mathbb{C}^d$, and $l^2(\mathbb{Z})$, the vector spaces with their inner products and norm satisfy two important properties:

(i) The inner product is strictly positive-definite, that is, $||X|| = 0$ implies $X = 0$

(ii) The vector space is complete, which by definition means that every Cauchy sequence in the norm converges to a limit in the vector space. An inner product

space with these two properties is called a Hilbert space (see for details [?]). If either of the conditions above fail, the space is called a pre-Hilbert space.
We now give an example of a pre-Hilbert space where both conditions (i) and (ii) fail.

Example 6. ([6]) Let $\mathcal{R}$ denote the set of complex-valued Riemann integrable functions on $[0, 2\pi]$. This is a vector space over $\mathbb{C}$. Addition is defined pointwise by

$$(f + g)(\theta) = f(\theta) + g(\theta).$$

Naturally, multiplication by a scalar $\lambda \in \mathbb{C}$ is given by

$$(\lambda f)(\theta) = \lambda \cdot f(\theta).$$

An inner product is defined on this vector space by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\overline{g(\theta)}\, d\theta.$$

The norm of f is then

$$||f|| = (\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta)^{1/2}.$$

One needs to check that the analogue of the Cauchy-Schwarz and triangle inequalities hold in this example; that is $|(f, g)| \leq ||f||\ ||g||$. We first observe that $2AB \leq (A^2 + B^2)$ for any two real numbers A and B. If we set $A = \lambda^{1/2}|f(\theta)|$ and $B = \lambda^{-1/2}|g(\theta)|$ with $\lambda > 0$, we get

$$|f(\theta)\overline{g(\theta)}| \leq \frac{1}{2}(\lambda|f(\theta)|^2 + \lambda^{-1}|g(\theta)|^2).$$

We then integrate this in θ to obtain

$$|(f, g)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|\overline{g(\theta)}|d\theta \leq \frac{1}{2}(\lambda||f|| + \lambda^{-1}||g||^2).$$

Then, put $\lambda = ||g||/||f||$ to get Cauchy-Schwarz inequality. In our choice of λ we must assume that $||f|| \neq 0$ and $||g|| \neq 0$.

This leads us to the following observation.

Lemma 11. *The vector space $\mathcal{R}$ is a pre-Hilbert space.*

Proof. In $\mathcal{R}$, condition (i) for a Hilbert space fails, since $||f|| = 0$ implies only that f vanishes at its points of continuity.
To prove condition (ii) for a Hilbert space fails, we now want to show that the space $\mathcal{R}$ is not complete. To see this, define a sequence of functions in $\mathcal{R}$ by

$$f_n(\theta) = \begin{cases} 0 & \text{for} \quad 0 \leq \theta \leq 1/n, \\ f(\theta) & \text{for} \quad 1/n < \theta \leq 2\pi, \end{cases}$$

where

$$f(\theta) = \begin{cases} 0 & \text{for} \quad \theta = 0, \\ \log(1/\theta\ ) & \text{for} \quad 0 < \theta \leq 2\pi. \end{cases}$$

Then we can easily show that $\{f_n\}$ is a Cauchy sequence in $\mathcal{R}$, since

$$\begin{aligned} ||f_n(\theta) - f_m(\theta)|| &= \Big(\frac{1}{2\pi}\int_a^b (\log\theta)^2 d\theta\Big)^{1/2} \\ &= \Big(\frac{1}{2\pi}[\theta(\log\theta)^2 - 2\theta\log\theta + 2\theta]_a^b\Big)^{1/2} \end{aligned}$$

converges to 0 for all $0 < a < b$ and $b \to 0$. However, this sequence does not converge to an element in $\mathcal{R}$ since f is not bounded. □

We now turn to the proof of Theorem 10 ([6]).

Lemma 12. (**Best approximation**) *If f is integrable on the circle with Fourier coefficients a_n, then*

$$||f - S_N(f)|| \;\leq\; ||f - \sum_{|n|\leq N} c_n e_n||$$

for any complex numbers c_n. Moreover, equality holds precisely when $c_n = a_n$ for all $|n| \leq N$.

Proof. This follows immediately by applying the Pythagorean theorem to

$$f - \sum_{|n|\leq N} c_n e_n = f - S_N(f) + \sum_{|n|\leq N} b_n e_n,$$

where $b_n = a_n - c_n$. □

Proof of Theorem 10. Suppose that f is continuous on the circle. Then, given $\epsilon > 0$, there exists a trigonometric polynomial P, say of degree M, such that

$$|f(\theta) - P(\theta)| < \epsilon \quad \text{for all} \quad \theta.$$

This leads to us that

$$||f - P|| = \Big(\frac{1}{2\pi}\int_0^{2\pi} \big|f(\theta) - P(\theta)\big|^2 d\theta\Big)^{1/2} < \epsilon.$$

Then by the best approximation lemma 12, we conclude that

$$||f - S_N(f)|| < \epsilon \quad \text{whenever} \quad N \geq M.$$

If f is merely integrable, we apply the approximation lemma in [6], p.285 and choose a continuous function g on the circle which satisfies

$$sup_{\theta\in[0,2\pi]}\ |g(\theta)| \;\leq\; sup_{\theta\in[0,2\pi]}\,|f(\theta)|,$$

and

$$\int_0^{2\pi} |f(\theta) - g(\theta)|\, d\theta \;<\; \epsilon^2/4.$$

Since f is integrable, we set $|f(\theta)| = B$, and $|f(\theta) - g(\theta)| \leq |f(\theta)| + |g(\theta)| \leq 2B$. Then we get

$$\begin{aligned} ||f-g||^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)-g(\theta)||f(\theta)-g(\theta)|\, d\theta \\ &\leq \frac{B}{\pi}\int_0^{2\pi} |f(\theta)-g(\theta)|\, d\theta \\ &\leq C\epsilon^2. \end{aligned}$$

Now we may approximate g by a trigonometric polynomial P so that $||g - P|| < \epsilon/2$. Then

$$||f - P|| \leq ||f - g|| + ||g - P|| \leq C\,\epsilon.$$

If we apply the best approximation lemma 12, we complete the proof. □

We now summarize the results of this Chapter.

Theorem 11. *Let f be an integrable function on the circle with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Then we have*

(i) *Mean-square convergence of the Fourier series*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to \; 0 \quad as\ N \to \; \infty.$$

(ii) *Parseval's identity*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta.$$

Remark 3. We may associate to every integrable function the sequence $\{a_n\}$ formed by its Fourier coefficients. Parseval's identity guarantees that $\{a_n\} \in l^2(\mathbb{Z})$. Since $l^2(\mathbb{Z})$ is a Hilbert space, the failure of $\mathcal{R}$ to be complete, discussed earlier, may be understood as follows: there exist sequences $\{a_n\}_{n\in(\mathbb{Z})}$ such that $\sum_{n\in(\mathbb{Z})} |a_n|^2 < \infty$, yet no Riemann integrable function F has n^{th} Fourier coefficients equal to a_n for all n. We have an example.

Example 7. Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by

$$a_k = \begin{cases} 1/k & \text{if } k \geq 1, \\ 0 & \text{if } k \leq 0. \end{cases}$$

Note that $\{a_k\} \in l^2(\mathbb{Z})$, but that no Riemann integrable function has k^{th} Fourier coefficient equal to a_k for all k.

Finally, we study that if a function is differentiable at a point, then its Fourier series converges to that point (see [6]).

Proposition 3. *Let f be a bounded function on the compact interval $[a, b]$. If $c \in (a, b)$, and if for all small $\delta > 0$ the function f is integrable on the intervals $[a, c-\delta]$ and $[c+\delta, b]$, then f in integrable on $[a, b]$.*

Theorem 12. *Let f be an integrable function on the circle which is differentiable at a point θ_0. Then $S_N(f)(\theta_0) \to f(\theta_0)$ as N tends to infinity.*

Theorem 13. *Suppose f and g are two integrable functions defined on the circle, and for some θ_0 there exist an open interval I containing θ_0 such that*

$$f(\theta) = g(\theta) \quad \textit{for all} \quad \theta \in I.$$

Then $S_N(f)(\theta_0) - S_N(g)(\theta_0) \to 0$ as N tends to infinity.

Proof. . The function $f - g$ is 0 in I, so it is differentiable at θ_0, and we may apply the previous theorem 12 to conclude the proof. ☐

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