2007年 2月
博士學位 論文

ZEROS OF SELF-RECIPROCAL POLYNOMIALS

朝鮮大學校 大學院

數 學 科

朴　昌　雨

ZEROS OF SELF-RECIPROCAL POLYNOMIALS

자기상반다항식의 근들

2007年 2月 23日

朝鮮大學校 大學院

數 學 科

朴 昌 雨

ZEROS OF SELF-RECIPROCAL POLYNOMIALS

지도교수 홍 성 금

이 論文을 理學博士學位申請 論文으로 제출함

2006年 10月

朝鮮大學校 大學院

數 學 科

朴 昌 雨

朴昌雨의 博士學位 論文을 認准함

委員長 朝鮮大學校 敎授 李成伯

委 員 朝鮮大學校 敎授 鄭潤巻

委 員 朝鮮大學校 敎授 安永濬

委 員 淑明女子大學校 敎授 金善弘

委 員 朝鮮大學校 敎授 洪成金

2006年 12月

朝鮮大學校 大學院

CONTENTS

국문초록

자기상반다항식의 근들

박 창 우
지도교수 : 홍 성 금
조 선 대 학 교 대 학 원
수 학 과

자기상반다항식의 근은 단위원 위에 있거나 쌍들이 단위원에 대하여 공액으로 나타나며, 모든 근이 단위원 위에 있는 n차 다항식 $Q(z)$는 어떤 $|\mu|=1$인 μ에 대하여 $Q(z)=\mu z^{n}Q(1/z)$을 만족한다. 따라서 자기상반다항식의 모든 근들이 단위원 위에 있을 충분조건을 알아보는 것은 흥미롭다.

가운데 항의 계수의 절대값이 다른 항의 계수들의 합과 같으며 실수를 계수로 갖는 삼항 자기상반다항식의 모든 근들은 단위원 위에 존재한다. 이 논문에서는 위의 다항식을 항이 다섯 개인 경우로 확장하여 이 다항식의 근들이 단위원을 중심으로 어떻게 분포하는지를 연구하였다. 한편 연구의 과정으로부터 새로운 부등식과 Eneström-Kakeya 유형의 결과를 부가적으로 얻었다.

1. Introduction and statements of results

1.1 Introduction

Throughout this thesis, U denotes the unit circle and n is a positive integer. A polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is said to be a self-inversive polynomial of degree n if it satisfies $a_n \neq 0$ and $P(z) = \mu P^*(z)$, where $|\mu| = 1$ and

$$P^*(z) = z^n \overline{P(1/\bar{z})} = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \cdots + \overline{a_n}.$$

In particular, if $P(z) = z^n P(1/z)$, $P(z)$ is called be a self-reciprocal polynomial. Thus the zeros of a self-reciprocal polynomial either lie on U or occur in pairs conjugate to U. By Cohn's theorem ([4]), a polynomial $Q(z)$ of degree n with all its zeros on U must be of the form $Q(z) = \mu z^n Q(1/z)$ for some μ, $|\mu| = 1$. Hence it is interesting to mention the condition for a self-reciprocal polynomial with all zeros on U. Besides problems for the zeros, a self-reciprocal polynomial $P(z)$ of degree n has the remarkable relation (see p. 153 of [11])

$$\max_{z \in U} |P'(z)| = \frac{n}{2} \max_{z \in U} |P(z)|.$$

Furthermore, every point of maximum modulus of $P(z)$ on U is also a point of maximum modulus of $P'(z)$. In particular, this relation holds for all polynomials of degree n whose zeros lie on U.

A useful tool for showing a self-inversive polynomial having all its zeros on U is due to Cohn ([4]).

Theorem 1.1. **(Cohn)** *Let $P(z)$ be a self-inversive polynomial of degree n. Suppose that $P(z)$ has exactly τ zeros on U (counted according to multiplicity) and exactly ν critical points in the closed unit disc (counted according to multiplicity). Then*

$$\tau = 2(\nu + 1) - n.$$

Thus a necessary and sufficient condition for all zeros of a self-inversive polynomial $P(z)$ to lie on U is that all zeros of $P'(z)$ lie inside or on U. Another approach to show a self-inversive polynomial having all its zeros on U is by using Chebyshev transformation. Studying the spectral properties of the Coxeter transformation, Lakatos ([8]) found sufficient conditions for self-reciprocal polynomials having all their zeros on U by using Chebyshev transformation.

This transformation was also used for the study of zeros of certain sums of two self-reciprocal polynomials. A convex combination of two self-reciprocal polynomials that are products with same degree of finitely many finite geometric series with each having even degree does not always have all its zeros on U. However Kim ([7]) showed that, if a polynomial is obtained by adding a finite geometric series multiplied by a large constant to such a convex combination, it has all its zeros on U by using the transformation. Schinzel ([10]) generalized Lakatos's result ([9]) above for self-inversive polynomials by proving that all zeros of the polynomials $P(z) = \sum_{k=0}^{n} a_k z^k$ where

$$a_n \neq 0, \;\; a_k \in \mathbb{C}, \;\; \text{and} \; \epsilon \bar{a_k} = a_{n-k} \; (0 \leq k \leq n) \; \text{with} \; \epsilon \in \mathbb{C}, \;\; |\epsilon| = 1$$

lie on U provided that

$$|a_n| \geq \inf_{c,d \in \mathbb{C}, |d|=1} \sum_{k=0}^{n} \left| c a_k - d^{m-k} a_n \right|$$

holds. Schinzel's proof was based on a theorem of Cohn ([4]) and on the estimate

$$\min_{z \in \mathbb{C}, |z|=1} \left| \sum_{k=1}^{n} k z^{n-k} \right| \geq \frac{n}{2}.$$

Lakatos and Losonczi ([9]) again improved Lakatos's previous result and above Schinzel's result for polynomials of odd degrees. However their method didn't work for even degree polynomials.

Our first goal in this thesis is to investigate the zero distribution of certain self-reciprocal polynomials. More precisely, we will have some results about the zero distribution of real self-reciprocal polynomials of even degrees with five terms whose absolute values of middle coefficients are equal to the sum of all other coefficients. While studying this, we will get a new inequality and Eneström-Kakeya ([6]) types of results. We end this section with introducing one of the most fundamental theorems, Rouché's theorem (p. 48 of [11]), related to this study in complex analysis. It will be often used in the sequel.

Theorem 1.2. **(Rouché)** *Suppose $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C, with $|g(z)| < |f(z)|$ on C. Then $f(z) + g(z)$ has the same number of zeros as $f(z)$ inside C.*

1.2 Statements of results

All real self-reciprocal polynomials of even degrees with three terms whose absolute values of middle coefficients are equal to the sum of all other coefficients are of the form

$$Az^{2m} \pm 2Az^m + A = A(z^m \pm 1)^2,$$

and have all their zeros on U. This induces our attention naturally how zeros of the same kinds of self-reciprocal polynomials with five terms are located around U. In fact, there are exactly four types of such polynomials as follows. For integers m, n with $m > n > 0$ and positive real numbers a, b, we let

$$\begin{aligned}
P^+(z) &= az^{2m} - bz^{m+n} + 2(a-b)z^m - bz^{m-n} + a, \\
P^-(z) &= az^{2m} - bz^{m+n} - 2(a-b)z^m - bz^{m-n} + a, \\
Q^+(z) &= az^{2m} + bz^{m+n} + 2(a+b)z^m + bz^{m-n} + a, \\
Q^-(z) &= az^{2m} + bz^{m+n} - 2(a+b)z^m + bz^{m-n} + a.
\end{aligned}$$

The first purpose of this thesis is to investigate zero distribution of above four polynomials around U. In Section 2.1, we will show that for $a \geq b > 0$ all zeros of $P^{+}(z)$ and $P^{-}(z)$ lie on U. However these polynomials for $b > a > 0$ will be studied for some special cases. In Section 2.2, we will see that the number of zeros of $Q^{+}(z)$ and $Q^{-}(z)$, respectively on U relies on the greatest common divisor of $2m$, $m+n$, and $m-n$. All results for $P^{+}(z)$, $P^{-}(z)$, $Q^{+}(z)$ and $Q^{-}(z)$ are given below. Throughout these theorems, we assume that m, n are integers with $m > n > 0$ and a, b are positive real numbers as above. We will prove Theorems 1.3, 1.4 and 1.5 in Chapter 2.

Theorem 1.3. *For $a \geq b > 0$, all zeros of $P^{+}(z)$ and $P^{-}(z)$ lie on U.*

Theorem 1.4. *Let d be the greatest common divisor of $2m$, $m+n$, and $m-n$. Then we have the following.*

(a) *If $d \mid m$, $Q^{+}(z)$ has no zero on U.*

(b) *If $d \nmid m$ and $m = dk+r$ for some integers k, r with $1 \leq r \leq d-1$, $Q^{+}(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1.*

Theorem 1.5. *Let d be the greatest common divisor of $2m$, $m+n$, and $m-n$. Then we have the following.*

(a) *If $d \mid m$, $Q^{-}(z)$ has exactly d zeros on U without counting multiplicities. Such zeros are the d-th roots of -1.*

(b) *If $d \nmid m$ and $m = dk+r$ for some integers k, r with $1 \leq r \leq d-1$, $Q^{-}(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1.*

The polynomials $P^{+}(z)$ and $P^{-}(z)$ are of interest. For example, a special case of $P^{+}(z)$ with $a = m^2$ and $b = n^2$, i.e.,

$$T(z) := m^2 z^{2m} - n^2 z^{m+n} + 2(m^2 - n^2) z^m - n^2 z^{m-n} + m^2,$$

plays a role in the study of inequalities. In fact, it follows from

$$T(z) > 0 \quad \text{for } 0 < z < 1$$

that we have a new inequality.

Theorem 1.6. *For $y > x > 0$ and $0 < \lambda < 1$, we have an inequality*

$$x^2 + 2(1-\lambda^2)xy + y^2 > \lambda^2 (x^{1+\lambda} y^{1-\lambda} + x^{1-\lambda} y^{1+\lambda}). \tag{1.1}$$

We observe that, in (1.1), if $\lambda = 0$, $(x+y)^2 > 0$, and if $\lambda = 1$, the both sides are equal to $x^2 + y^2$. The proof of Theorem 1.6 will be given in Section 3.1. Also the zeros of a special case of $P^-(z)$ with $a = n^2$ and $b = m^2$ are those of the equation

$$z^n \left(\frac{1}{m} \frac{z^m - 1}{z - 1} \right)^2 = z^m \left(\frac{1}{n} \frac{z^n - 1}{z - 1} \right)^2 .$$

This is interesting because $\frac{z^m-1}{z-1}$ is an analogue of m. Finally some self-reciprocal polynomials that are factors of $P^-(z)$ seem to be related to Eneström-Kakeya ([6]) types of problems.

Theorem 1.7. (Eneström-Kakeya) *Let $a_0, a_1, \cdots, a_n$ be real numbers satisfying*

$$a_0 \geq a_1 \geq \cdots \geq a_n > 0.$$

Then the polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ has no zeros inside U.

For the proof of Eneström-Kakeya Theorem above, see p. 12 of [2]. In fact, if $m - n = 1$, then the first half terms starting with constant terms of the self-reciprocal polynomials

$$U(z) = \frac{P^-(z)}{(z-1)^2} = \sum_{k=0}^{2m-2} a_k z^k,$$

where $a_k = b + k(b-a)$, seem to have their coefficients in increasing order. From the increment of the order, it is natural to investigate how zeros of $U(z)$ are located around U. We will prove in Section 3.2 the theorem below about the zero distribution of the generalized polynomial of $U(z)$ above.

Theorem 1.8. *Let $P(z) = \sum_{k=0}^{2m+1} a_k z^k$ be a real self-reciprocal polynomial and $a_k = 1 + kr, k = 1, 2, \cdots, m$. Then we have the following.*

(a) *If $r < -\frac{2}{m}$, then $P(z)$ has $2m-1$ zeros on U and a zero in $[0,1]$,*

(b) *If $-\frac{2}{m} < r \leq 2$, then $P(z)$ has all its zeros on U,*

(c) *If $2 < r < 2+\frac{2}{m}$ and m is even, then $P(z)$ has all its zeros on U,*

(d) *If $r = 2 + \frac{2}{m}$ and m is even, then $P(z)$ has all its zeros on U,*

(e) *If $r > 2 + \frac{2}{m}$, then $P(z)$ has $2m-1$ zeros on U.*

In above theorem, the three cases "$2 < r < 2 + \frac{2}{m}$ and m is odd", "$r = -\frac{2}{m}$", "$r = 2 + \frac{2}{m}$ and m is odd" remain open problems.

2. Four types of self-reciprocal polynomials

We often use Theorems 1.1 and 1.2 to prove our results Theorems 1.3, 1.4 and 1.5 in Section 1.2.

2.1 The zeros of $P^{+}(z)$ and $P^{-}(z)$

We first prove Theorem 1.3.

Proof of Theorem 1.3. We use Theorem 1.1 and Theorem 1.2. Let

$$p(z) := \frac{[P^{+}(z)]'}{z^{m-n-1}} = 2amz^{m+n} - b(m+n)z^{2n} + 2(a-b)mz^{n} - b(m-n).$$

For $\epsilon > 0$, we define the polynomial

$$p_{\epsilon}(z) := (2am+\epsilon)z^{m+n} - b(m+n)z^{2n} + 2(a-b)mz^{n} - b(m-n).$$

Since $a \geq b > 0$, for $|z| = 1$, we have

$$\begin{aligned}|(2am+\epsilon)z^{m+n}| &= (2am+\epsilon) > 2am \\ &= b(m+n) + 2(a-b)m + b(m-n) \\ &\geq |-b(m+n)z^{2n} + 2(a-b)mz^{n} - b(m-n)|.\end{aligned}$$

By Theorem 1.2 $p_\epsilon(z)$ has all its zeros strictly inside U which implies that all zeros of $p(z)$ lie inside or on U. Now Theorem 1.1 completes the proof. The result for $P^-(z)$ can be proved in the same way. $\square$

We can also prove all zeros of $P^-(z)$ lying on U by using the following theorem ([3]).

Theorem 2.1. **(Chen)** *A necessary and sufficient condition for all the zeros of* $P_n(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ *with complex coefficients to lie on the unit circle is that there is a polynomial* $q_{n-l}(z)$ *with all its zeros in or on the unit circle such that*

$$P_n(z) = z^l q_{n-l}(z) + e^{i\theta} q^*_{n-l}(z), \text{ where } q^*_{n-l}(z) = z^{n-l}\overline{q_{n-l}(1/\bar{z})}$$

for some nonnegative integer l *and real* θ*.*

The suitable $q_{n-l}(z)$ in Theorem 2.1 for $P^-(z)$ is

$$az^m - bz^n - (a-b).$$

In fact, we have

$$P^-(z) = z^m[az^m - bz^n - (a-b)] + [-(a-b)z^m - bz^{m-n} + a]$$
$$= z^m[az^m - bz^n - (a-b)] + [az^m - bz^n - (a-b)]^*$$

and $az^m - bz^n - (a-b)$ has all its zeros in or on U. See p. 227 of [1] for the zero distribution of $az^m - bz^n - (a-b)$. The polynomials $P^+(z)$ and $P^-(z)$ for $b > a > 0$ are studied for some special cases. The problems about the number of zeros on U for these polynomials in the general case remain open.

Proposition 2.2. *If $n|m$, and $0 < b < a$, then $P^+(z)$ has at least $2n$ zeros on U.*

Proof. We note that

$$z^nP^+(z) = az^n(z^{2m} + 2z^m + 1) - bz^m(z^{2n} + 2z^n + 1)$$
$$= az^n(z^m + 1)^2 - bz^m(z^n + 1)^2.$$

For $n|m$, we have

$$
\begin{aligned}
&z^n P^+(z)\\
&= az^n(z^n+1)^2(z^{m-n}-z^{m-2n}+\cdots-z^n+1)^2 - bz^m(z^n+1)^2\\
&= (z^n+1)^2[az^n(z^{m-n}-z^{m-2n}+\cdots-z^n+1)^2 - bz^m],
\end{aligned}
$$

which proves the proposition. □

Proposition 2.3. *If $m-n=b-a=1$ and $a>n$, then all zeros of $P^-(z)$ lie on U.*

Proof. It follows from $m-n=b-a=1$ that

$$
\begin{aligned}
P^-(z) &= az^{2n+2}-(a+1)z^{2n+1}+2z^{n+1}-(a+1)z+a\\
&= z^{n+1}(az^{n+1}-(a+1)z^n+1)+(z^{n+1}-(a+1)z+a)\\
&= z^{n+1}(az^{n+1}-(a+1)z^n+1)+(az^{n+1}-(a+1)z^n+1)^*.
\end{aligned}
$$

By Theorem 2.1, it suffices to show that

$$az^{n+1}-(a+1)z^n+1$$

has all its zeros inside or on U. On the other hand,

$$az^{n+1}-(a+1)z^n+1=(z-1)(az^n-z^{n-1}-z^{n-2}-\cdots-z-1).$$

Since $a > n$, $az^n - z^{n-1} - z^{n-2} - \cdots - z - 1$ has all its zeros inside U by Theorem 1.2 and the assumption $a > n$. This completes the proof. □

Proposition 2.4. *If $n|m$ and $0 < b < a$, then $P^-(z)$ has at least $2n$ zeros on U.*

Proof. Note that

$$z^n P^-(z) = az^n(z^m - 1)^2 - bz^m(z^n - 1)^2.$$

For $n|m$, we have

$$\begin{aligned}
&z^n P^-(z) \\
&= az^n(z^n - 1)^2(z^{m-n} + z^{m-2n} + \cdots + z^n + 1)^2 - bz^m(z^n - 1)^2 \\
&= (z^n - 1)^2[az^n(z^{m-n} + z^{m-2n} + \cdots + z^n + 1)^2 - bz^m],
\end{aligned}$$

which proves the proposition. □

2.2 The zeros of $Q^+(z)$ and $Q^-(z)$

We prove Theorems 1.4 and 1.5 in this section.

Proof of Theorem 1.4. For $\epsilon > 0$, we define the polynomial

$$Q_\epsilon^+(z) = az^{2m} + bz^{m+n} + (2(a+b)+\epsilon)z^m + bz^{m-n} + a.$$

For $|z| = 1$,

$$|Q_\epsilon^+(z)| \geq 2(a+b) + \epsilon - 2a - 2b = \epsilon > 0,$$

which implies that $Q_\epsilon^+(z)$ does not have a zero on U. Also it follows from Theorem 1.2 that $Q_\epsilon^+(z)$ has exactly m zeros strictly inside U, say $\alpha_1, \cdots, \alpha_m$. Suppose that, as $\epsilon \to 0$, some of these tend to U, say

$$\alpha_j \to e^{i\theta_j}, \quad \theta_j \in \mathbb{R}.$$

Since $Q^+(z_j) = 0$, where $z_j = e^{i\theta_j}$, we have

$$|az_j^{2m} + bz_j^{m+n} + bz_j^{m-n} + a| = 2(a+b).$$

This equality holds only if the four points z_j^{2m}, z_j^{m+n}, z_j^{m-n} and 1 have the same argument, so

$$(2m)\theta_j \equiv (m+n)\theta_j \equiv (m-n)\theta_j \equiv 0 \pmod{2\pi}.$$

Hence $e^{i\theta_j}$ is a d-th root of unity, where

$$d = \gcd(2m, m+n, m-n). \tag{2.1}$$

If $d|m$, then

$$\begin{aligned} Q^+(w) &= aw^{2m} + bw^{m+n} + 2(a+b)w^m + bw^{m-n} + a \\ &= a + b + 2(a+b)w^m + b + a \\ &= 2(a+b)(1+w^m) = 4(a+b) \neq 0. \end{aligned}$$

Thus $Q^+(z)$ has no zero on U. We now suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \leq r \leq d-1$, then d must be even since, for d odd, $d \mid 2m$ and so $d \mid m$. Also $d \mid 2m$ implies that $d \mid 2r$. Letting $2r = du$ for some positive integer u, we have $du/2 < d$ and so $u = 1$, i.e., $d = 2r$. Now

$$\begin{aligned} Q^+(w) &= aw^{2m} + bw^{m+n} + 2(a+b)w^m + bw^{m-n} + a \\ &= a + b + 2(a+b)w^m + b + a \\ &= 2(a+b)(1+w^m) = 2(a+b)(1+w^{d/2}). \end{aligned}$$

Since the $d/2$-th roots of -1 are contained in the $d/2$-th roots of unity, $Q^+(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1. $\square$

Proof of Theorem 1.5. For $\epsilon > 0$, we define the polynomial

$$Q_\epsilon^-(z) = az^{2m} + bz^{m+n} - (2(a+b)+\epsilon)z^m + bz^{m-n} + a.$$

For $Q_\epsilon^-(z)$, we follow exactly same procedure of the proof of Theorem 1.4 until (2.1). Then we can see that if the zeros of $Q_\epsilon^-(z)$ on U exist, they must be d-th roots of unity as in proof of Theorem 1.4. If $d|m$, then

$$\begin{aligned} Q^-(w) &= aw^{2m} + bw^{m+n} - 2(a+b)w^m + bw^{m-n} + a \\ &= a + b - 2(a+b)w^m + b + a \\ &= 2(a+b)(1-w^m). \end{aligned}$$

Hence $Q^-(w)$ has exactly d zeros on U that are the d-th roots of unity. We now suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \le r \le d-1$, then $d = 2r$ as in the proof of Theorem 1.4.

Now

$$\begin{aligned}
Q^-(w) &= aw^{2m} + bw^{m+n} - 2(a+b)w^m + bw^{m-n} + a \\
&= a + b - 2(a+b)w^m + b + a \\
&= 2(a+b)(1-w^m) = 2(a+b)(1-w^r).
\end{aligned}$$

Since the $d/2$-th roots of -1 are contained in the $d/2$-th roots of unity, $Q^+(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1. $\square$

3. An inequality and Eneström-Kakeya types of problems

3.1 A new inequality

Now we prove Theorem 1.6 in section 1.2.

Proof of Theorem 1.6. Let, for $0 < z < 1$, where $m > n > 0$,

$$T(z) := m^2 z^{2m} - n^2 z^{m+n} + 2(m^2 - n^2) z^m - n^2 z^{m-n} + m^2.$$

Then we have

$$T(z) > 0 \quad \text{for } 0 < z < 1.$$

This is because

$$T(z) = z^{-n}((m^2 z^n (z^m + 1)^2) - n^2 z^m (z^n + 1)^2)), \quad z^n > z^m,$$

and

$$m^2 (z^m + 1)^2 > n^2 (z^n + 1)^2.$$

In fact

$$(3.1) \qquad mz^m - nz^n + m - n > nz^m - mz^n + m - n > 0.$$

For the second inequality of (3.1), see pp. 39-42 of [5].

We replace z by $\left(\frac{x}{y}\right)^{\frac{1}{m}}$ (with $y > x$) so that we obtain

$$m^2\left(\frac{x}{y}\right)^2 - n^2\left(\frac{x}{y}\right)^{1+\frac{n}{m}} + 2(m^2 - n^2)\left(\frac{x}{y}\right) - n^2\left(\frac{x}{y}\right)^{1-\frac{n}{m}} + m^2 > 0.$$

Letting $n/m = \lambda$ and dividing m^2y^{-2} of each side give

$$x^2 - \lambda^2 x^{1+\lambda}y^{1-\lambda} + 2(1-\lambda^2)xy - \lambda^2 x^{1-\lambda}y^{1+\lambda} + y^2 > 0. \quad \square$$

3.2 Eneström-Kakeya types of problems

In this section, we prove Theorem 1.8.

Proof of Theorem 1.8. We first prove (a). Observe that

$$P(z) = \frac{(z^{m+1}-1)(1+rz+rz^2+\cdots+rz^m+z^{m+1})}{z-1}.$$

So $P(z)$ has at least m zeros, $(m+1)$-th roots of unity except 1, on U. Let

$$u(z) := 1+rz+rz^2+\cdots+rz^m+z^{m+1}.$$

Then we observe that

$$(z-1)u(z) = z^{m+2}+(r-1)z^{m+1}-(r-1)z-1.$$

To use Theorem 1.1, we show that $(z-1)u(z)$ has exactly m critical points inside or on U. Differentiating $(z-1)u(z)$ with respect to z gives

$$(m+2)z^{m+1}+(r-1)(m+1)z^m-(r-1).$$

Let

$$f(z) := (m+2)z^{m+1}+(r-1)(m+1)z^m-(r-1).$$

Then, for $|z| = 1$,

$$\begin{aligned}|(r-1)(m+1)z^m| &= (1-r)(m+1)\\ &> (m+2)+(1-r)\\ &\geq |(m+2)z^{m+1}-(r-1)|.\end{aligned}$$

So $f(z)$ has m zeros inside U by Theorem 1.2. It follows from $f(1) = 2 + rm < 0$ and $f(z) \to \infty$ as $z \to \infty$ that $f(z)$ has at least one zero in $(1, \infty)$. Thus $f(z)$ has m zeros inside or on U, and so $(z-1)u(z)$ has exactly m critical points inside or on U. Now the fact $P(0) = 1$ and $P(1) = (m+1)(2+mr) < 0$ deduce that $P(z)$ has at least one zero in $(0,1)$. Next we prove (b). The cases for $r = 0, 1$ are trivial. For $-\frac{2}{m} < r < 0$, it suffices to show that

$$u'(z) = r + 2rz + 3rz^2 + \cdots + mrz^{m-1} + (m+1)z^m$$

has all its zeros inside or on U by Theorem 1.1. For $|z| = 1$,

$$\begin{aligned}&|r + 2rz + 3rz^2 + \cdots + mrz^{m-1}|\\ \leq& -r - 2r - 3r - \cdots - mr = -r\frac{m(m+1)}{2}\\ <& m+1 = |(m+1)z^m|.\end{aligned}$$

Thus Theorem 1.2 proves the result. In case $0 < r < 1$, we consider

$$(z-1)^2 P(z) = q(z)(z^{m+1} - 1), \tag{3.1}$$

where

$$q(z) = z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1.$$

Then $q(z)$ has all its zeros on U. In fact, $q(z)$ is self-inversive and all the zeros of

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1) \tag{3.2}$$

lie inside or on U since, for $|z| = 1$,

$$\begin{aligned} &|(r-1)(m+1)z^m - (r-1)| \\ &< (1-r)(m+1) + (1-r) = (1-r)(m+2) \\ &< (m+2) = |(m+2)z^{m+1}|. \end{aligned}$$

Theorem 1.1 and Theorem 1.2 prove the case. It remains the case $1 < r \leq 2$ to complete the proof of (b). A proof for the case $1 < r < 2$ is very similar to that of the case $0 < r < 1$ above.

From (3.1), it is enough to show that all zeros of

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1)$$

lie inside or on U by Theorem1.1. For $|z| = 1$,

$$\begin{aligned}
&|(r-1)(m+1)z^m - (r-1)| \\
&< (r-1)(m+1) + (r-1) = (r-1)(m+2) \\
&< (m+2) = |(m+2)z^{m+1}|.
\end{aligned}$$

By Theorem 1.2, $q'(z)$ has all its zeros inside U. The final case $r = 2$ is easily checked from

$$\begin{aligned}
(z-1)^2 P(z) &= (z^{m+1} - 1)(z^{m+2} + z^{m+1} - z - 1) \\
&= (z^{m+1} - 1)(z+1)(z^{m+1} - 1).
\end{aligned}$$

Now we prove (c). For $2 < r < 2 + \frac{2}{m}$ and m is even, it suffices to show that $q(z)$ has all its zeros on U in (3.1). We consider the zeros of $q(-z)$ instead of $q(z)$. Then we have

$$q(-z) = (z-1)r(z),$$

where $r(z) = z^{m+1} + (2-r)z^m + (2-r)z^{m-1} + \cdots + (2-r)z + 1$.

The $r(z)$ has all its zeros on U. In fact, $r(z)$ is a self-inversive polynomial and all zeros of

$$r'(z) = (m+1)z^m + m(2-r)z^{m-1} + \cdots + 2(2-r)z + (2-r)$$

lie inside or on U, since for $|z| = 1$,

$$\begin{aligned}
|(m+1)z^m| &= m+1 > (r-2)m(m+1)/2 \\
&= m(r-2) + (m-1)(r-2) + \cdots + 2(r-2) + (r-2) \\
&= |m(2-r)z^{m-1}| + |(m-1)(2-r)z^{m-2}| + \cdots \\
&\quad + |2(2-r)z| + |(2-r)| \\
&\geq |m(2-r)z^{m-1} + (m-1)(2-r)z^{m-2} + \cdots \\
&\quad + 2(2-r)z + (2-r)|.
\end{aligned}$$

The (c) is proved Theorem 1.1 and Theorem 1.2. Now we prove (d). For $r = 2 + \frac{2}{m}$ and m is even, we follow exactly the same procedure of the proof (b) until (3.2). In (3.2), we observe

$$\begin{aligned}
q'(-z) &= (1+1/n)[-2nz^{2n+1} + (2n+1)z^{2n} - 1] \\
&= -(1+1/n)(z-1)^2(2nz^{2n-1} + (2n-1)z^{2n-2} + \cdots + 2z + 1),
\end{aligned}$$

where $m = 2n$, $n = 1, 2, \cdots$.

By Theorem 1.7, $2nz^{2n-1} + (2n-1)z^{2n-2} + \cdots + 2z + 1$ has all its zeros inside U. So all zeros of $q'(z)$ lie inside or on U. Now we prove (e). In (3.1), it is enough to show that

$$q'(z) = (m+2)z^{m+1} + (r-1)(m+1)z^m - (r-1)$$

has exactly m zeros inside or on U by Theorem 1.1. For $|z| = 1$,

$$\begin{aligned}|(m+2)z^{m+1} - (r-1)| &\leq m+2+r-1 \\ &< (m+1)(r-1) = |(m+1)(r-1)z^m|.\end{aligned}$$

By Theorem 1.2, $q'(z)$ has m zeros inside U. If m is odd, then $q'(-1) = 2(m+2) - r(m+2) < 0$ and $q'(z) \to \infty$ as $z \to -\infty$. If m is even, then $q'(-1) = -2m - 2 + rm > 0$ and $q'(z) \to -\infty$ as $z \to -\infty$. Hence $q'(z)$ has at least a zero $(-\infty, -1)$. Finally $q'(z)$ has exactly m zeros inside or on U. $\square$

It can be proved by Theorem 2.1 that the zeros of $P(z)$ has all its zeros on U if $0 \leq r \leq 2$. From (3.1) it suffices to show that

$$z^{m+2} + (r-1)z^{m+1} - (r-1)z - 1 \tag{3.3}$$

has all its zeros on U. The suitable $q_{n-l}(z)$ in Theorem 2.1 for (3.3) is $z+r-1$. In fact, we have

$$\begin{aligned}&z^{m+2}+(r-1)z^{m+1}-(r-1)z-1\\=&z^{m+1}(z+r-1)-[(r-1)z+1]\\=&z^{m+1}(z+r-1)-[z+r-1]^*\end{aligned}$$

and $z+r-1$ has all its zeros on U since $0\le r\le 2$.

3.3 An open problem

There have been a number of literatures about the relationship between the coefficients of a polynomial and the location of its zeros. One of the most beautiful results in this subject is maybe Theorem 1.7 which is Eneström-Kakeya's theorem. While studying the materials the previous sections, we encountered with the following Eneström-Kakeya type of problem. We end with introducing our conjecture.

Conjecture 3.1. *Let* $P(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial of degree* n*. If* $a_n \geq a_{n-1} \geq \cdots \geq a_2 \geq a_1 > 0$, $a_0 < 0$ *and* $P(-1) > 0$*, then all zeros of* $P(z)$ *lie in* $|z| < 1$*.*

As an example of the conjecture, we consider

$$\begin{aligned} P(z) =& 14z^{12} + \frac{23}{2}z^{11} + 11z^{10} + 10z^9 + \frac{19}{2}z^8 + 8z^7 \\ &+ \frac{15}{2}z^6 + 7z^5 + \frac{16}{3}z^4 + 5z^3 + 3z^2 + z - 2. \end{aligned}$$

By computer algebra, we can compute the absolute values of the zeros of $P(z)$ as following.

$$\begin{array}{llll} 0.90611\cdots, & 0.89104\cdots, & 0.89104\cdots, & 0.88194\cdots, \\ 0.88194\cdots, & 0.90514\cdots, & 0.90514\cdots, & 0.91057\cdots, \\ 0.91057\cdots, & 0.45139\cdots, & 0.91245\cdots, & 0.91245\cdots. \end{array}$$

REFERENCES

[1] Y. J. Ahn, S.-H. Kim, *Zeros of certain trinomial of equation*, Math. Inequal. Appl. **9** (2006), no. 2, 225–232.

[2] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics, Springer, New York, 1991.

[3] W. Chen, *On the polynomials with all their zeros on the unit circle*, J. Math. Anal. Appl. **190** (1995), 714–724.

[4] A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. **14** (1922), 110–148.

[5] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1975.

[6] S. Kakeya, *On the limits of the roots of an algebraic equation with positive coefficients*, Tôhoku Math. J. **2** (1912), 140–142.

[7] S.-H. Kim, *On zeros of certain sums of polynomials*, Bull. Korean Math. Soc. **41** (2004), no. 4, 641-646.

[8] P. Lakatos, *On zeros of reciprocal polynomials*, Publ. Math. Debrecen **61** (2002), 645–661.

[9] P. Lakatos, László Losonczi, *On zeros of reciprocal polynomials of odd degree*, J. Inequal. Pure Appl. Math **4** (2003), no. 3, 8–15.

[10] A. Schinzel, *Self-inversive polynomials with all zeros on the unit circle*, Ramanujan J. **9** (2005), 19–23.

[11] T. Sheil-Small, *Complex Polynomials*, Cambridge Studies in Advanced mathematics 73, Cambridge University Press, Cambridge, 2002.