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# DIFFERENTIAL EQUATION ON WARPED PRODUCT MANIFOLDS

朝 鮮 大 學 校 大 學 院

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- 휜곱다양체 위의 미분방정식-

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指導敎授 鄭 潤 泰

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# 李尙澈의 碩士學位 論文을 認准함

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李 尙 澈
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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다.

연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

본 논문에서는 상해와 하해의 방법을 이용하여 휜곱다양체 위의 편미분방정식의 해를 연구하고자 한다.

연구내용은 다음과 같다.

제2장에서는 휜곱다양체에 관한 기본적인 개념과 몇 가지 결과를 설명하였다.

제3장에서는 $N$이 상수 스칼라 곡률을 갖는 적당한 다양체일 때, $M=[a,\ \infty)\times_f N$ 위의 함수 $R(t,\ x)$가 $t$만의 함수이고 조건 $\frac{4n}{n+1}Bt^{\beta}\geq R(t)\geq\frac{4n}{n+1}\cdot\frac{C}{t^{\alpha}}$, $t\geq t_0$,(여기서

$t_0$, $0<\alpha<2$, $0<\beta<\frac{4n}{n+1}$ $C$, $B$ : 양의 정수)을 만족하면 $R(t)$를 스칼라 곡률로 갖는 $M$ 위의 휜곱거리가 존재함을 보였다.

# 1. INTRODUCTION

One of the well-known problems in differential geometry is that of whether a given smooth function on a compact Riemannian manifold is necessarily the scalar curvature of some metric. In order to study these kinds of problems, we need some analytic methods in differential geometry, because they have the forms of differential equations.

In particular, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifold $B \times F$ be a warped product construction is studied very widely in differential geometry. The warped product has been used in making some special examples in Riemannian geometry and Lorentzian geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [4], [5], [6], [7], [8], [10], [11], [14], [20], [21], [23]), and also in studying $L_2-$cohomology (cf. [27]).

For Lorentzian manifolds, warped products have been used widely in studying the space-times (cf. [1], [3], [12], [13], [15], [19], [16,17,18], [22], [25]). Since warped product have proven important in global Riemannian

geometry, it is thus not surprising that the equivalent Lorentzian concept is also quite useful.

Perhaps even more interestingly on physical grounds than purely Riemannian constructions employing warped products, many of known exact solutions of the Einstein field equations of General Relativity are warped product metrics of the form $B \times_f F$, where $(B, g_B)$ is a Lorentzian manifold and $(F, g_F)$ is a Riemannian manifold. A most notable class of examples are the Robertson-Walker space-time of cosmology theory as well as the Schwarzschild space-time. So, in Lorentzian geometry, the warped product is also widely used for studying space-times with various applications (cf. [1], [7], [9], [13], [22], etc.).

In recent work, some authors have considered the problem of scalar curvature functions on a warped product manifold and obtained partial results about the existence and nonexistence of a warped metric with some prescribed scalar curvature function (cf. [15], [16,17,18], [19]).

In this paper, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f F$ in terms of its warping function $f$ and the scalar curvatures of $B$ and $F$. Using upper solution and lower solution

methods, we treat the existence of a warping function $f$ such that the resulting metric admits the prescribed scalar curvature function.

In [18], authors showed that if $R(t,x) = R(t) \in C^{\infty}([a,\infty))$ is a positive function such that

$$b \geq R(t) \geq \frac{4n}{n+1} \cdot \frac{C}{t^{\alpha}} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $\alpha < 2$, $C$ and $b$ are positive constants, then equation (3.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

In this paper, we extend the results of [18]. That is, we show that if $R(t,x) = R(t) \in C^{\infty}([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1} B t^{\beta} \geq R(t) \geq \frac{4n}{n+1} \cdot \frac{C}{t^{\alpha}} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \alpha < 2$, $0 < \beta < \frac{4}{n+1}$, $C$ and $B$ are positive constants, then equation (3.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

This thesis is constituted as follows:

In Section 2, we introduce the basic concepts and some results about warped product manifolds.

In Section 3, we discuss the method of using warped products to construct complete warped product metrics on $M = B \times_f N$ with specific scalar curvatures when $N$ is a manifold with constant scalar curvature. It is shown that if the fiber manifold $N$ admits a metric of constant scalar curvature and if $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1} B t^\beta \geq R(t) \geq \frac{4n}{n+1} \cdot \frac{C}{t^\alpha} \quad for \quad t \geq t_0,$$

where $t_0 > a$, $0 < \alpha < 2$, $0 < \beta < \frac{4}{n+1}$, $C$ and $B$ are positive constants, then $M$ admits a Lorentzian metric with scalar curvature $R(t)$.

## 2. PRELIMINARIES ON A WARPED PRODUCT MANIFOLD

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

**Definition 2.1.** Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on $M$, and let $\mathfrak{F}(M)$ denote the ring of all smooth real-valued functions on $M$. A connection $\nabla$ on a smooth manifold $M$ is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

with the properties that

(D1) $\nabla_V(X+Y) = \nabla_V X + \nabla_V Y$

(D2) $\nabla_{fV+hW}(X) = f\nabla_V X + h\nabla_W X$

(D3) $\nabla_V(fW) = V(f)W + f\nabla_V W$

for all $f, h \in \mathfrak{F}(M)$ and all $X, Y, V, W \in \mathfrak{X}(M)$.

$\nabla_V W$ is called the *covariant derivative* of $W$ with respect to $V$ for the connection $\nabla$. The vector $\nabla_X Y|_p = \nabla_{X(P)} Y$ at the point $p \in M$ depends only on the connection $\nabla$, the value $X(p) = X_p$ of $X$ at $p$, and

the values of $Y$ along a smooth curve which passes through $p$ and has tangent $X(p)$ at $p$.

In particular, given a semi-Riemannian manifold $(M, g)$, there is a unique connection $\nabla$ on $M$ such that

$$[V, W] = \nabla_V W - \nabla_W V,$$

and

$$X(g(V, W)) = g(\nabla_X V, W) + g(V, \nabla_X W)$$

for all $X, V, W \in \mathfrak{X}(M)$, where $[\ ,\ ]$ is the Lie bracket. This connection $\nabla$ is called the *Levi – Civita connection* of $M$, which is characterized by the *Koszul formula* ([2], [24]).

**Definition 2.2.** The curvature $R$ of $\nabla$ is a function which assigns to each pair $X, Y \in \mathfrak{X}(M)$ the $f$-linear map $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all $Z \in \mathfrak{X}(M)$.

It is well-known that $R(X, Y)Z$ at $p$ depends only upon the values of $X, Y$, and $Z$ at $p$. In a local chart, we denote by $R^l_{kij}$ the $l$-th component

of $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$ i.e.,$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = \sum_{l=1}^{n} R^l_{kij}\frac{\partial}{\partial x^l}$, which is called the component of the *curvature tensor*, and

$$R^l_{kij}Z^k = \nabla_i\nabla_j Z^l - \nabla_j\nabla_i Z^l.$$

If $\{\Gamma^k_{ij}\}$ are Christoffel symbols of $\nabla$, it follows that

$$R^l_{kij} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}.$$

**Definition 2.3.** From the curvature tensor $R$, a nonzero tensor (or its negative) can be obtained by contraction, which is called the *Ricci tensor*. Its components are $R_{ij} = \sum_{k=1}^{n} R^k_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{i,j=1}^{n} R_{ij}g^{ij}$ is called the *scalar curvature*.

**Definition 2.4.** Let $\phi : M \longrightarrow N$ be a smooth mapping. If $A \in \mathfrak{F}^0_s(N)$ with $s \geq 1$, that is, an $(0, s)$ tensor over $T_{\phi(p)}(N)$, let

$$(\phi^* A)(v_1, v_2, \cdots, v_s) = A(d\phi(v_1), \cdots, d\phi(v_s))$$

for all $v_i \in T_p(M)$, $p \in M$. Then $\phi^*(A)$ is called the *pullback* of $A$ by $\phi$.

At each point $p$ in $M$, $\phi^*(A)$ gives an $\mathbb{R}$−multilinear function from $T_p(M)^s$ to $\mathbb{R}$, that is, an $(0, s)$ tensor over $T_p(M)$. In the special case

if a $(0,0)$ tensor $f \in \mathfrak{F}(N)$, is given the pullback to $M$ is defined to be $\phi^*(f) = f \circ \phi \in \mathfrak{F}(M)$. Note that $\phi^*(df) = d(\phi^* f)$.

We briefly recall some results on warped product manifolds. Complete details may be found in [2], or [24]. On a semi-Riemannian product manifold $B \times F$, let $\pi$ and $\sigma$ be the projections of $B \times F$ onto $B$ and $F$, respectively, and let $f > 0$ be a smooth function on $B$.

**Definition 2.5.** The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with the metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where $g_B$ and $g_F$ are metric tensors of $B$ and $F$, respectively. In other words, if $v$ is tangent to $M$ at $(p, q)$, then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here $B$ is called the base of $M$ and $F$ the fiber. We denote the metric $g$ by $\langle\ ,\ \rangle$. In view of Remark 2.6 (1) and Lemma 2.7, we may also denote the metric $g_B$ by $\langle\ ,\ \rangle$.

**Remark 2.6.** *Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows;*

(1) *For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B\times q}$ is an isometry onto $B$.*

(2) *For each $p \in B$, the map $\sigma|_{\pi^{-1}(p)=p\times F}$ is a positive homothetic map onto $F$ with homothetic factor $\frac{1}{f(p)}$.*

(3) *For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at $(p, q)$.*

(4) *The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of $M$ and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of $M$.*

(5) *If $\phi$ is an isometry of $F$, then $1 \times \phi$ is an isometry of $M$, and if $\psi$ is an isometry of $B$ such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of $M$.*

Recall that vectors tangent to leaves are called *horizontal* and vector tangent to fibers are called *vertical*. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If $X$ is a vector field on $B$, we define $\overline{X}$ at $(p, q)$

by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is $\pi-$related to $X$ and $\sigma-$related to the zero vector field on $F$. Similarly, if $Y$ is a vector field on $F$, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

**Lemma 2.7.** *If $h$ is a smooth function on $B$, then the gradient of the lift $h \circ \pi$ of $h$ to $M$ is the lift to $M$ of gradient of $h$ on $B$.*

*Proof.* See Lemma 7.34 in [24]. ■

In view of Lemma 2.7, we simplify the notations by writting $h$ for $h \circ \pi$ and $grad(h)$ for grad $(h \circ \pi)$. For a covariant tensor $A$ on $B$, its lift $\overline{A}$ to $M$ is just its pullback $\pi^*(A)$ under the projection $\pi : M \longrightarrow B$. That is, if $A$ is a $(1,s)-$tensor, and if $v_1, \cdots, v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, \cdots, v_s) = A(d\pi(v_1), \cdots, d\pi(v_s)) \in T_p(B)$. Hence if $v_k$ is vertical, then $\overline{A} = 0$ on $B$. For example, if $f$ is a smooth function on $B$, the lift to $M$ of the Hessian of $f$ is also denoted by $H^f$. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in [3].

Now we recall the formula for the Ricci curvature tensor $Ric$ if the warped product manifold $M = B \times_f F$. We write $Ric^B$ for the pullback by $\pi$ of the Ricci curvature of $B$ and similarly for $Ric^F$.

**Lemma 2.8.** *On a warped product manifold $M = B \times_f F$ with $n = \dim F > 1$, let $X, Y$ be horizontal and $V, W$ vertical.*

*Then*

(1) $Ric(X,Y) = Ric^B(X,Y) - \frac{n}{f}H^f(X,Y)$

(2) $Ric(X,V) = 0$

(3) $Ric(V,W) = Ric^F(V,W) - \langle V,W\rangle f^{\#}$,

*where* $f^{\#} = \frac{\Delta f}{f} + (n-1)\frac{\langle grad(f), grad(f)\rangle}{f^2}$, *and* $\Delta f = trace(H^f)$ *is the Laplacian on* $B$.

*Proof.* See Corollary 7.43 in [24] ■

On the given warped product manifold $M = B \times_f F$, we also write $S^B$ for the pullback by $\pi$ of the scalar curvature $S_B$ of $B$ and similarly for $S^F$. From now on, we denote $grad(f)$ by $\nabla f$.

**Theorem 2.9.** *If $S$ is the scalar curvature of $M = B \times_f F$ with $n = \dim F > 1$, then*

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2} \tag{2.1}$$

*where $\Delta$ is the Laplacian on $B$.*

*Proof.* For each $(p,q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for $T_pB$. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal

set in $T_{(p,q)}M$. We can choose $\{d_j\}$ on $T_qF$ such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 = \langle \overline{d_j}, \overline{d_j} \rangle = f(p)^2 (d_j, d_j) = (f(p)d_j, f(p)d_j)$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for $T_qF$.

By Lemma 2.8 (1) and (3), for each $i$ and $j$

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i}),$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(d_j, d_j)(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}).$$

Hence, for $\varepsilon_\alpha = g(e_\alpha, e_\alpha)$

$$\begin{aligned} S(p,q) &= \sum_\alpha \varepsilon_\alpha R_{\alpha\alpha} \\ &= \sum_i \varepsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_j \varepsilon_j Ric(\overline{d_j}, \overline{d_j}) \\ &= S^B(p,q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}. \end{aligned}$$

■

## 3. MAIN RESULTS

Let $(N, g)$ be a Riemannian manifold of dimension $n$ and let $f : [a, \infty) \to R^+$ be smooth function, where $a$ is a positive number. The Lorentzian warped product of $N$ and $[a, \infty)$ with warping function $f$ is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \tag{3.1}$$

Let $R(g)$ be the scalar curvature of $(N, g)$. Then Theorem 2.9 implies that the scalar curvature $R(t, x)$ of $g'$ is given by the equation

$$R(t, x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$ (for details, cf. [6] or [12]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u'' - R(t, x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

In this paper, we assume that the fiber manifold $N$ is nonempty, connected and a compact Riemannian $n$-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [12], we have the following proposition.

**Proposition 3.1.** *If the scalar curvature of the fiber manifold $N$ is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.*

However, the results of [12] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

**Remark 3.2.** By Remark 2.58 in [2] and Corollary 5.6 in [25], if $(a, b)$ is a finite interval and $n = 3$, then all nonspacelike geodesics are incomplete. But on $(-\infty, +\infty)$ there exists a warping function so that all nonspacelike geodesics are complete. For Theorem 5.5 in [25] implies that all timelike geodesics are future (resp.past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) and Remark 2.58 in [2] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$(resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) (cf. Theorem 4.1 and Remark 4.2 in [3]).

We assume that the fiber manifold $N$ of $M = [a, \infty) \times_f N$ has a positive scalar curvature, where $a$ is a positive number. If we let $u(t) =$

$t^\alpha$, where $\alpha \in (0,1)$ is a constant, then we have

$$R(t,x) > -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} \geq -\frac{4n}{n+1}\cdot\frac{1}{4}\cdot\frac{1}{t^2}, \quad t > a.$$

By the similar proof like as Theorem 2.4 in [17], we have the following:

**Theorem 3.3.** *If $R(g)$ is positive, then there is no positive solution to equation (3.3) with*

$$R(t) \leq \frac{-4n}{n+1}\cdot\frac{c}{4}\cdot\frac{1}{t^2} \quad for \quad t \geq t_0,$$

*where $c > 1$ and $t_0 > a$ are constants.*

If $N$ ahs a positive scalar curvature, then any smooth function on $N$ is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric $g$ on $N$ with scalar curvature $R(g) = \frac{4n}{n+1}k^2$ , where $k$ is a positive constant. Then equation (3.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0. \tag{3.4}$$

If $R(t,x)$ is the function of only $t$-variable, then we have the following theorem.

**Theorem 3.4.** *Suppose that $R(g) = \frac{4n}{n+1}k^2$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$*

*and a lower solution $u_-(t)$ of equation (3.4) such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (3.4) such that for $t > t_0 \quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.*

*Proof.* . We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty) \quad \tilde{u}_-(t) \leq \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a, \infty))$, there exists a positive constant $d$ such that $|R(t)| \leq \frac{4n}{n+1} d^2$ for $t \in [a, t_0]$. We assume that $u_+(t) \geq 1$ for $t \in [a, t_0]$. Then we have

$$\begin{aligned}
&\frac{4n}{n+1} u_+''(t) + \frac{4n}{n+1} k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t) u_+(t) \\
&\leq \frac{4n}{n+1} u_+''(t) + \frac{4n}{n+1} k^2 u_+(t) + \frac{4n}{n+1} d^2 u_+(t) \\
&= \frac{4n}{n+1} [u_+''(t) + (k^2 + d^2) u_+(t)].
\end{aligned}$$

And if we devide the given interval $[a, t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval $I_i$ we have an upper solution $u_+^i(t)$ by parallel transporting $c_1 cos(\sqrt{k^2 + d^2} t)$ such that $u_+^i(t) \geq 1$ for some constant $c_1$. That is to say, for each interval $I_i$, $\frac{4n}{n+1} u_+^i(t)'' + \frac{4n}{n+1} k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t) u_+^i(t) \leq \frac{4n}{n+1} (u_+^i(t)'' + (k^2 + d^2) u_+^i(t)) = 0$, which means that $u_+^i(t)$ is an upper solution for each interval $I_i$. Then put $\tilde{u}_+(t) = u_+^i(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution on $[a, b)$ such that $\tilde{u}_+(t) \geq 1$ for all $t \in [a, t_0]$.

Put $\tilde{u}_-(t) = e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive $\alpha$, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$, $\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \geq \frac{4n}{n+1}(u_-''(t) - d^2 u_-(t)) = \frac{4n}{n+1}e^{-\alpha t}(\alpha^2 - d^2) \geq 0$ for large $\alpha$. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$ $\quad 0 < \tilde{u}_-(t) \leq \tilde{u}_+(t)$.

**Theorem 3.5.** *Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that*

$$\frac{4n}{n+1}Bt^\beta \geq R(t) \geq \frac{4n}{n+1} \cdot \frac{C}{t^\alpha} \quad for \quad t \geq t_0,$$

*where $t_0 > a$, $0 < \alpha < 2$, $0 < \beta < \frac{4}{n+1}$, $C$ and $B$ are positive constants. Then equation (3.4) has a positive solution on $[a, \infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.*

*Proof.* We let $u_+ = t^m$ and $u_- = t^{-\delta}$ , where $m$ and $\delta$ are positive numbers. If we take $m$ large enough so that $m\frac{4}{n+1} > 2$, then we have, $t \geq t_0$ for some large $t_0$,

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1} \cdot \frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m \left[\frac{m(m-1)}{t^2} + \frac{k^2}{t^{m\frac{4}{n+1}}} - \frac{C}{t^\alpha}\right] \\
&\leq 0,
\end{aligned}$$

which is possible for large fixed $m$ since $\alpha < 2$. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t) \leq \frac{4n}{n+1} Bt^{\beta}$ , if we take $0 < \delta < 1$ so that $\delta \frac{4}{n+1} > \beta$, then

$$\begin{aligned}
&\frac{4n}{n+1} u_-''(t) + \frac{4n}{n+1} k^2 u_-(t)^{1-\frac{4}{n+1}} - R(t) u_-(t) \\
&\geq \frac{4n}{n+1} u_-''(t) + \frac{4n}{n+1} k^2 u_-(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1} Bt^{\beta} u_-(t) \\
&= \frac{4n}{n+1} t^{-\delta} \left[ \delta(\delta+1) t^{-2} + k^2 t^{\delta \frac{4}{n+1}} - Bt^{\beta} \right] \geq 0
\end{aligned}$$

for large t. Since $t > t_0 > a > 0$, we can take the lower solution $u_-(t) = t^{-\delta}$ so that $0 < u_-(t) < u_+(t)$. So by the upper and lower solution method, we obtain a positive solution $u(t) = f(t)^{\frac{n+1}{2}}$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$ . Hence

$$\begin{aligned}
&\int_{t_0}^{+\infty} \left( \frac{f(t)}{1+f(t)} \right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \left( \frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}} \right)^{\frac{1}{2}} dt \\
&\geq \int_{t_0}^{+\infty} \left( \frac{t^{-\delta \frac{2}{n+1}}}{1+t^{-\delta \frac{2}{n+1}}} \right)^{\frac{1}{2}} dt \geq \frac{1}{2} \int_{t_0}^{\infty} t^{-\frac{\delta}{n+1}} dt = \infty
\end{aligned}$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \geq \int_{t_0}^{+\infty} t^{-\frac{\delta}{n+1}} dt = +\infty$$

which, by Remark 3.2, implies that the resulting warped product metric is a future geodesically complete one.

**Theorem 3.6.** *Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that*

$$\frac{4n}{n+1}Bt^\beta \geq R(t) \geq \frac{C}{t^2} \quad for \quad t \geq t_0,$$

*where $t_0 > a$, $0 < \beta < \frac{4}{n+1}$, $B$ and $C$ are positive constants. If $C > n(n-1)$, then equation (3.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped metric is a future nonspacelike geodesically complete metric of positive scalar curvature outside a compact set.*

*Proof.* In case $C > n(n-1)$, we may take $u_+ = C_+ t^{\frac{n+1}{2}}$ , where $C_+$ is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq C_+ \frac{4n}{n+1} t^{\frac{n-3}{2}} \left[\frac{n^2-1}{4} + k^2 C_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C\right] \leq 0,
\end{aligned}$$

which is possible if we take $C_+$ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C_+ < 0$. And we take $u_-(t)$ as in the proof of Theorem 3.5. In this case, we also obtain a positive solution as in Theorem 3.5. Hence

$$\begin{aligned}
&\int_{t_0}^{+\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \\
&\geq \int_{t_0}^{+\infty} \left(\frac{t^{-\delta\frac{2}{n+1}}}{1+t^{-\delta\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \geq \frac{1}{2}\int_{t_0}^{\infty} t^{-\frac{\delta}{n+1}} dt = \infty
\end{aligned}$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \geq \int_{t_0}^{+\infty} t^{-\frac{\delta}{n+1}} dt = +\infty$$

which, by Remark 3.2, implies that the resulting warped product metric is a future nonspacelike geodesically complete one.

**Remark 3.7.** By Theorem 3.4 and Corollary 3.5 in [15], the result in Theorem 3.5 is almost sharp as we can get as close to $\frac{n(n-1)}{t^2}$ as possible. For example, let $R(g) = \frac{4n}{n+1}k^2$ and $f(t) = tlnt$ for $t > a$. Then we have

$$R = \frac{1}{t^2}\left[\frac{4n}{n+1} \cdot \frac{k^2}{(lnt)^2} + \frac{2n}{lnt} + n(n-1)(1+\frac{1}{lnt})^2\right],$$

which converges to $\frac{n(n-1)}{t^2}$ as $t$ goes to $\infty$.

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