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Master's Thesis

Derivation of Modulation Equations for Resonant Interaction of the Waves in a Beach

Graduate School of Chosun University

Department of Naval Architecture & Ocean Engineering

Venu Vasudevan

Derivation of Modulation Equations for Resonant Interaction of the Waves in a Beach

해변에서 발생하는 파도의 공명 간섭에 대한 변조 식 유도

2005-08-20

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Advisor: Prof. Lee, Kwi Joo

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of the requirement for the Award of
the degree of
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Nomenclature

Wave elevation	η
Slope	s, θ (Degree)
Angular frequency	ω
Amplitude (Standing wave)	A_0
Amplitude (Edge wave)	A_e
Wave Number	k
Group Velocity	Cg
Wave Length	λ
Distance between Wave Breaker	D
Resonance Coefficient	L
Potential Function for Standing Wave	Φ_0
Potential Function for Standing Wave	Φ_e

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ABSTRACT

Edge Wave 는 연근해에서 주요한 역할을 하는 것으로 인식된다. 그리고 여러 학자들에 의해 지난 30 년 동안 광범위하게 연구되어왔다. Slow amplitude method 를 이용한 다른 접근 방식이 edge waves 의 공진에 대한 modulation equations 을 설명하는데 사용된다. Incoming swell 에 의해 경사진 해변에서 edge wave 의 발생은 shallow water model 을 기초로 한다. 공진현상은 multy – scaled expansion asymptotic techniques 에 의해 분석된다. 그리고 파 발생에 대한 유동 간의 경계가 제시된다. 이것은 wave breakers 사이의 edge wave 발생을 분석하는데 적용된다. 경사진 해변, incoming waves 의 진폭, 파장과 같은 다른 변수들의 영향이 wave breakers 사이의 거리를 결정하는데 여러 그래프를 이용해 연구 삽입된다.Edge waves 상호작용의 nonlinear wave solutions 은 잘 알려진 sine – Gordon model 로 일반화된다.

CHAPTER1
INTRODUCTION & BACKGROUND

1.1 INTRODUCTION

Edge waves are gravity waves that progress along the shoreline. These waves, often difficult to visualize are coastal trapped, i.e. their amplitude is maximal at the shoreline and decays rapidly offshore. They produce on the beach beautiful run-up patterns although propagation is along the straight shoreline and the waveform is sinusoidal in the long shore. Edge waves are produced by the variability of wave energy reaching shore. Waves tend to come in groups, especially when waves come from distant storms. For several minutes breakers may be smaller than average, then a few very large waves will break. The minute-to-minute variation in the height of breakers produces low-frequency variability in the along-shore current. This, in turn, drives a low-frequency edge wave attached to the beach. Study of edge waves is a rapidly growing area in near shore hydrodynamics. A sample picture of edge wave along the shore is shown below.

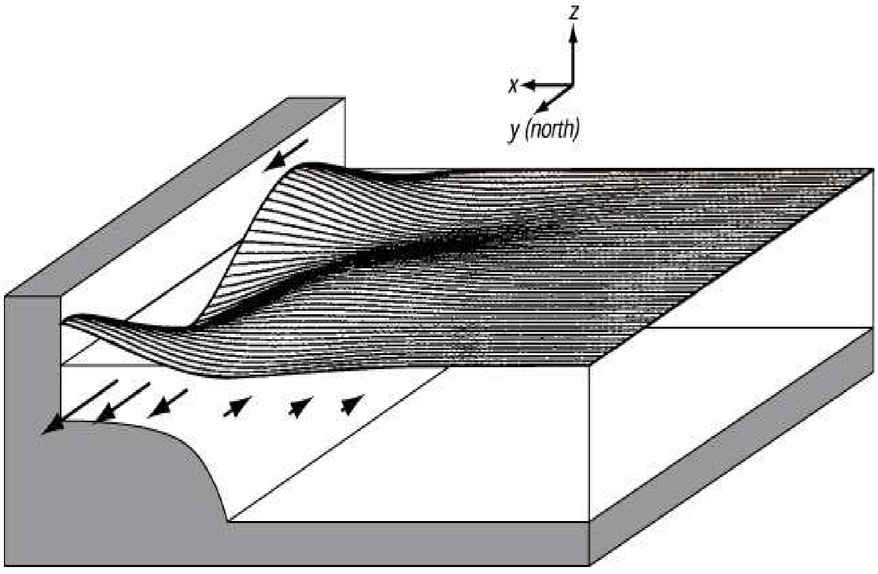

Fig: 1.1 Edge Wave Pattern

Several mechanisms for generating edge waves are possible in nature. On a large scale edge waves can be excited by wind stress directly above the water. Munk, Snodgrass, Carrier and Green Span (1956) have studied the effect of pressure deviation in storm surges. Smaller scale edge waves can be excited by a nonlinear mechanism of sub harmonic resonance. Medium-scale edge waves can also be excited by a long group of short swells through a nonlinear mechanism

1.2 STOKES SOLUTION

The first analytic evidence for the existence of waves, which propagate parallel to and are trapped against a shoaling beach, was provided by Stokes (1846). He found the following solutions to the inviscid linear equations of motion applied to a wedge-like fluid domain with a constant angle β. The Potential function and surface elevation of edge waves are derived as shown below respectively.

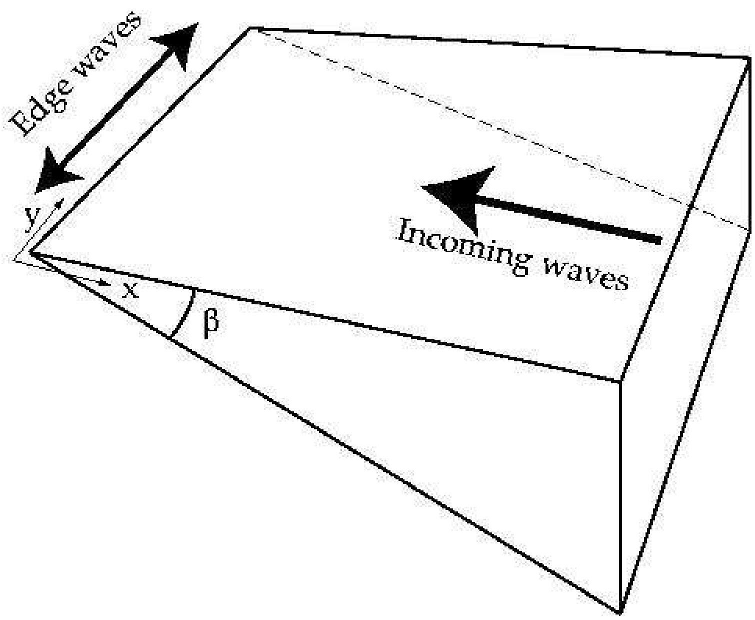

Fig: 1.2 Incoming Wave angle to the Shore

$$\phi_e(x,y,z,t) = \frac{gA_e}{\omega}\sin(\beta)\exp(-k_e x\cos(\beta) + k_e z\sin(\beta))\sin(k_e y - \omega_e t)$$

$$\eta_e(x,z,t) = A_e\sin(\beta)\exp(-k_e x\cos(\beta))\cos(k_e y - \omega_e t)$$

Where the subscript "e" now refers to the properties of a wave traveling parallel to the shore. For a confined beach, edge waves with an integral number of half wavelengths along the beach may form standing edge waves. If the beach is infinite in extent, then edge waves will propagate in both directions along the shoreline at a range of frequencies.

1.3. OTHER MAJOR DEVELOPMENTS IN THE STUDY OF EDGE WAVES

Bowen and Inman (1969) found field evidence of standing edge waves of periods comparable in order of magnitude to the period of the incoming swell. The amplified edge waves cause long shore modulation of the incident swell, which may be sufficiently short to break near the shore. The periodic cells of currents, which, in turn, lead to beach cusps. Motivated by these interests, Guza and Davis (1974) made a systematic examination of the non-linear mechanism of sub harmonic resonance in which a standing edge wave of frequency ω was resonated by a normally incoming and reflected wave of frequency 2ω. Guza and Bowen (1974) employed Airy's shallow-water approximation as the basis of their theory. In addition to the initial instability of edge waves, the incident and reflected waves were found to leak energy by radiation due to quadratic nonlinearity. Considering the cubic nonlinearity and of radiation damping enabled them to predict both the initial resonant growth and the final equilibrium amplitude. Their own experiments strongly supported these findings.

1.4 LINEAR INCOMING AND REFLECTING WAVES ON A SLOPING BEACH

For finding the resonance first we need the equation of standing wave. Here we consider the special case of a standing wave in a sloped beach, which can be derives as shown below.
In terms of horizontal velocity u = u (x, t) and free surface elevation η = η (x, t), equations for shallow water waves over a non uniform bottom y = -h (x) are

$$u_t + uu_x + g\eta_x = 0 \qquad (1.4.1)$$

$$H_t + (uH)_x = 0 \qquad (1.4.2)$$

Where the total depth $H(x, t) = h(x) + \eta(x, t)$.

In the linear theory, disturbances are assumed to be small. Hence we also assume that the derivatives are of same order. The equation (1.4.2) can be rewritten in terms of h and η as given below.

$$\eta_t + uh'(x) + hu_x + \eta u_x + u\eta_x = 0 \qquad (1.4.3)$$

And assuming first order assumption to the equation (1.4.1), (1.4.2) we can write

$$u_t + g\eta_x = 0, \quad \eta_t + uh'(x) + hu_x = 0.,$$

By eliminating the term "u" from these equation

$$\eta_{tt} - gh'(x)\eta_x - gh\eta_{xx} = 0 \qquad (1.4.4)$$

We now consider the waves on sloping beach that is inclined at angle β with the horizontal. For using this theory we have to assume that the angle β is small.

For small β, $h = sx$

And the equation (5.4) becomes

$$\eta_{tt} - gs\,(\eta_x - x\eta_{xx}) = 0 \qquad (1.4.5)$$

The simplest solution of this equation is $\eta = e^{(iwt)} f(x)$

Substituting in the above equation will reduce the equation to

$$f''(x) + (1/x) f'(x) + (\omega^2 / gs)(1/x) f(x) = 0,$$

Where $0 < x < \infty$. This equation has regular singular point at $x = 0$ and an irregular singular point at $x = \infty$. The transformation $x = (gs / \omega^2)(X/2)^2$ can be reduce the above equation to zero order Bessel equation.

$$f''(X) + (1/X) f'(X) + f(X) = 0$$

Where general solution of this equation is given by

$$f(X) = AJ_0(\beta\sqrt{X}) - iBY_0(\beta\sqrt{X}) \qquad (1.4.6)$$

Where A and B are constants and X and β are as given below.

$$X = \beta\sqrt{x} \quad \text{and} \quad \beta = (4\omega^2 / gs)^{1/2}$$

Thus the final solution is

$$\eta(x, t) = e^{-i\omega t}[AJ_0(\beta\sqrt{x}) - iBY_0(\beta\sqrt{x})]. \qquad (1.4.7)$$

Using the asymptotic representation of Bessel function for $x \to \infty$

$$\eta \sim (4 / \pi^2\beta^2 x)^{1/4} (A+B)/2) \exp\{-i(\beta\sqrt{x} + \omega t + \pi/4)\} + (A+B)/2) \exp\{-i(\beta\sqrt{x} - \omega t - \pi/4)\}$$

The first term of this result represents an incoming wave and the second term corresponds to outgoing wave. The amplitude of the former wave depends on x.

The wave number and the frequency of the outgoing waves given by

$$K(x, t) = \theta_x = \omega / \sqrt{gsx}$$

We need only first term of the Bessel final solution to find resonance in our problem. So in general we can take the equation of standing waves as

$$\Phi_0 = A_0 J_0(\beta\sqrt{x})e^{-2iwt} \tag{1.4.8}$$

1.5 EQUATION OF EDGE WAVES

For getting simplified equation for edge wave for using in this problem, we consider a straight and long beach with constant slope. Let the mean shoreline coincide with the y-axis and the water be in the region x> 0. The bottom is described by

$$Z = -h = -sx \quad x > 0,\ s = \text{const.}$$

Because the coefficients are constant in y and t, we try the solution

$$\zeta = \eta(x)\, e^{i(ky - \omega t)}$$

From the linearized long wave theory considering the mass and momentum conservation and using the above equation we will get

$$X\eta'' + \eta' + \left(\frac{\omega^2}{sg} - k^2 x\right)\eta = 0 \tag{1.5.1}$$

By applying the following transformation

$$\zeta = 2kx$$

$$\eta = e^{-(\xi/2)} f(\xi)$$

We will get the following equation

$$\xi f'' + (1-\xi) f' + [\omega^2/2ksg - 1/2] f = 0 \tag{1.5.2}$$

This is similar to Kummer's equation. In general there are 2 homogeneous solutions, one of which is singular at the shoreline ξ = 0 and it can be discarded.

Non Trivial Solutions which render η finite at ξ = 0 and zero as ξ ⟶ 0 exist when ω corresponds to the following discrete values.

ω2 /2β sg = n + ½, n = 0,1,2,3…

The associated Eigen functions are proportional to Laguerre polynomials

$$L_n(\xi) = \frac{(-)^n}{n!}\begin{bmatrix} \xi^n - \frac{n^2}{1!}\xi^{n-1} + \frac{n^2(n-1)^2}{2!}\xi^{n-2} - \frac{n^2(n-1)^2(n-2)^2}{3!}\xi^{n-3} \\ + \ldots + (-)^n n! \end{bmatrix};$$

The first few modes of edge waves can be plotted as shown below.

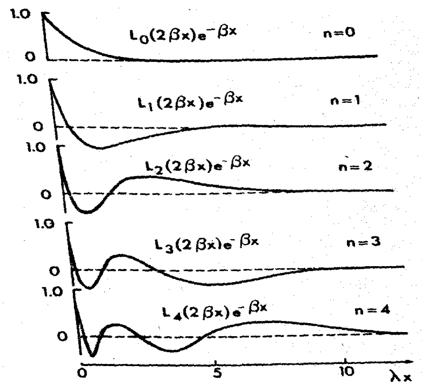

Fig: 1.3 Different Modes of Edge Wave

ecause these Eigen functions correspond to modes, which are applicable, only near the shore they are called edge waves. These Eigen functions are ortho normal in the following sense.

$$\int_0^\infty e^{-\xi} L_n L_m d\xi = \delta_{nm}$$

For our problem we use only the simplest mode of edge wave equation that is given as below

$$\Phi_e = A_e e^{-kx} e^{i(ky \pm wt)} \qquad (1.5.3)$$

Considering the edge waves in both directions we can write the 2 equations as

$$\Phi_{e1} = A_{e1}\ e^{-kx} e^{i(ky-wt)} \qquad \Phi_{e2} = A_{e2}\ e^{-kx} e^{i(ky+wt)}$$

CHAPTER 2
PROCEDURE AND DERIVATION

2.1. SUBHARMONIC RESONANCE OF EDGE WAVES

It is already mentioned that progressive waves may be generated by a storm traveling along the coast at a speed close to the phase velocity of an edge wave mode. The typical period of this kind of edge wave is related to the spatial extent of the storm area and is of the order of several hours. If the coastline has an indentation, a linear resonance is possible. Here we are trying to analyze the Resonance interactions between the amplitudes of edge waves with frequency ω and a standing wave of frequency 2ω also considering the spatial variations for a beach with small slope.

Since the slopes are assumed to be small due to the small inclination in the beach we can use the equations [Guza and Bowen]

$$\zeta_t + [(sx+\zeta)\Phi_x]_x + [(sx+\zeta)\Phi_y]_y = 0 \qquad (2.1.1)$$

$$\Phi_t + \frac{1}{2}(\Phi_x^2 + \Phi_y^2) + g\zeta = 0 \qquad (2.1.2)$$

2.2. DERIVATION OF MODULATION EQUATION

Eliminating ζ from the above equation we can write the single non linear equation as shown below.

$$\Im\Phi \equiv -\Phi_{tt} + sg[(x\Phi_x)_x + x\Phi_{yy}] = 2(\Phi_x\Phi_{xt} + \Phi_y\Phi_{yt}) + \Phi_t(\Phi_{xx} + \Phi_{yy})$$
$$+\frac{1}{2}(\Phi_x{}^2 + \Phi_y{}^2)(\Phi_{xx} + \Phi_{yy}) + \Phi_x^2\Phi_{xx} + \Phi_y^2\Phi_{yy} + 2\Phi_x\Phi_y\Phi_{xy}$$

Here we can substitute our equation for Φ as a sum of the above 3 equations we mentioned before.

$$\Phi = \Phi_0 + \Phi_{e1} + \Phi_{e2} + Conjugate(\Phi_0 + \Phi_{e1} + \Phi_{e2}) \qquad (2.2.1)$$

$$\Phi = A_0 J_0(\beta\sqrt{x})e^{-2iwt} + A_{e1}e^{-kx}e^{i(ky-iwt)} + A_{e2}e^{-kx}e^{i(ky+iwt)} +$$
$$A^*{}_0 J_0(\beta\sqrt{x})e^{2iwt} + A^*_{e1}e^{-kx}e^{i(-ky+iwt)} + A^*_{e2}e^{-kx}e^{i(-ky-iwt)}$$

We can avoid the non-linear terms on the right side of the equation. We need to consider only first 2 terms.

Here we make an assumption that the amplitudes are slowly varying functions of y and time. Taking ε as a small parameter we can write A_0 & A_e as A_0 (εy εt) and A_e (εy εt)

In that case the above equation can be indicated as shown.

$$\Phi = A_0[\varepsilon y,\varepsilon t]J_0(\beta\sqrt{x})e^{-2iwt} + A_{e1}[\varepsilon y,\varepsilon t]e^{-kx}e^{i(ky-iwt)} +$$
$$A_{e2}[\varepsilon y,\varepsilon t]e^{-kx}e^{i(ky+iwt)} + A^*{}_0[\varepsilon y,\varepsilon t]J_0\beta(\sqrt{x})e^{2iwt} +$$
$$A^*_{e1}[\varepsilon y,\varepsilon t]e^{-kx}e^{i(-ky+iwt)} + A^*_{e2}[\varepsilon y,\varepsilon t]e^{-kx}e^{i(-ky-iwt)}$$

After substituting these amplitudes and evaluating LHS and RHS in the above equation we have to integrate the whole equation with respect to dx from 0 to $\propto$ to eliminate x terms. The terms are integrated with weight. Both sides are multiplied with e^{-kx} for this purpose before integration. Finally we equate the similar terms on both sides of the equation. We will get 3 sets of equations corresponding to

1) $e^{i(ky-iwt)}$

2) $e^{i(ky+iwt)}$

3) e^{-2iwt}

These terms were taken because of the possibility of resonance interaction exist only in these. For example e^{-2iwt} can interact with $e^{i(ky+iwt)}$ and will result waves correspond to $e^{i(ky-iwt)}$. Similarly there are different ways of interaction possible by the combination of the above three terms. Hence our idea is to separate these terms after integrating and equate the terms in RHS and LHS.

The analysis was carried out using the software MATHEMATICA 4.1 and the details are as shown in APPENDIX: B.

While integrating derivative of Bessel function we will have problems due to x in the denominator. Hence for integrating such terms we have to use the following transformation. The problem can comes only from right side of equation with terms of second x derivative for Bessel function.

Let us rewrite x derivatives in quadratic terms of the right side in the following manner:

$$(\Phi_t \Phi_x)_x = \Phi_{tx}\Phi_x + \Phi_t \Phi_{xx} \tag{2.2.2}$$

We have to integrate this term with exponential weight, but this integral will be finite due to following integration by parts:

$$\int_0^\infty e^{-kx}(\Phi_t \Phi_x)_x\,dx = \Phi_t \Phi_x e^{-kx}\Big|_0^\infty + k\int_0^\infty e^{-kx}\Phi_t \Phi_x\,dx$$

And both terms in the right side are constricted and easily calculated in Mathematica.

We can avoid the higher order terms on the right side .So now we will have three parts on the right side as

1) $\Phi_x \Phi_{xt} + 2(\Phi_y \Phi_{yt}) + \Phi_t \Phi_{yy}$

2) Two terms from the above transformation

Cubic terms are considered here, but terms involving quadratic terms of slow amplitudes are avoided.

After equating resonance terms as mentioned above we will get the final equations as shown below.

$$\frac{\varepsilon\omega}{k}(A_{e1})_t + \frac{gs\varepsilon}{2k}[(A_{e1})_y] = \left[4ke^{-\beta^2/8k} - 4k \;\; + 0.25\beta^2\right](\omega)A_0 A_{e2}$$

$$-\frac{\omega\varepsilon}{k}(A_{e2})_t + \frac{gs\varepsilon}{2k}[(A_{e2})_y] = -\left[4ke^{-\beta^2/8k} - 4k \;\; + 0.25\beta^2\right](\omega)(A_0)^* A_{e1}$$

$$(\varepsilon)[(A_0)_t] = -2/3\,(k^2 e^{1/4k})A_{e1}(A_{e2})^*$$

2.3 GROUP VELOCITY

When Waves are generated by a local disturbance such as the dropping of large stone into a lake or the motion of wave through water, the successive waves with different wavelength propagates and hence they travel with different phase velocities. So we might expect that the wave trains would be sorted out as time goes on into different groups of waves such that each group would consist of waves of approximately the same wavelength.

First let us consider a one dimensional progressive plane wave of the form

$$\eta(x,t) = a\exp[(i(kx-\omega t)] \qquad (2.3.1)$$

Where a is the amplitude, and the frequency ω and wave number k are related by dispersion relation

$$\omega = \omega(k)$$

We now suppose two such waves with the same amplitude, but the wave numbers and frequencies are slightly different so that

$$\eta = a\cos(\omega t - kx) + a\cos[(\omega + \Delta\omega)t - (k+\Delta k)x]$$

$$= \text{Acos}\ \{\ \omega + 1/2\omega)t - (k + 1/2\Delta k)x \qquad (2.3.2)$$

Where $\Delta\omega$ and Δk are small and A = 2acos{(1/2 $\Delta\omega$) t – (1/2 Δk) x} is a slowly varying amplitude for the rapidly varying mean wave with much larger frequency, $\omega + \frac{1}{2}\Delta\omega$ and wave number, $k + \frac{1}{2}\Delta k$, so that the above equation can be interpreted as a series of wave traveling with the velocity

$$\text{Cg} = \frac{\Delta\omega}{\Delta k} \qquad (2.3.3)$$

This is the final equitation for group velocity for any kind of waves in general.

For an edge wave if the incident wave have the frequency ω we have the dispersion relation fro edge waves as shown below.

$$\omega^2 = gks(2n+1) \qquad (2.3.4)$$

Since in our problem we are dealing with the lowest mode of edge wave we can take the dispersion relation of edge wave as

$$\omega^2 = gks \tag{2.3.5}$$

Where k is the wave number and s is the slop of the beach.

Substituting this in the above equation of group velocity we will get

$$\Delta\omega / \Delta k = 1/2 \sqrt{gs/k} \tag{2.3.6}$$

This is the equation of Group velocity for edge waves

In the set of equations we got from our analysis, dividing by the coefficients of *t* derivative and substituting the above formula for group velocity in it we will get three final equations of resonance as

$$(A_{e1})_t + C_g[(A_{e1})_y] = k\left[4ke^{-\beta^2/8k} - 4k \quad + 0.25\beta^2\right]A_0 A_{e2} \tag{2.3.7}$$

$$(A_{e2})_t - C_g[(A_{e2})_y] = k\left[4ke^{-\beta^2/8k} - 4k \quad + 0.25\beta^2\right](A_0)^* A_{e1} \tag{2.3.8}$$

$$[(A_0)_t] = -2/3\,(k^2 e^{1/4k})A_{e1}(A_{e2})^* \tag{2.3.9}$$

These are the Modulation equations for edge waves due to resonance of a standing wave in a sloping beach.

CHAPTER 3

ANALYSIS

3.1. CHECKING FOR GROUP VELOCITY

We will analyze first the simple case that A_0 is fixed. We can recheck our equation by the following method.

$$(A_{e1})_t + Cg(A_{e1})_y = \chi A_0 A_{e2} \tag{3.1.1}$$

$$(A_{e2})_t - Cg(A_{e2})_y = \chi A^*{}_0 A_{e1} \tag{3.1.2}$$

Here we have substituted χ for the term of k in the right side of the equation.

From the first equation we will get

$$A_{e2} = \left[\frac{Cg}{A_0\chi}(A_{e1})_t\right] + \frac{(A_{e1})_t}{\chi A_0} \tag{3.1.3}$$

Substituting this in the second equation

$$\left[\frac{Cg}{A_0\chi}(A_{e1})_{yt}\right] + \frac{(A_{e1})_{tt}}{A_0\chi} - \left[\frac{(Cg)^2}{A_0\chi}(A_{e1})_{yy}\right] + Cg\,\frac{(A_{e1})_{ty}}{\chi A_0} = \chi A^*{}_0 A_{e1}$$

The equation of wave envelope is given by

$$A_{e1} = ae^{i(\kappa y - \Omega t)} \tag{3.1.4}$$

Since we are assuming there is no interaction between waves by substituting and equating to zero.

$$(Cg)\Omega\kappa - \Omega^2 + (Cg)^2\kappa^2 - Cg\Omega\kappa = 0$$

And we will get, $Cg = \Omega / \kappa$ (3.1.5)

That means, that envelope of waves propagates with its group or energy velocity in the mean order, it is the classical result of nonlinear waves propagation theory.

3.2. CONDITION OF NO RESONANCE

Based on these analyses the influence of variables like beach slope, incoming wavelength and Amplitude of incoming waves in the generation of edge waves along the beach can be evaluated.

Coefficient in the Right side of the edge wave modulation equation is plotted as a function of **k** as shown below. Here slop is assumed to be a constant, 0.3 there exist different values of k for different values of beach slope.

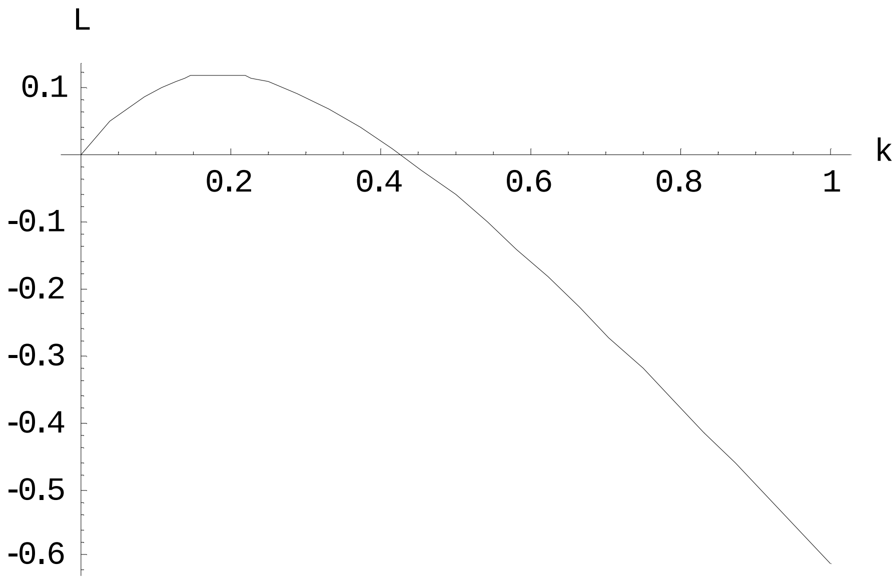

Fig: 3.1 L Vs Wave Number(s=0.3)

From the above graph we can see that a value of **k** exist where there is no resonance of edge waves and incoming waves. Here for k = 0.42, There will not be any resonance.

By substituting for equation of **k** in terms of wave length as

$$k = \frac{2\pi}{\lambda}$$

We can draw the graph showing variation of the resonance factor with respect to wavelength.

For slope of 0.3

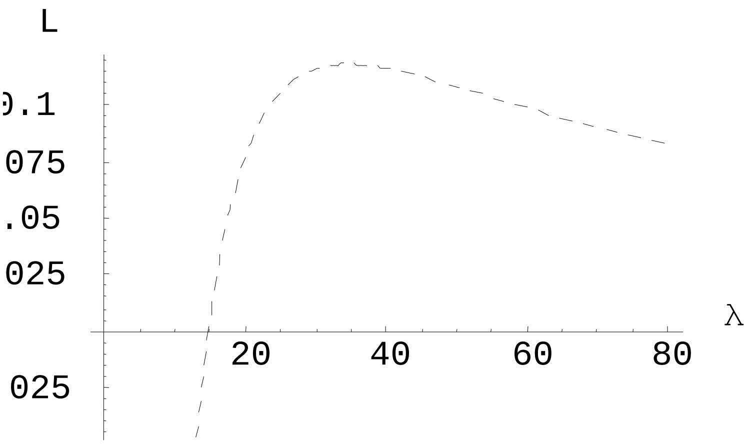

Fig: 3.2 L Vs Wave Length (s=0.3)

Here as discussed above for λ = 14.5 we have no resonance of edge waves and the incoming and reflected waves. This is corresponding to the value of k =0.42 as we already mentioned.

By changing the value of slope to 0.2

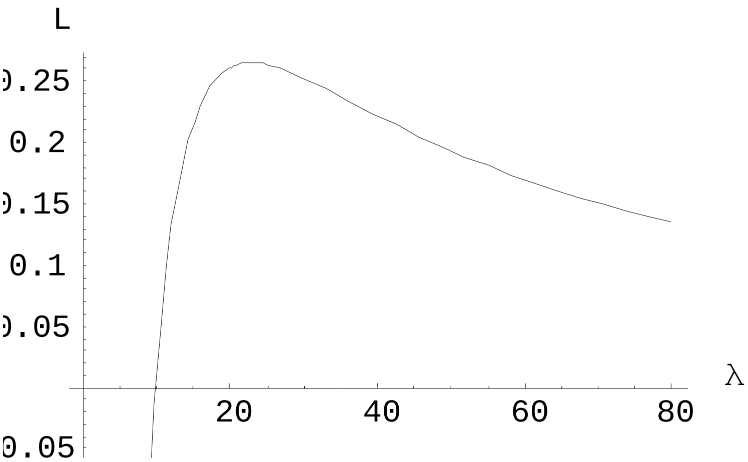

Fig: 3.3 L Vs Wave Length (s=0.2)

It is evident that both the graph shows at a particular value of wavelength for every slope a point is reached where there is no resonant interaction between the edge wave and incoming and reflected wave.

Angular frequency variation also affects the resonant interaction between edge wave and incoming and reflected waves. The variations with respect to angular frequency for a fixed wavelength at two different slopes are shown below. (Dashed line indicates slope=0.3, and continuous line indicates slope of 0.2)

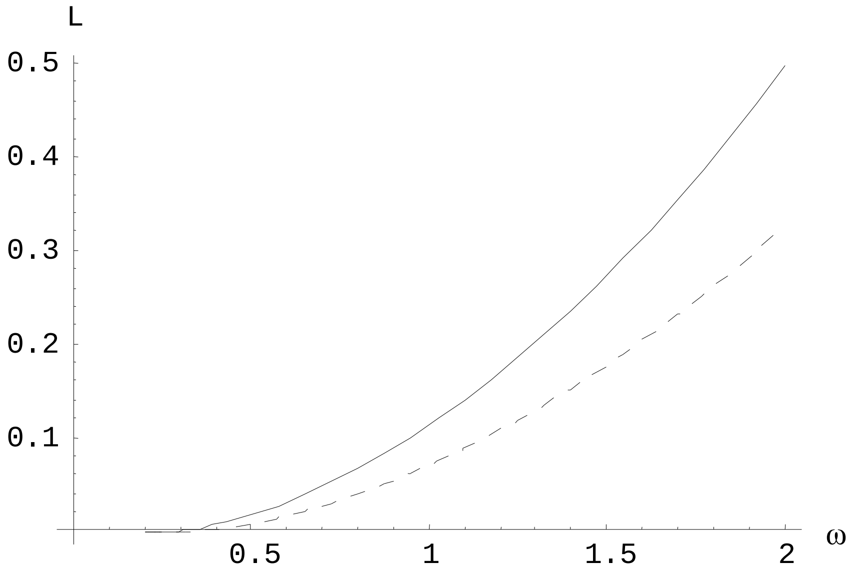

Fig: 3.4 L Vs Angular Frequency

Here Wave Length of incoming wave is assumed to be 100 m and the amplitude of incoming waves are assumed to be 1.As shown in the graph for a fixed wavelength χ increases as angular frequency increases and there exist a particular value of angular frequency at which there will be no resonance interaction for every beach angle and wavelength. This is higher for higher beach angles.

The Dependability of Resonant interactions on the beach slope are evident from the above graphs. For a variation in slope at a constant value of wavelength (λ=100) and angular frequency we can draw the variation in Coefficient L as shown below.

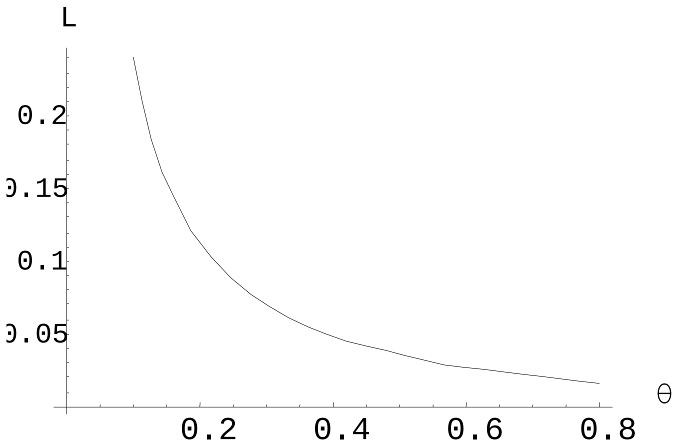

Fig: 3.5 L Vs Slope

It is evident from the graph that in a beach when slope is increased the resonant interaction reduces.

3.3. APPLYING SPATIAL CONSTRICTION

From the derived equations of modulation it's possible to analyze the generation of edge waves with in a constrained space (in y direction).

This is important to study various coastal mechanisms occurred due to edge waves, like sand transportation in long shore direction.

The first 2 modulation equations are given by

$$(A_{e1})_t + C_g[(A_{e1})_y] = LA_0 A_{e2} \qquad (3.3.1)$$

$$(A_{e2})_t - C_g[(A_{e2})_y] = L(A_0)^* A_{e1} \qquad (3.3.2)$$

Where L = $k\left[4ke^{-\beta^2/8k} - 4k + 0.25\beta^2\right]$ (3.3.3)

Considering only the variation in y direction we can write rewrite the above equations as

$$C_g(A_{e1})_y = LA_0 A_{e2} \qquad (3.3.4)$$

From the above equation $A_{e2} = Cg\, (A_{e1})_y / LA_0$

The Second Equation is given by

$$-C_g(A_{e2})_y = LA^*{}_0 A_{e1} \qquad (3.3.5)$$

$$(A_{e2})_y = -\frac{L}{Cg}(A^*{}_0 A_{e1})$$

Substituting the value of A_{e2} in the above equation

$$(A_{e1})_{yy} = -\left(\frac{L}{Cg}\right)^2 A_0 A^*{}_0 A_{e1} \qquad (3.3.6)$$

We will substitute $\left(\frac{L}{Cg}\right)^2 A_0 A^*{}_0 = \chi^2$

$$A_0 A^*{}_0 = |A_0|^2$$

Substituting for χ^2 we will get

$$(A_{e1})_{yy} + \chi^2 A_{e1} = 0 \qquad (3.3.7)$$

This equation is similar to the equation of simple harmonic motion. We can solve this differential equation in standard format given as

$$C_1 e^{ixy} + C_2 e^{-ixy} = A_{e1}$$

And $A_{e1} = cCos(\chi y + C_0)$ (3.3.8)

This result can be used for predicting the possibility of edge waves in a beach with in a wave breaker in y direction. From the property of the constant χ , which depends on the wave number and slop we can predict approximately how far wave breakers should be placed for not generating the edge waves. Edge wave cannot be generated between wave breakers spaced at a distance of D, if the following zero boundary conditions are satisfied:

$$A_{e1}(y=0) = A_{e1}(y=D) = 0$$

That means, that exists threshold or minimum distance for the possibility of edge waves generation by incoming swell:

$$D = \pi / \chi$$

Its obvious that for different values of χ , which is defined by the beach slope, wave Length and Amplitude of incoming waves there exist an appropriate D, placed at which there will be minimum possibility of edge waves between wave breaker. As we already discussed edge waves are the key factor for different beach phenomenon like sedimentation and sand transportation it is important to know the dependability of edge eave envelope on the distance between wave breakers. We can check the generation of edge waves by varying these parameters. Two slopes for beach is considered in general. As s=0.3(dashed line) and s = 0.2 (continuous line)

1.) <u>Change of wave breaker distance for different wave length</u>

(Continuous line indicates slope of 0.2 and dashed line shows slope of 0.3)

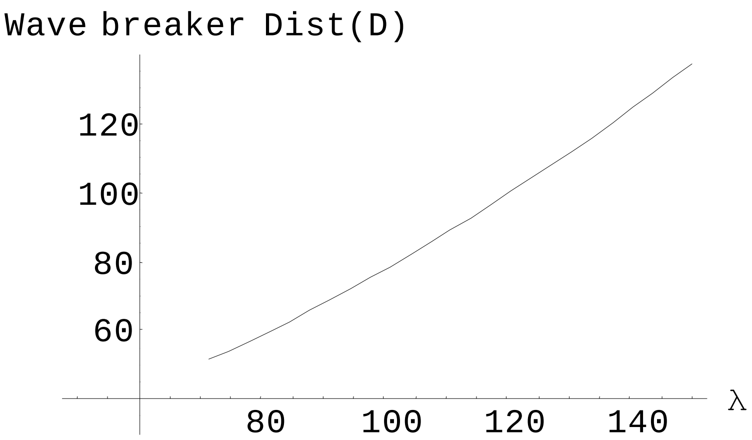

Fig: 3.6 Wave Breaker Dist. Vs Wave Length (s=0.2)

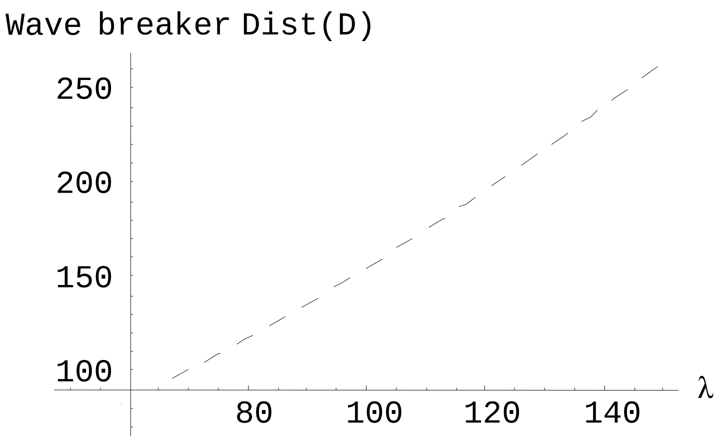

Fig: 3.7 Wave Breaker Dist. Vs Wave Length (s=0.3)

These Graphs can be used for finding the exact value of Water breaker distance for a particular value of slope, amplitude of incoming wave and wavelength, such a way that the edge wave envelope generated in between them is minimal. For example for an incoming wave length of

100 m at a beach with a slope of 0.3 for an amplitude of incoming wave of 1m the wave breakers have to be placed at around 150 m (from the graph) to reduce the edge wave envelope generated in between them. Less generation of edge waves are desired due to the threat of sand transportation and other phenomenon corresponds to long shore currents.

2) <u>Change of wave breaker distance for different Amplitude of incoming waves</u>

(Continuous line indicates slope of 0.2 and dashed line shows slope of 0.3)

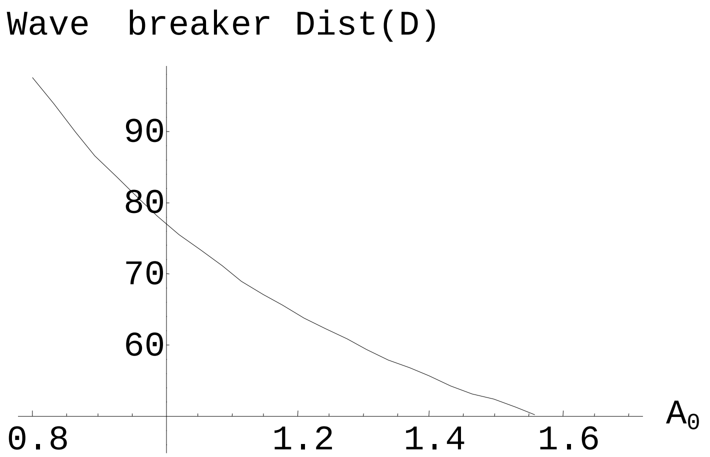

Fig: 3.8 Wave Breaker Dist. Vs Amplitude (s=0.2)

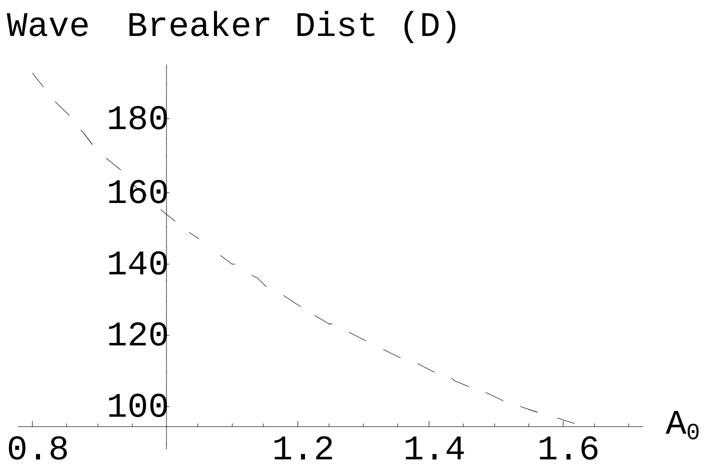

Fig: 3.9 Wave Breaker Dist. Vs Amplitude (s=0.3)

As the Graphs indicates the Distance of Water breakers decreases as Amplitude of incoming waves increases.

3) Change of wave breaker distance for different Beach slope

For different beach slope for a constant wave length and constant amplitude of incoming waves we will get the variations of distance between break waters as shown below

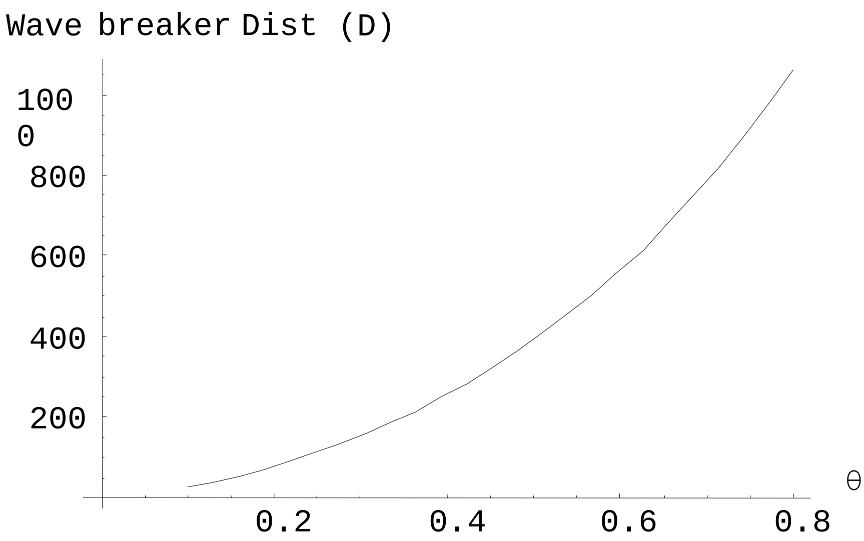

Fig: 3.10 Wave Breaker Dist. Vs Slope

This graph shows the constant increase in the wave breaker distance for constant increase of slope. Which means for a same incoming wave length and amplitude of wave at a fixed angular frequency we need water breakers placed at a higher distances for lesser edge waves and corresponding coastal processes at higher angles of slope.

1) <u>Change of wave breaker distance for different Angular frequency</u>

(Continuous line indicates slope of 0.2 and dashed line shows slope of 0.3)

For constant amplitude of incoming wave and constant wavelength we can draw the graph of angular frequency at 2 different slops as shown below.

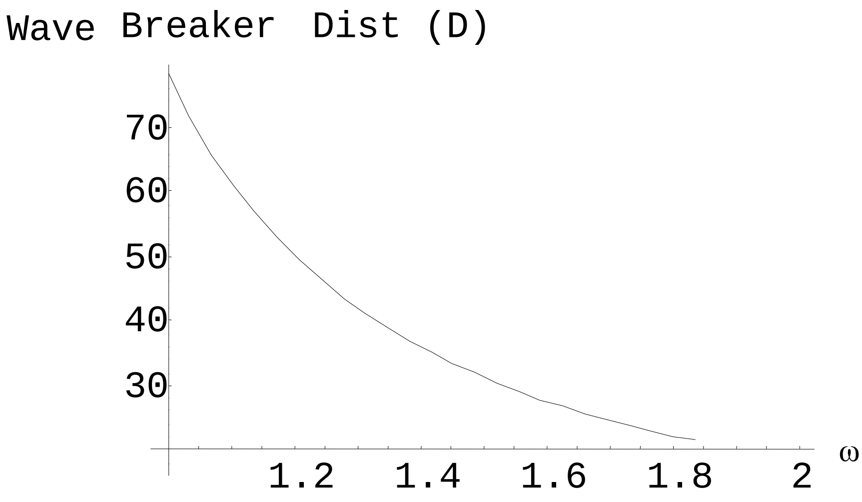

Fig: 3.11 Wave Breaker Dist. Vs Angular Freq. (s=0.2)

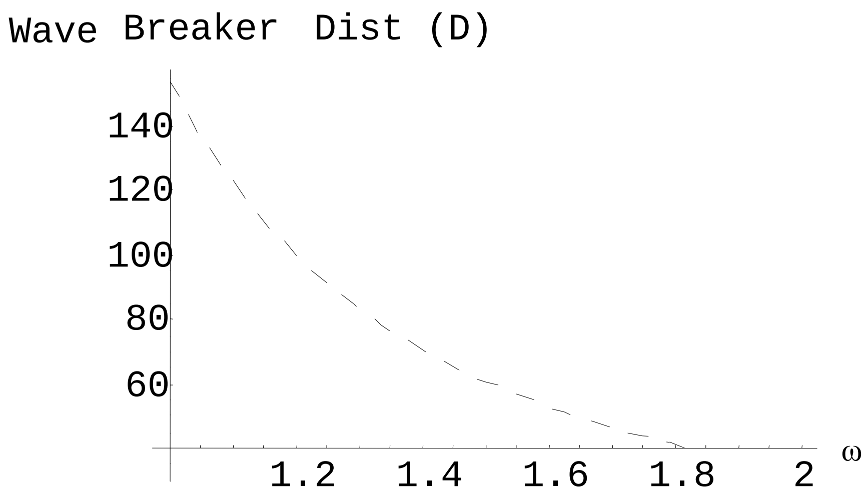

Fig: 3.12 Wave Breaker Dist. Vs Angular Freq. (s=0.3)

As angular frequency increased distance between wave breakers decrease.

CHAPTER 4
NONLINEAR WAVE INTERACTION

Let us rewrite the general system of resonance interactions between edge waves and incoming swell (1,2) by following way:

$$A_{e1t} + C_g A_{e1y} = LA_0 A_{e2} \tag{4.1}$$

$$A_{e2t} - C_g A_{e2y} = LA_0^* A_{e1} \tag{4.2}$$

$$A_{0t} = -SA_{e1} A_{e2}^* \tag{4.3}$$

Where $L = \quad k \left[4ke^{-\beta^2/8k} - 4k \quad + 0.25\beta^2 \right]$, $S = 2/3 (k^2 e^{1/4k})$.

Introducing new moving variables:

$$\xi = C_G t + y, \eta = C_G t - y,$$

$$\frac{\partial}{\partial t} = C_G \frac{\partial}{\partial \xi} + C_G \frac{\partial}{\partial \eta}; \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$$

We can rewrite system of equations (4.1)-(4.3) :

$$\frac{\partial A_{e1}}{\partial \xi} = \frac{L}{2C_G} A_0 A_{e2};$$

$$\frac{\partial A_{e2}}{\partial \eta} = \frac{L}{2C_G} A_0 A_{e1}^*;$$

$$\frac{\partial A_0}{\partial \xi} + \frac{\partial A_0}{\partial \eta} = -\frac{S}{2C_G} A_{1e} A_{e2}^*$$

Renormalization of unknown functions:

$$a_0 = \frac{L}{2C_G} A_0; a_{1e} = \sqrt{\frac{LS}{2C_G}} A_{1e}; a_{2e} = \sqrt{\frac{LS}{2C_G}} A_{2e}$$

Gives the final system of modulation equations:

$$\begin{aligned} &a_{1e\xi} = a_0 a_{2e}; \\ &a_{2e\xi} = a_0^* a_{1e}; \\ &a_{0\xi} + a_{0\eta} = -a_{1e} a_{2e}^*. \end{aligned} \tag{4.4}$$

Third equation for interacting modes (4.3) includes only time derivative of the amplitude of incoming wave, so further we'll assume that A_0 is slow function of time only: $A_0 = A_0(\varepsilon t)$, or $a_0 = a_0(\xi + \eta)$ and $a_{0\xi} = a_{0\eta}$.

System of equations (4.4) has two first integrals of motion in this case:

$$
\begin{aligned}
&|a_{1e}|^2 + 2|a_0|^2 = g(\eta); \\
&|a_{2e}|^2 + 2|a_0|^2 = f(\xi)
\end{aligned}
\tag{4.5}
$$

Phase synchronism of waves is a typical regime for wave generation problems due to resonance interactions, so we'll assume it here to be also, and correspondingly consider all wave amplitudes as real functions of space and time.

Following substitution of variables and unknown functions:

$$
a_{1e} = \sqrt{g(\eta)}Sin(\varphi), a_{2e} = \sqrt{f(\xi)}Sin(\psi), \hat{\xi} = \frac{1}{2}\int f(\xi)d\xi, \hat{\eta} = \frac{1}{2}\int g(\eta)d\eta,
$$

Where *g* and *f* are arbitrary functions of it's arguments, gives the following system of equations for new ψ, φ functions:

$$
\begin{aligned}
&\varphi_{\hat{\xi}} = Sin(\psi); \\
&\psi_{\hat{\eta}} = Sin(\varphi).
\end{aligned}
\tag{4.6}
$$

System of equations (4.6) has a very beautiful property: sum and difference of unknown functions are satisfied to well-known nonlinear Sin-Gordon equation:

$$
(\varphi \pm \psi)_{\hat{\xi}\hat{\eta}} = Sin(\varphi \pm \psi) \tag{4.7}
$$

So, it is possible construct a large (infinite) set of solutions for our model by using different known solutions of Sin – Gordon equation: If *S1* and *S2* are two of them, then equations (4.6) due to (4.7) will be evidently satisfied by the following functions:

$$
\begin{aligned}
&\varphi = \frac{S1 + S2}{2}; \\
&\psi = \frac{S1 - S2}{2}.
\end{aligned}
\tag{4.8}
$$

And even more: if we'll take two different solutions of (4.6): $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$ than sum and difference of them will satisfy to (4.7):

$$(\varphi_1 + \psi_1), (\varphi_2 + \psi_2), (\varphi_1 - \psi_1), (\varphi_2 - \psi_2) \tag{4.9}$$

And using (4.8) we can construct subset of new solutions of the system (4.6):

$$\varphi = (\varphi_1 + \psi_1 + \varphi_2 + \psi_2)/2, \psi = (\varphi_1 + \psi_1 - \varphi_2 - \psi_2)/2;$$
$$\varphi = (\varphi_1 + \psi_1 + \varphi_2 - \psi_2)/2, \psi = (\varphi_1 + \psi_1 - \varphi_2 + \psi_2)/2.$$

By repeating this many times we can construct infinite number solutions of equations (4.6).

CHAPTER 5
CONCLUSION

From the discussions in this thesis it is evident that the resonant interactions between incoming waves and edge waves significantly depend on factors like wave Length of incoming wave, angular frequency of incoming waves, Amplitude of the incoming wave, slope of beach. From the graphs discussed above chapter we have seen that for every angular frequency and every value of slope there exist a particular wavelength for which there is no resonant interaction between the edge waves and incoming and reflected waves. These wavelengths for no resonant interaction or angular frequency of no resonant interaction can be find out as from the respective graphs. Applying spatial constrictions enables us to study the generation of edge wave in between a wave breaker. Less edge wave envelope formed between the wave breakers is considered as a favorable condition. Exists threshold or minimum distance for the possibility of edge waves generation by incoming swell. The distance between wave breakers can be adjusted for less generation of edge waves. As wave Length increases the distance between wave breakers is increased As the Amplitude of incoming wave increases the wave breaker distance decreases As Angular frequency of incoming wave increases the distance between wave breaker decrease. As Beach slope increases the distance of wave breaker increases. These are the results of this analysis. The exact prediction of generation of edge waves in a beach is very important due to the increasing concerns over coastal processes including sand transportation and beach erosion. More researches shall be conducted considering complex phenomenon including wave breaking and interaction of sand for getting more accurate results.

APPENDIX: A

(*Mathematical Models of Waves)*

1.) 3-D model of Edge waves created using the equation used in this discussion. This Model shows the exponential Decay of edge waves along the offshore direction.

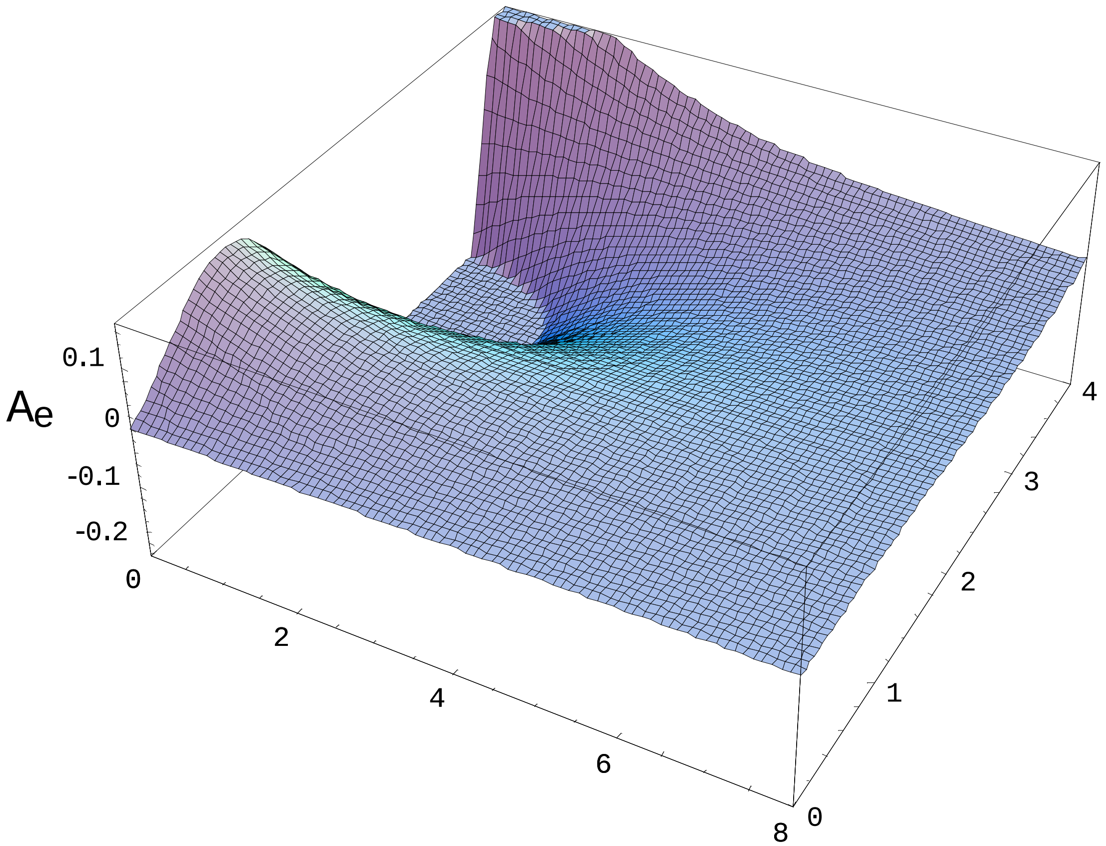

Fig: A1

2) Mathematical Model Of Standing Waves near a sloping Bottom Made using the Bessel function

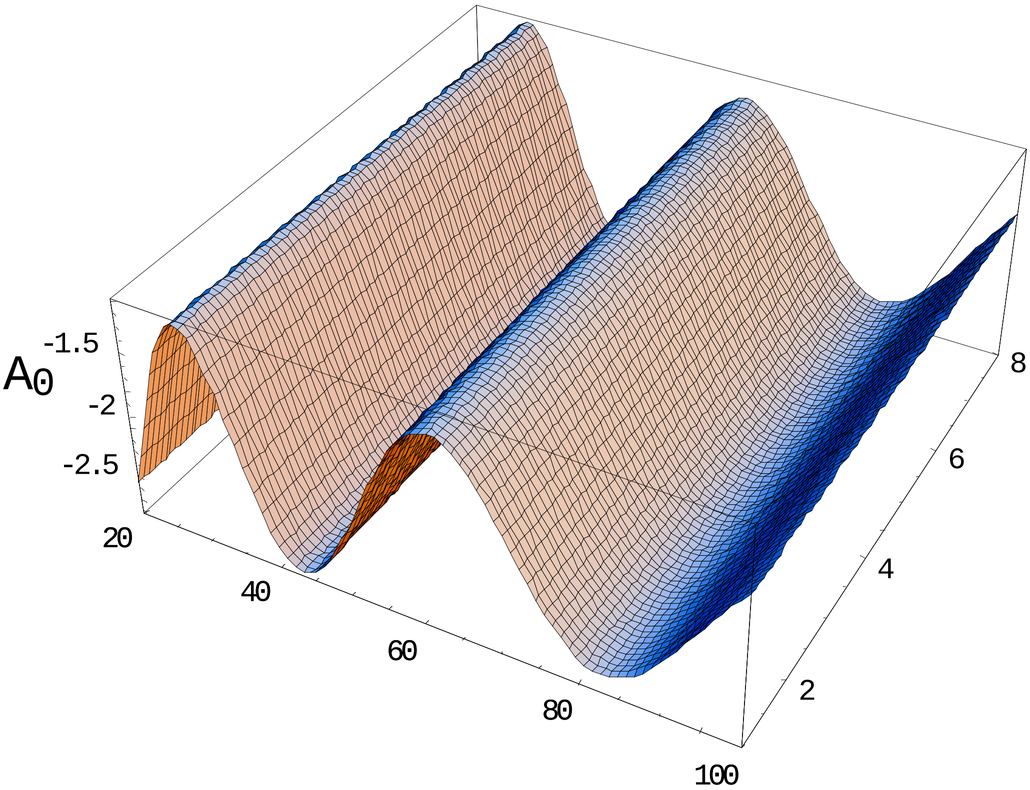

Fig: A2

3) Resonance of edge wave of frequency ω with Standing waves of frequency 2ω

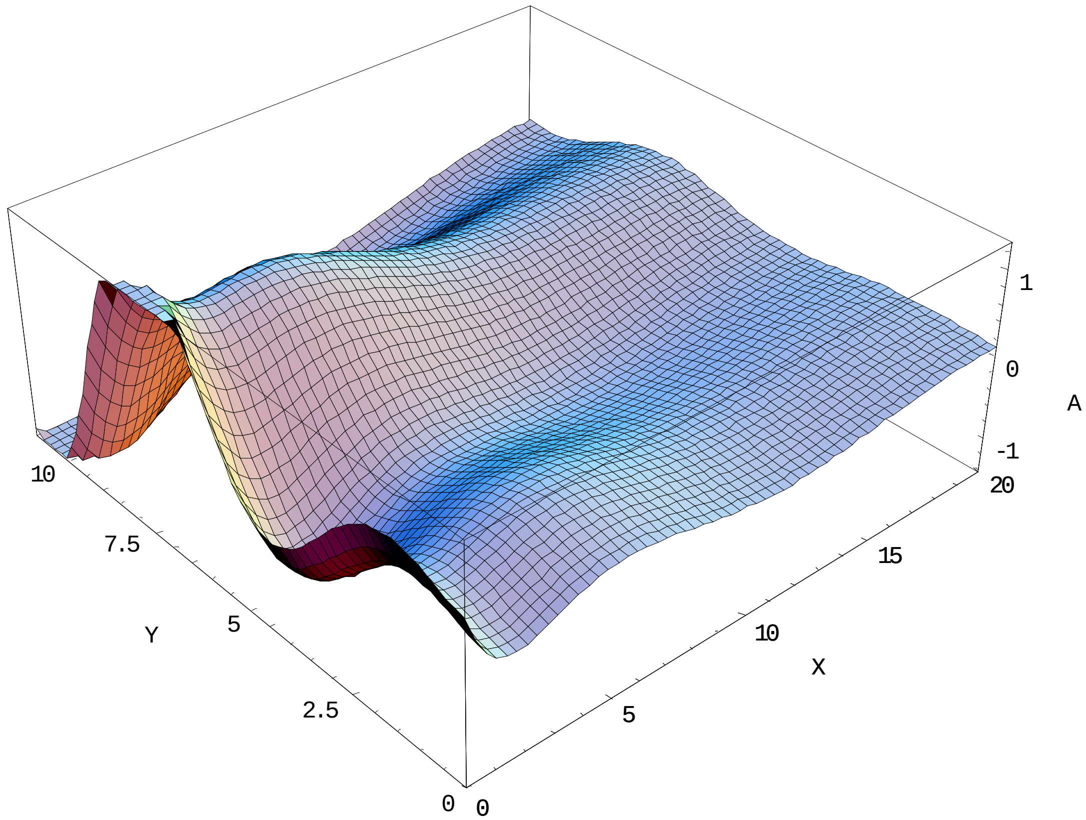

Fig: A3

APPENDIX: B

(*Detailed Analysis Using Mathematica 4.1*)

THE DETAILED ANALYSIS USING MATHEMATICA 4.1

THE INPUT FUNCTION IS

$$\Psi = A[\epsilon y, \epsilon t]\left(\text{BesselJ}\left[0, \beta\sqrt{x}\right]\right)\text{Exp}[-2 i \omega t] + A_{\epsilon 1}[\epsilon y, \epsilon t]\,\text{Exp}[-k x]\,\text{Exp}[i k y - i \omega t] + A_{\epsilon 2}[\epsilon y, \epsilon t]\,\text{Exp}[-k x]\,\text{Exp}[i k y + i \omega t] + \left((A)^{*}[\epsilon y, \epsilon t]\,\text{BesselJ}\left[0, \beta\sqrt{x}\right]\right)\text{Exp}[2 i \omega t] + (A_{\epsilon 1})^{*}[\epsilon y, \epsilon t]\,\text{Exp}[-k x]\,\text{Exp}[-i k y + i \omega t] + (A_{\epsilon 2})^{*}[\epsilon y, \epsilon t]\,\text{Exp}[-k x]\,\text{Exp}[-i k y - i \omega t];$$

THE LEFT SIDE EQUATION

$$-\partial_{t,t}\Psi + s\,g\,(\partial_x(x(\partial_x\Psi)) + (x(\partial_{Y,Y}\Psi)))$$

WE ARE NOT CONSIDERING THE HIGHER ORDER TERMS OF ϵ HERE SINCE IT IS TOO SMALL.

INTEGRATING WITH WEIGHT AFTER AVOIDING HIGHER ϵ TERMS WILL RESULTIN THE BELOW GIVE EQUATION

$$\text{Integrate}\left[\left(e^{-k x}\right) - \partial_{t,t}\Psi + s\,g\,(\partial_x(x(\partial_x\Psi)) + (x(\partial_{Y,Y}\Psi))), \{x, 0, \infty\}, \text{Assumptions} \to k > 0\right]$$

$$
\begin{aligned}
&e^{-2 i t \omega}\Bigg(-\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \beta^2\, A[y\epsilon, t\epsilon]}{4k}+\frac{4\, e^{-\frac{\beta^2}{4k}}\, \omega^2\, A[y\epsilon, t\epsilon]}{k}+\frac{4\, i\, e^{-\frac{\beta^2}{4k}}\, \epsilon\, \omega\, A^{(0,1)}[y\epsilon, t\epsilon]}{k}-\\
&\quad\frac{e^{-\frac{\beta^2}{4k}}\, \epsilon^2\, A^{(0,2)}[y\epsilon, t\epsilon]}{k}+\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \epsilon^2\, A^{(2,0)}[y\epsilon, t\epsilon]}{k^2}-\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \beta^2\, \epsilon^2\, A^{(2,0)}[y\epsilon, t\epsilon]}{4k^3}\Bigg)+\\
&e^{2 i t \omega}\Bigg(-\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \beta^2\, A^*[y\epsilon, t\epsilon]}{4k}+\frac{4\, e^{-\frac{\beta^2}{4k}}\, \omega^2\, A^*[y\epsilon, t\epsilon]}{k}-\frac{4\, i\, e^{-\frac{\beta^2}{4k}}\, \epsilon\, \omega\, (A^*)^{(0,1)}[y\epsilon, t\epsilon]}{k}-\\
&\quad\frac{e^{-\frac{\beta^2}{4k}}\, \epsilon^2\, (A^*)^{(0,2)}[y\epsilon, t\epsilon]}{k}+\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \epsilon^2\, (A^*)^{(2,0)}[y\epsilon, t\epsilon]}{k^2}-\frac{e^{-\frac{\beta^2}{4k}}\, g\, s\, \beta^2\, \epsilon^2\, (A^*)^{(2,0)}[y\epsilon, t\epsilon]}{4k^3}\Bigg)+\\
&e^{i t \omega}\Bigg(-\frac{1}{2}\, e^{i k Y}\, g\, s\, A_{e2}[y\epsilon, t\epsilon]+\frac{e^{i k Y}\, \omega^2\, A_{e2}[y\epsilon, t\epsilon]}{2k}-\frac{1}{2}\, e^{-i k Y}\, g\, s\, (A_{e1})^*[y\epsilon, t\epsilon]+\\
&\quad\frac{e^{-i k Y}\, \omega^2\, (A_{e1})^*[y\epsilon, t\epsilon]}{2k}-\frac{i\, e^{i k Y}\, \epsilon\, \omega\, A_{e2}^{(0,1)}[y\epsilon, t\epsilon]}{k}-\frac{i\, e^{-i k Y}\, \epsilon\, \omega\, ((A_{e1})^*)^{(0,1)}[y\epsilon, t\epsilon]}{k}-\\
&\quad\frac{e^{i k Y}\, \epsilon^2\, A_{e2}^{(0,2)}[y\epsilon, t\epsilon]}{2k}-\frac{e^{-i k Y}\, \epsilon^2\, ((A_{e1})^*)^{(0,2)}[y\epsilon, t\epsilon]}{2k}+\frac{i\, e^{i k Y}\, g\, s\, \epsilon\, A_{e2}^{(1,0)}[y\epsilon, t\epsilon]}{2k}-\\
&\quad\frac{i\, e^{-i k Y}\, g\, s\, \epsilon\, ((A_{e1})^*)^{(1,0)}[y\epsilon, t\epsilon]}{2k}+\frac{e^{i k Y}\, g\, s\, \epsilon^2\, A_{e2}^{(2,0)}[y\epsilon, t\epsilon]}{4k^2}+\frac{e^{-i k Y}\, g\, s\, \epsilon^2\, ((A_{e1})^*)^{(2,0)}[y\epsilon, t\epsilon]}{4k^2}\Bigg)+\\
&e^{-i t \omega}\Bigg(-\frac{1}{2}\, e^{i k Y}\, g\, s\, A_{e1}[y\epsilon, t\epsilon]+\frac{e^{i k Y}\, \omega^2\, A_{e1}[y\epsilon, t\epsilon]}{2k}-\frac{1}{2}\, e^{-i k Y}\, g\, s\, (A_{e2})^*[y\epsilon, t\epsilon]+\\
&\quad\frac{e^{-i k Y}\, \omega^2\, (A_{e2})^*[y\epsilon, t\epsilon]}{2k}+\frac{i\, e^{i k Y}\, \epsilon\, \omega\, A_{e1}^{(0,1)}[y\epsilon, t\epsilon]}{k}+\frac{i\, e^{-i k Y}\, \epsilon\, \omega\, ((A_{e2})^*)^{(0,1)}[y\epsilon, t\epsilon]}{k}-\\
&\quad\frac{e^{i k Y}\, \epsilon^2\, A_{e1}^{(0,2)}[y\epsilon, t\epsilon]}{2k}-\frac{e^{-i k Y}\, \epsilon^2\, ((A_{e2})^*)^{(0,2)}[y\epsilon, t\epsilon]}{2k}+\frac{i\, e^{i k Y}\, g\, s\, \epsilon\, A_{e1}^{(1,0)}[y\epsilon, t\epsilon]}{2k}-\\
&\quad\frac{i\, e^{-i k Y}\, g\, s\, \epsilon\, ((A_{e2})^*)^{(1,0)}[y\epsilon, t\epsilon]}{2k}+\frac{e^{i k Y}\, g\, s\, \epsilon^2\, A_{e1}^{(2,0)}[y\epsilon, t\epsilon]}{4k^2}+\frac{e^{-i k Y}\, g\, s\, \epsilon^2\, ((A_{e2})^*)^{(2,0)}[y\epsilon, t\epsilon]}{4k^2}\Bigg)
\end{aligned}
$$

ELIMINATING ZERO TERMS WILL RESULT THE BELOW GIVEN THREE EQUATIONS

$$e^{-2 i t \omega}\left(\frac{4\, i\, e^{-\frac{1}{4k}}\,\epsilon\,\omega\, A^{(0,1)}[y\,\epsilon,\ t\,\epsilon]}{k}\right)$$

$$e^{i t \omega}\left(-\frac{i\, e^{i k Y}\,\epsilon\,\omega\, A_{e2}^{(0,1)}[y\,\epsilon,\ t\,\epsilon]}{k}+\frac{i\, e^{i k Y}\, g\, s\,\epsilon\, A_{e2}^{(1,0)}[y\,\epsilon,\ t\,\epsilon]}{2\,k}\right)$$

$$e^{-i t \omega}\left(\frac{i\, e^{i k Y}\,\epsilon\,\omega\, A_{e1}^{(0,1)}[y\,\epsilon,\ t\,\epsilon]}{k}+\frac{i\, e^{i k Y}\, g\, s\,\epsilon\, A_{e1}^{(1,0)}[y\,\epsilon,\ t\,\epsilon]}{2\,k}\right)$$

****

THE RIGHT SIDE EQUATIONS

(β IS APPPLIED ONLY IN THE LAST)

FIRST PART OF THE RIGHT SIDE EQUATION

$$\partial_x \Psi\, \partial_{x,t} \Psi + 2\,(\partial_Y \Psi\ \partial_{Y,t} \Psi) +\ \partial_t \Psi\,(\partial_{Y,Y} \Psi)$$

INTEGRATING WITH WEIGHT

$$\text{Integrate}\left[e^{-k\,x}\, \partial_x \Psi\, \partial_{x,t} \Psi + 2\,(\partial_Y \Psi\ \partial_{Y,t} \Psi) +\ \partial_t \Psi\,(\partial_{Y,Y} \Psi),\ \{x,\ 0,\ \infty\},\ \text{Assumptions} \to k > 0\right]$$

COLLECTING SIMILAR TERMS THE RESUT IS

$$
\begin{aligned}
&\frac{1}{4}\,\epsilon\,A^{*}[y\,\epsilon,\,t\,\epsilon]\,A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,\text{BesselI}\left[0,\,\frac{1}{2\,k}\right]A^{*}[y\,\epsilon,\,t\,\epsilon]\,A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\\
&\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,\text{BesselI}\left[1,\,\frac{1}{2\,k}\right]A^{*}[y\,\epsilon,\,t\,\epsilon]\,A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]+\\
&e^{-4\,i\,t\,\omega}\left(-\frac{1}{2}\,i\,\omega\,A[y\,\epsilon,\,t\,\epsilon]^{2}+\frac{1}{2}\,i\,e^{-\frac{1}{2k}}\,\omega\,A[y\,\epsilon,\,t\,\epsilon]^{2}\,\text{BesselI}\left[0,\,\frac{1}{2\,k}\right]+\frac{1}{2}\,i\,e^{-\frac{1}{2k}}\,\omega\,A[y\,\epsilon,\,t\,\epsilon]^{2}\,\text{BesselI}\left[1,\,\frac{1}{2\,k}\right]+\right.\\
&\quad\frac{1}{4}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,\text{BesselI}\left[0,\,\frac{1}{2\,k}\right]A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\\
&\quad\left.\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,\text{BesselI}\left[1,\,\frac{1}{2\,k}\right]A^{(0,1)}[y\,\epsilon,\,t\,\epsilon]\right)-\frac{2}{3}\,e^{2\,i\,k\,y}\,k\,\epsilon\,A_{e2}[y\,\epsilon,\,t\,\epsilon]\,A_{e1}^{(0,1)}[y\,\epsilon,\,t\,\epsilon]+\\
&\frac{2}{3}\,k\,\epsilon\,(A_{e1})^{*}[y\,\epsilon,\,t\,\epsilon]\,A_{e1}^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\frac{2}{3}\,e^{2\,i\,k\,y}\,k\,\epsilon\,A_{e1}[y\,\epsilon,\,t\,\epsilon]\,A_{e2}^{(0,1)}[y\,\epsilon,\,t\,\epsilon]+\\
&\frac{2}{3}\,k\,\epsilon\,(A_{e2})^{*}[y\,\epsilon,\,t\,\epsilon]\,A_{e2}^{(0,1)}[y\,\epsilon,\,t\,\epsilon]+\frac{1}{4}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,(A^{*})^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\\
&\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,\text{BesselI}\left[0,\,\frac{1}{2\,k}\right](A^{*})^{(0,1)}[y\,\epsilon,\,t\,\epsilon]-\\
&\frac{1}{4}\,e^{-\frac{1}{2k}}\,\epsilon\,A[y\,\epsilon,\,t\,\epsilon]\,\text{BesselI}\left[1,\,\frac{1}{2\,k}\right](A^{*})^{(0,1)}[y\,\epsilon,\,t\,\epsilon]+
\end{aligned}
$$

$$
\begin{aligned}
&e^{4 i t \omega} \Big(\frac{1}{2} i \omega A^*[y \epsilon, t \epsilon]^2 - \frac{1}{2} i e^{-\frac{1}{2k}} \omega \,\mathrm{BesselI}\Big[0, \frac{1}{2k}\Big] A^*[y \epsilon, t \epsilon]^2 - \frac{1}{2} i e^{-\frac{1}{2k}} \omega \,\mathrm{BesselI}\Big[1, \frac{1}{2k}\Big] A^*[y \epsilon, t \epsilon]^2 + \\
&\quad \frac{1}{4} \epsilon A^*[y \epsilon, t \epsilon] (A^*)^{(0,1)}[y \epsilon, t \epsilon] - \frac{1}{4} e^{-\frac{1}{2k}} \epsilon \,\mathrm{BesselI}\Big[0, \frac{1}{2k}\Big] A^*[y \epsilon, t \epsilon] (A^*)^{(0,1)}[y \epsilon, t \epsilon] - \\
&\quad \frac{1}{4} e^{-\frac{1}{2k}} \epsilon \,\mathrm{BesselI}\Big[1, \frac{1}{2k}\Big] A^*[y \epsilon, t \epsilon] (A^*)^{(0,1)}[y \epsilon, t \epsilon]\Big) + \frac{2}{3} k \epsilon A_{e1}[y \epsilon, t \epsilon] ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - \\
&\frac{2}{3} e^{-2 i k Y} k \epsilon (A_{e2})^*[y \epsilon, t \epsilon] ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] + \frac{2}{3} k \epsilon A_{e2}[y \epsilon, t \epsilon] ((A_{e2})^*)^{(0,1)}[y \epsilon, t \epsilon] - \\
&\frac{2}{3} e^{-2 i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon] ((A_{e2})^*)^{(0,1)}[y \epsilon, t \epsilon] - \frac{2}{3} e^{2 i k Y} \epsilon \omega A_{e2}[y \epsilon, t \epsilon] A_{e1}^{(1,0)}[y \epsilon, t \epsilon] - \\
&\frac{2}{3} \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon] A_{e1}^{(1,0)}[y \epsilon, t \epsilon] + \frac{2}{3} e^{2 i k Y} \epsilon \omega A_{e1}[y \epsilon, t \epsilon] A_{e2}^{(1,0)}[y \epsilon, t \epsilon] + \\
&\frac{2}{3} \epsilon \omega (A_{e2})^*[y \epsilon, t \epsilon] A_{e2}^{(1,0)}[y \epsilon, t \epsilon] - \frac{2}{3} \epsilon \omega A_{e1}[y \epsilon, t \epsilon] ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon] - \\
&\frac{2}{3} e^{-2 i k Y} \epsilon \omega (A_{e2})^*[y \epsilon, t \epsilon] ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon] +
\end{aligned}
$$

$$
\begin{aligned}
&e^{-i t \omega} \Big(-i e^{i k Y} k \omega A[y \epsilon, t \epsilon] A_{e2}[y \epsilon, t \epsilon] + 2 i e^{-\frac{1}{8k} + i k Y} k \omega A[y \epsilon, t \epsilon] A_{e2}[y \epsilon, t \epsilon] - \\
&\quad i e^{-i k Y} k \omega A[y \epsilon, t \epsilon] (A_{e1})^*[y \epsilon, t \epsilon] + 2 i e^{-\frac{1}{8k} - i k Y} k \omega A[y \epsilon, t \epsilon] (A_{e1})^*[y \epsilon, t \epsilon] + \\
&\quad e^{i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon] - \frac{3}{2} e^{-\frac{1}{8k} + i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{-i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon] - \frac{3}{2} e^{-\frac{1}{8k} - i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{i k Y} k \epsilon A[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon] - e^{-\frac{1}{8k} + i k Y} k \epsilon A[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{-i k Y} k \epsilon A[y \epsilon, t \epsilon] ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - e^{-\frac{1}{8k} - i k Y} k \epsilon A[y \epsilon, t \epsilon] ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{-\frac{1}{8k} + i k Y} \epsilon \omega A_{e2}[y \epsilon, t \epsilon] A^{(1,0)}[y \epsilon, t \epsilon] - e^{-\frac{1}{8k} - i k Y} \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon] A^{(1,0)}[y \epsilon, t \epsilon] + \\
&\quad 2 e^{-\frac{1}{8k} + i k Y} \epsilon \omega A[y \epsilon, t \epsilon] A_{e2}^{(1,0)}[y \epsilon, t \epsilon] - 2 e^{-\frac{1}{8k} - i k Y} \epsilon \omega A[y \epsilon, t \epsilon] ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon]\Big) +
\end{aligned}
$$

$$
\begin{aligned}
&e^{3 i t \omega} \Big(3 i\, e^{i k Y} k \omega A_{e2}[y \epsilon, t \epsilon]\, A^*[y \epsilon, t \epsilon] - 4 i\, e^{-\frac{1}{8k} + i k Y} k \omega A_{e2}[y \epsilon, t \epsilon]\, A^*[y \epsilon, t \epsilon] + \\
&\quad 3 i\, e^{-i k Y} k \omega A^*[y \epsilon, t \epsilon]\, (A_{e1})^*[y \epsilon, t \epsilon] - 4 i\, e^{-\frac{1}{8k} - i k Y} k \omega A^*[y \epsilon, t \epsilon]\, (A_{e1})^*[y \epsilon, t \epsilon] + \\
&\quad e^{i k Y} k \epsilon A^*[y \epsilon, t \epsilon]\, A_{e2}^{(0,1)}[y \epsilon, t \epsilon] - e^{-\frac{1}{8k} + i k Y} k \epsilon A^*[y \epsilon, t \epsilon]\, A_{e2}^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon]\, (A^*)^{(0,1)}[y \epsilon, t \epsilon] - \frac{3}{2}\, e^{-\frac{1}{8k} + i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon]\, (A^*)^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{-i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon]\, (A^*)^{(0,1)}[y \epsilon, t \epsilon] - \frac{3}{2}\, e^{-\frac{1}{8k} - i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon]\, (A^*)^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad e^{-i k Y} k \epsilon A^*[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - e^{-\frac{1}{8k} - i k Y} k \epsilon A^*[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - \\
&\quad 2\, e^{-\frac{1}{8k} + i k Y} \epsilon \omega A^*[y \epsilon, t \epsilon]\, A_{e2}^{(1,0)}[y \epsilon, t \epsilon] - 3\, e^{-\frac{1}{8k} + i k Y} \epsilon \omega A_{e2}[y \epsilon, t \epsilon]\, (A^*)^{(1,0)}[y \epsilon, t \epsilon] + \\
&\quad 3\, e^{-\frac{1}{8k} - i k Y} \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon]\, (A^*)^{(1,0)}[y \epsilon, t \epsilon] + 2\, e^{-\frac{1}{8k} - i k Y} \epsilon \omega A^*[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon] \Big)
\end{aligned}
$$

$$
\begin{aligned}
&e^{2 i t \omega} \Big(-\frac{2}{3}\, i\, e^{2 i k Y} k \omega A_{e2}[y \epsilon, t \epsilon]^2 + \frac{4}{3}\, i k \omega A_{e2}[y \epsilon, t \epsilon]\, (A_{e1})^*[y \epsilon, t \epsilon] - \frac{2}{3}\, i\, e^{-2 i k Y} k \omega (A_{e1})^*[y \epsilon, t \epsilon]^2 - \\
&\quad \frac{2}{3}\, e^{2 i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon]\, A_{e2}^{(0,1)}[y \epsilon, t \epsilon] + \frac{2}{3}\, k \epsilon (A_{e1})^*[y \epsilon, t \epsilon]\, A_{e2}^{(0,1)}[y \epsilon, t \epsilon] + \\
&\quad \frac{2}{3}\, k \epsilon A_{e2}[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - \frac{2}{3}\, e^{-2 i k Y} k \epsilon (A_{e1})^*[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(0,1)}[y \epsilon, t \epsilon] - \\
&\quad 2\, e^{2 i k Y} \epsilon \omega A_{e2}[y \epsilon, t \epsilon]\, A_{e2}^{(1,0)}[y \epsilon, t \epsilon] + \frac{2}{3}\, \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon]\, A_{e2}^{(1,0)}[y \epsilon, t \epsilon] - \\
&\quad \frac{2}{3}\, \epsilon \omega A_{e2}[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon] + 2\, e^{-2 i k Y} \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon]\, ((A_{e1})^*)^{(1,0)}[y \epsilon, t \epsilon] \Big) + \\
&\frac{2}{3}\, \epsilon \omega A_{e2}[y \epsilon, t \epsilon]\, ((A_{e2})^*)^{(1,0)}[y \epsilon, t \epsilon] + \frac{2}{3}\, e^{-2 i k Y} \epsilon \omega (A_{e1})^*[y \epsilon, t \epsilon]\, ((A_{e2})^*)^{(1,0)}[y \epsilon, t \epsilon] +
\end{aligned}
$$

$$
\begin{aligned}
&e^{-3\mathrm{i}t\omega}\Big(-3\,\mathrm{i}\,e^{\mathrm{i}kY}k\,\omega\,A[y\epsilon,t\epsilon]\,A_{e1}[y\epsilon,t\epsilon]+4\,\mathrm{i}\,e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\omega\,A[y\epsilon,t\epsilon]\,A_{e1}[y\epsilon,t\epsilon]-\\
&\quad 3\,\mathrm{i}\,e^{-\mathrm{i}kY}k\,\omega\,A[y\epsilon,t\epsilon]\,(A_{e2})^{*}[y\epsilon,t\epsilon]+4\,\mathrm{i}\,e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\omega\,A[y\epsilon,t\epsilon]\,(A_{e2})^{*}[y\epsilon,t\epsilon]+\\
&\quad e^{\mathrm{i}kY}k\,\epsilon\,A_{e1}[y\epsilon,t\epsilon]\,A^{(0,1)}[y\epsilon,t\epsilon]-\frac{3}{2}\,e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\epsilon\,A_{e1}[y\epsilon,t\epsilon]\,A^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{-\mathrm{i}kY}k\,\epsilon\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,A^{(0,1)}[y\epsilon,t\epsilon]-\frac{3}{2}\,e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\epsilon\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,A^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{\mathrm{i}kY}k\,\epsilon\,A[y\epsilon,t\epsilon]\,A_{e1}^{(0,1)}[y\epsilon,t\epsilon]-e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\epsilon\,A[y\epsilon,t\epsilon]\,A_{e1}^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{-\mathrm{i}kY}k\,\epsilon\,A[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(0,1)}[y\epsilon,t\epsilon]-e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\epsilon\,A[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad 3\,e^{-\frac{1}{8k}+\mathrm{i}kY}\epsilon\,\omega\,A_{e1}[y\epsilon,t\epsilon]\,A^{(1,0)}[y\epsilon,t\epsilon]-3\,e^{-\frac{1}{8k}-\mathrm{i}kY}\epsilon\,\omega\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,A^{(1,0)}[y\epsilon,t\epsilon]+\\
&\quad 2\,e^{-\frac{1}{8k}+\mathrm{i}kY}\epsilon\,\omega\,A[y\epsilon,t\epsilon]\,A_{e1}^{(1,0)}[y\epsilon,t\epsilon]-2\,e^{-\frac{1}{8k}-\mathrm{i}kY}\epsilon\,\omega\,A[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(1,0)}[y\epsilon,t\epsilon]\Big)+\\
&e^{\mathrm{i}t\omega}\Big(\mathrm{i}\,e^{\mathrm{i}kY}k\,\omega\,A_{e1}[y\epsilon,t\epsilon]\,A^{*}[y\epsilon,t\epsilon]-2\,\mathrm{i}\,e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\omega\,A_{e1}[y\epsilon,t\epsilon]\,A^{*}[y\epsilon,t\epsilon]+\\
&\quad \mathrm{i}\,e^{-\mathrm{i}kY}k\,\omega\,A^{*}[y\epsilon,t\epsilon]\,(A_{e2})^{*}[y\epsilon,t\epsilon]-2\,\mathrm{i}\,e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\omega\,A^{*}[y\epsilon,t\epsilon]\,(A_{e2})^{*}[y\epsilon,t\epsilon]+\\
&\quad e^{\mathrm{i}kY}k\,\epsilon\,A^{*}[y\epsilon,t\epsilon]\,A_{e1}^{(0,1)}[y\epsilon,t\epsilon]-e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\epsilon\,A^{*}[y\epsilon,t\epsilon]\,A_{e1}^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{\mathrm{i}kY}k\,\epsilon\,A_{e1}[y\epsilon,t\epsilon]\,(A^{*})^{(0,1)}[y\epsilon,t\epsilon]-\frac{3}{2}\,e^{-\frac{1}{8k}+\mathrm{i}kY}k\,\epsilon\,A_{e1}[y\epsilon,t\epsilon]\,(A^{*})^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{-\mathrm{i}kY}k\,\epsilon\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,(A^{*})^{(0,1)}[y\epsilon,t\epsilon]-\frac{3}{2}\,e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\epsilon\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,(A^{*})^{(0,1)}[y\epsilon,t\epsilon]+\\
&\quad e^{-\mathrm{i}kY}k\,\epsilon\,A^{*}[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(0,1)}[y\epsilon,t\epsilon]-e^{-\frac{1}{8k}-\mathrm{i}kY}k\,\epsilon\,A^{*}[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(0,1)}[y\epsilon,t\epsilon]-\\
&\quad 2\,e^{-\frac{1}{8k}+\mathrm{i}kY}\epsilon\,\omega\,A^{*}[y\epsilon,t\epsilon]\,A_{e1}^{(1,0)}[y\epsilon,t\epsilon]-e^{-\frac{1}{8k}+\mathrm{i}kY}\epsilon\,\omega\,A_{e1}[y\epsilon,t\epsilon]\,(A^{*})^{(1,0)}[y\epsilon,t\epsilon]+\\
&\quad e^{-\frac{1}{8k}-\mathrm{i}kY}\epsilon\,\omega\,(A_{e2})^{*}[y\epsilon,t\epsilon]\,(A^{*})^{(1,0)}[y\epsilon,t\epsilon]+2\,e^{-\frac{1}{8k}-\mathrm{i}kY}\epsilon\,\omega\,A^{*}[y\epsilon,t\epsilon]\,((A_{e2})^{*})^{(1,0)}[y\epsilon,t\epsilon]\Big)+
\end{aligned}
$$

$$e^{-2 i t \omega}\Big(\frac{2}{3}\, i\, e^{2 i k Y} k\,\omega\, A_{e1}[y\epsilon, t\epsilon]^2 - \frac{4}{3}\, i\, k\,\omega\, A_{e1}[y\epsilon, t\epsilon]\,(A_{e2})^*[y\epsilon, t\epsilon] + \frac{2}{3}\, i\, e^{-2 i k Y} k\,\omega\,(A_{e2})^*[y\epsilon, t\epsilon]^2 -$$

$$\frac{2}{3}\, e^{2 i k Y} k\,\epsilon\, A_{e1}[y\epsilon, t\epsilon]\, A_{e1}^{(0,1)}[y\epsilon, t\epsilon] + \frac{2}{3}\, k\,\epsilon\,(A_{e2})^*[y\epsilon, t\epsilon]\, A_{e1}^{(0,1)}[y\epsilon, t\epsilon] +$$

$$\frac{2}{3}\, k\,\epsilon\, A_{e1}[y\epsilon, t\epsilon]\,((A_{e2})^*)^{(0,1)}[y\epsilon, t\epsilon] - \frac{2}{3}\, e^{-2 i k Y} k\,\epsilon\,(A_{e2})^*[y\epsilon, t\epsilon]\,((A_{e2})^*)^{(0,1)}[y\epsilon, t\epsilon] +$$

$$2\, e^{2 i k Y}\epsilon\,\omega\, A_{e1}[y\epsilon, t\epsilon]\, A_{e1}^{(1,0)}[y\epsilon, t\epsilon] - \frac{2}{3}\,\epsilon\,\omega\,(A_{e2})^*[y\epsilon, t\epsilon]\, A_{e1}^{(1,0)}[y\epsilon, t\epsilon] +$$

$$\frac{2}{3}\,\epsilon\,\omega\, A_{e1}[y\epsilon, t\epsilon]\,((A_{e2})^*)^{(1,0)}[y\epsilon, t\epsilon] - 2\, e^{-2 i k Y}\epsilon\,\omega\,(A_{e2})^*[y\epsilon, t\epsilon]\,((A_{e2})^*)^{(1,0)}[y\epsilon, t\epsilon]\Big)$$

THE SECOND PART OF THE EQUATION IS FROM THE TRANSFORMATION TERM $(\partial_t \Psi\, \partial_x \Psi)\, e^{-k x}$

$$-\frac{1}{2\sqrt{x}}\left(e^{-k x}\epsilon\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A^*[y\epsilon, t\epsilon]\, A^{(0,1)}[y\epsilon, t\epsilon]\right) +$$

$$e^{-4 i t \omega}\left(\frac{i\, e^{-k x}\omega\, A[y\epsilon, t\epsilon]^2\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]}{\sqrt{x}} - \frac{1}{2\sqrt{x}}\left(e^{-k x}\epsilon\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A^{(0,1)}[y\epsilon, t\epsilon]\right)\right) -$$

$$e^{-3 k x + 2 i k Y} k\,\epsilon\, A_{e2}[y\epsilon, t\epsilon]\, A_{e1}^{(0,1)}[y\epsilon, t\epsilon] - e^{-3 k x} k\,\epsilon\,(A_{e1})^*[y\epsilon, t\epsilon]\, A_{e1}^{(0,1)}[y\epsilon, t\epsilon] - e^{-3 k x + 2 i k Y} k\,\epsilon\, A_{e1}[y\epsilon, t\epsilon]\, A_{e2}^{(0,1)}[y\epsilon, t\epsilon] -$$

$$e^{-3 k x} k\,\epsilon\,(A_{e2})^*[y\epsilon, t\epsilon]\, A_{e2}^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{2\sqrt{x}}\left(e^{-k x}\epsilon\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]\,(A^*)^{(0,1)}[y\epsilon, t\epsilon]\right) +$$

$$e^{4 i t \omega}\left(-\frac{i\, e^{-k x}\omega\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A^*[y\epsilon, t\epsilon]^2}{\sqrt{x}} - \frac{1}{2\sqrt{x}}\left(e^{-k x}\epsilon\,\mathrm{BesselJ}[0, \sqrt{x}]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A^*[y\epsilon, t\epsilon]\,(A^*)^{(0,1)}[y\epsilon, t\epsilon]\right)\right) -$$

$$e^{-3 k x} k\,\epsilon\, A_{e1}[y\epsilon, t\epsilon]\,((A_{e1})^*)^{(0,1)}[y\epsilon, t\epsilon] - e^{-3 k x - 2 i k Y} k\,\epsilon\,(A_{e2})^*[y\epsilon, t\epsilon]\,((A_{e1})^*)^{(0,1)}[y\epsilon, t\epsilon] +$$

$$e^{-i t \omega}\Bigg(2\, i\, e^{-2 k x + i k Y} k\,\omega\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[0, \sqrt{x}]\, A_{e2}[y\epsilon, t\epsilon] - \frac{i\, e^{-2 k x + i k Y}\omega\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A_{e2}[y\epsilon, t\epsilon]}{2\sqrt{x}} +$$

$$2\, i\, e^{-2 k x - i k Y} k\,\omega\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[0, \sqrt{x}]\,(A_{e1})^*[y\epsilon, t\epsilon] - \frac{i\, e^{-2 k x - i k Y}\omega\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[1, \sqrt{x}]\,(A_{e1})^*[y\epsilon, t\epsilon]}{2\sqrt{x}} -$$

$$e^{-2 k x + i k Y} k\,\epsilon\,\mathrm{BesselJ}[0, \sqrt{x}]\, A_{e2}[y\epsilon, t\epsilon]\, A^{(0,1)}[y\epsilon, t\epsilon] - e^{-2 k x - i k Y} k\,\epsilon\,\mathrm{BesselJ}[0, \sqrt{x}]\,(A_{e1})^*[y\epsilon, t\epsilon]\, A^{(0,1)}[y\epsilon, t\epsilon] -$$

$$\frac{e^{-2 k x + i k Y}\epsilon\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[1, \sqrt{x}]\, A_{e2}^{(0,1)}[y\epsilon, t\epsilon]}{2\sqrt{x}} - \frac{1}{2\sqrt{x}}\left(e^{-2 k x - i k Y}\epsilon\, A[y\epsilon, t\epsilon]\,\mathrm{BesselJ}[1, \sqrt{x}]\,((A_{e1})^*)^{(0,1)}[y\epsilon, t\epsilon]\right)\Bigg) +$$

$$
\begin{aligned}
&e^{i t \omega}\Bigg(-2\, i\, e^{-2 k x+i k y} k\, \omega\, \mathrm{BesselJ}\left[0, \sqrt{x}\right] A_{e1}[y \epsilon, t \epsilon]\, A^{*}[y \epsilon, t \epsilon] + \frac{i\, e^{-2 k x+i k y} \omega\, \mathrm{BesselJ}\left[1, \sqrt{x}\right] A_{e1}[y \epsilon, t \epsilon]\, A^{*}[y \epsilon, t \epsilon]}{2 \sqrt{x}} - \\
&\quad 2\, i\, e^{-2 k x-i k y} k\, \omega\, \mathrm{BesselJ}\left[0, \sqrt{x}\right] A^{*}[y \epsilon, t \epsilon]\, (A_{e2})^{*}[y \epsilon, t \epsilon] + \frac{i\, e^{-2 k x-i k y} \omega\, \mathrm{BesselJ}\left[1, \sqrt{x}\right] A^{*}[y \epsilon, t \epsilon]\, (A_{e2})^{*}[y \epsilon, t \epsilon]}{2 \sqrt{x}} - \\
&\quad \frac{e^{-2 k x+i k y} \epsilon\, \mathrm{BesselJ}\left[1, \sqrt{x}\right] A^{*}[y \epsilon, t \epsilon]\, A_{e1}^{(0,1)}[y \epsilon, t \epsilon]}{2 \sqrt{x}} - e^{-2 k x+i k y} k\, \epsilon\, \mathrm{BesselJ}\left[0, \sqrt{x}\right] A_{e1}[y \epsilon, t \epsilon]\, (A^{*})^{(0,1)}[y \epsilon, t \epsilon] - \\
&\quad e^{-2 k x-i k y} k\, \epsilon\, \mathrm{BesselJ}\left[0, \sqrt{x}\right] (A_{e2})^{*}[y \epsilon, t \epsilon]\, (A^{*})^{(0,1)}[y \epsilon, t \epsilon] - \frac{1}{2 \sqrt{x}}\left(e^{-2 k x-i k y} \epsilon\, \mathrm{BesselJ}\left[1, \sqrt{x}\right] A^{*}[y \epsilon, t \epsilon]\, ((A_{e2})^{*})^{(0,1)}[y \epsilon, t \epsilon]\right)\Bigg) + \\
&e^{-2 i t \omega}\Big(i\, e^{-3 k x+2 i k y} k\, \omega\, A_{e1}[y \epsilon, t \epsilon]^{2} + 2\, i\, e^{-3 k x} k\, \omega\, A_{e1}[y \epsilon, t \epsilon]\, (A_{e2})^{*}[y \epsilon, t \epsilon] + i\, e^{-3 k x-2 i k y} k\, \omega\, (A_{e2})^{*}[y \epsilon, t \epsilon]^{2} - \\
&\quad e^{-3 k x+2 i k y} k\, \epsilon\, A_{e1}[y \epsilon, t \epsilon]\, A_{e1}^{(0,1)}[y \epsilon, t \epsilon] - e^{-3 k x} k\, \epsilon\, (A_{e2})^{*}[y \epsilon, t \epsilon]\, A_{e1}^{(0,1)}[y \epsilon, t \epsilon] - e^{-3 k x} k\, \epsilon\, A_{e1}[y \epsilon, t \epsilon]\, ((A_{e2})^{*})^{(0,1)}[y \epsilon, t \epsilon] - \\
&\quad e^{-3 k x-2 i k y} k\, \epsilon\, (A_{e2})^{*}[y \epsilon, t \epsilon]\, ((A_{e2})^{*})^{(0,1)}[y \epsilon, t \epsilon]\Big)
\end{aligned}
$$

SUBSTITUTE ONLY 0, AS AT INFINITY THIS EQUATION IS ZERO DUE TO THE NEGATIVE EXPONENTIAL TERM.

NOW THE FINAL INTEGRAL TERM OF THE TRANSFORMATION EQATION IS AS SHOWN BELOW

$$\mathrm{Integrate}\left[k\left(e^{-k x}\right)(\partial_t \Psi\, \partial_x \Psi), \{x, 0, \infty\}, \mathrm{Assumptions} \to k > 0\right]$$

INTEGRATING AND EXPANDING THE EQUATION,

$$-\frac{1}{2} k \epsilon A^{*}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon]+\frac{1}{2} e^{-\frac{1}{2 k}} k \epsilon \operatorname{BesselI}\left[0, \frac{1}{2 k}\right] A^{*}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon]+$$

$$e^{-4 i t \omega}\left(i k \omega A[y \epsilon, t \epsilon]^{2}-i e^{-\frac{1}{2 k}} k \omega A[y \epsilon, t \epsilon]^{2} \operatorname{BesselI}\left[0, \frac{1}{2 k}\right]-\frac{1}{2} k \epsilon A[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon]+\right.$$

$$\left.\frac{1}{2} e^{-\frac{1}{2 k}} k \epsilon A[y \epsilon, t \epsilon] \operatorname{BesselI}\left[0, \frac{1}{2 k}\right] A^{(0,1)}[y \epsilon, t \epsilon]\right)-\frac{1}{3} e^{2 i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon] A_{e1}^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\frac{1}{3} k \epsilon\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon] A_{e1}^{(0,1)}[y \epsilon, t \epsilon]-\frac{1}{3} e^{2 i k Y} k \epsilon A_{e1}[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\frac{1}{3} k \epsilon\left(A_{e2}\right)^{*}[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]-\frac{1}{2} k \epsilon A[y \epsilon, t \epsilon]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+$$

$$\frac{1}{2} e^{-\frac{1}{2 k}} k \epsilon A[y \epsilon, t \epsilon] \operatorname{BesselI}\left[0, \frac{1}{2 k}\right]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+$$

$$e^{4 i t \omega}\left(-i k \omega A^{*}[y \epsilon, t \epsilon]^{2}+i e^{-\frac{1}{2 k}} k \omega \operatorname{BesselI}\left[0, \frac{1}{2 k}\right] A^{*}[y \epsilon, t \epsilon]^{2}-\frac{1}{2} k \epsilon A^{*}[y \epsilon, t \epsilon]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+\right.$$

$$\left.\frac{1}{2} e^{-\frac{1}{2 k}} k \epsilon \operatorname{BesselI}\left[0, \frac{1}{2 k}\right] A^{*}[y \epsilon, t \epsilon]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]\right)-\frac{1}{3} k \epsilon A_{e1}[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\frac{1}{3} e^{-2 i k Y} k \epsilon\left(A_{e2}\right)^{*}[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+$$

$$e^{-i t \omega}\left(-i e^{i k Y} k \omega A[y \epsilon, t \epsilon] A_{e2}[y \epsilon, t \epsilon]+2 i e^{-\frac{1}{8 k}+i k Y} k \omega A[y \epsilon, t \epsilon] A_{e2}[y \epsilon, t \epsilon]-\right.$$

$$i e^{-i k Y} k \omega A[y \epsilon, t \epsilon]\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon]+2 i e^{-\frac{1}{8 k}-i k Y} k \omega A[y \epsilon, t \epsilon]\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon]-$$

$$\frac{1}{2} e^{-\frac{1}{8 k}+i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon]-\frac{1}{2} e^{-\frac{1}{8 k}-i k Y} k \epsilon\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon] A^{(0,1)}[y \epsilon, t \epsilon]-$$

$$e^{i k Y} k \epsilon A[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]+e^{-\frac{1}{8 k}+i k Y} k \epsilon A[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\left.e^{-i k Y} k \epsilon A[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+e^{-\frac{1}{8 k}-i k Y} k \epsilon A[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]\right)+$$

$$e^{3 i t \omega}\left(-i e^{i k Y} k \omega A_{e2}[y \epsilon, t \epsilon] A^{*}[y \epsilon, t \epsilon]-i e^{-i k Y} k \omega A^{*}[y \epsilon, t \epsilon]\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon]-\right.$$

$$e^{i k Y} k \epsilon A^{*}[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]+e^{-\frac{1}{8 k}+i k Y} k \epsilon A^{*}[y \epsilon, t \epsilon] A_{e2}^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\frac{1}{2} e^{-\frac{1}{8 k}+i k Y} k \epsilon A_{e2}[y \epsilon, t \epsilon]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]-\frac{1}{2} e^{-\frac{1}{8 k}-i k Y} k \epsilon\left(A_{e1}\right)^{*}[y \epsilon, t \epsilon]\left(A^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]-$$

$$\left.e^{-i k Y} k \epsilon A^{*}[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]+e^{-\frac{1}{8 k}-i k Y} k \epsilon A^{*}[y \epsilon, t \epsilon]\left(\left(A_{e1}\right)^{*}\right)^{(0,1)}[y \epsilon, t \epsilon]\right)+$$

$$e^{2 i t \omega} \Big(-\frac{1}{3} i\, e^{2 i k y} k \omega A_{c2}[y\epsilon, t\epsilon]^2 - \frac{2}{3} i k \omega A_{c2}[y\epsilon, t\epsilon] (A_{c1})^*[y\epsilon, t\epsilon] - \frac{1}{3} i\, e^{-2 i k y} k \omega (A_{c1})^*[y\epsilon, t\epsilon]^2 -$$

$$\frac{1}{3} e^{2 i k y} k \epsilon A_{c2}[y\epsilon, t\epsilon] A_{c2}^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{3} k \epsilon (A_{c1})^*[y\epsilon, t\epsilon] A_{c2}^{(0,1)}[y\epsilon, t\epsilon] -$$

$$\frac{1}{3} k \epsilon A_{c2}[y\epsilon, t\epsilon] ((A_{c1})^*)^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{3} e^{-2 i k y} k \epsilon (A_{c1})^*[y\epsilon, t\epsilon] ((A_{c1})^*)^{(0,1)}[y\epsilon, t\epsilon] \Big) -$$

$$\frac{1}{3} k \epsilon A_{c2}[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{3} e^{-2 i k y} k \epsilon (A_{c1})^*[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] +$$

$$e^{-3 i t \omega} \Big(i\, e^{i k y} k \omega A[y\epsilon, t\epsilon] A_{c1}[y\epsilon, t\epsilon] + i\, e^{-i k y} k \omega A[y\epsilon, t\epsilon] (A_{c2})^*[y\epsilon, t\epsilon] -$$

$$\frac{1}{2} e^{-\frac{1}{8k} + i k y} k \epsilon A_{c1}[y\epsilon, t\epsilon] A^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{2} e^{-\frac{1}{8k} - i k y} k \epsilon (A_{c2})^*[y\epsilon, t\epsilon] A^{(0,1)}[y\epsilon, t\epsilon] -$$

$$e^{i k y} k \epsilon A[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] + e^{-\frac{1}{8k} + i k y} k \epsilon A[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] -$$

$$e^{-i k y} k \epsilon A[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] + e^{-\frac{1}{8k} - i k y} k \epsilon A[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] \Big) +$$

$$e^{i t \omega} \Big(i\, e^{i k y} k \omega A_{c1}[y\epsilon, t\epsilon] A^*[y\epsilon, t\epsilon] - 2 i\, e^{-\frac{1}{8k} + i k y} k \omega A_{c1}[y\epsilon, t\epsilon] A^*[y\epsilon, t\epsilon] +$$

$$i\, e^{-i k y} k \omega A^*[y\epsilon, t\epsilon] (A_{c2})^*[y\epsilon, t\epsilon] - 2 i\, e^{-\frac{1}{8k} - i k y} k \omega A^*[y\epsilon, t\epsilon] (A_{c2})^*[y\epsilon, t\epsilon] -$$

$$e^{i k y} k \epsilon A^*[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] + e^{-\frac{1}{8k} + i k y} k \epsilon A^*[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] -$$

$$\frac{1}{2} e^{-\frac{1}{8k} + i k y} k \epsilon A_{c1}[y\epsilon, t\epsilon] (A^*)^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{2} e^{-\frac{1}{8k} - i k y} k \epsilon (A_{c2})^*[y\epsilon, t\epsilon] (A^*)^{(0,1)}[y\epsilon, t\epsilon] -$$

$$e^{-i k y} k \epsilon A^*[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] + e^{-\frac{1}{8k} - i k y} k \epsilon A^*[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] \Big) +$$

$$e^{-2 i t \omega} \Big(\frac{1}{3} i\, e^{2 i k y} k \omega A_{c1}[y\epsilon, t\epsilon]^2 + \frac{2}{3} i k \omega A_{c1}[y\epsilon, t\epsilon] (A_{c2})^*[y\epsilon, t\epsilon] + \frac{1}{3} i\, e^{-2 i k y} k \omega (A_{c2})^*[y\epsilon, t\epsilon]^2 -$$

$$\frac{1}{3} e^{2 i k y} k \epsilon A_{c1}[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{3} k \epsilon (A_{c2})^*[y\epsilon, t\epsilon] A_{c1}^{(0,1)}[y\epsilon, t\epsilon] -$$

$$\frac{1}{3} k \epsilon A_{c1}[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] - \frac{1}{3} e^{-2 i k y} k \epsilon (A_{c2})^*[y\epsilon, t\epsilon] ((A_{c2})^*)^{(0,1)}[y\epsilon, t\epsilon] \Big)$$

NOW WE HAVE TO COLLECT THE SIMILAR TERMS ON THIS SIDE AS GIVEN BELOW

(ADDING FIRST AND THIRD TERM AND SUBSTRACTING SECOND TERM)

TERMS OF $e^{-i t \omega}$

$$0.25\,\beta^2\, i\, e^{i k y - i t \omega}\,\omega\, A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon] - 4\, i\, e^{i k y - i t \omega}\, k\,\omega\, A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon] +$$

$$4\, i\, e^{-\frac{\beta^2}{8k} + i k y - i t \omega}\, k\,\omega\, A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon]$$

TERMS OF $e^{i t \omega}$

$$-0.25\,\beta^2\, i\, e^{i k y + i t \omega}\,\omega\, A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon] + 2\, i\, e^{i k y + i t \omega}\, k\,\omega\, A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon] +$$

$$2\, i\, e^{i k y + i t \omega}\, k\,\omega\, A_{e1}[y\epsilon, t\epsilon]\, A^*[y\epsilon, t\epsilon] - 4\, i\, e^{-\frac{\beta^2}{8k} + i k y + i t \omega}\, k\,\omega\, A_{e1}[y\epsilon, t\epsilon]\, A^*[y\epsilon, t\epsilon]$$

TERMS OF $e^{-2 i t \omega}$

$$-\frac{8}{3}\, i\, e^{-2 i t \omega}\, k\,\omega\, A_{e1}[y\epsilon, t\epsilon]\, (A_{e2})^*[y\epsilon, t\epsilon]$$

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REARRANGING WE WILL GET THREE TERMS RESPECTIVELY AS

$$\left(0.25\,\beta^2\, i\,\omega - 4\, i\, k\,\omega + 4\, i\, e^{-\frac{\beta^2}{8k}}\, k\,\omega\right) A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon]$$

$$\left(-0.25\,\beta^2\, i\,\omega + 2\, i\, k\,\omega + 2\, i\, k\,\omega\, A_{e1}[y\epsilon, t\epsilon]\, A^*[y\epsilon, t\epsilon] - 4\, i\, e^{-\frac{\beta^2}{8k}}\, k\,\omega\right) A[y\epsilon, t\epsilon]\, A_{e2}[y\epsilon, t\epsilon]$$

$$\left(-\frac{8}{3}\, i\, k\,\omega\right) A_{e1}[y\epsilon, t\epsilon]\, (A_{e2})^*[y\epsilon, t\epsilon]$$