# THE STRUCTURE JACOBI OPERATOR ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

- 複素空間型에서 實超曲面의
Jacobi 構造作用素 -

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# THE STRUCTURE JACOBI OPERATOR ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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## - 複素空間型에서 實超曲面의 Jacobi 構造作用素 -

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複素空間型의 實超曲面에는 자연스럽게 概接触計量構造 $(\phi, \xi, \eta, g)$가 유도된다. 이 실초곡면의 제 2 기본텐서를 $A$로 나타낸다. 실초곡면의 분류문제를 연구하는 것은 위의 $(\phi, \xi, \eta, g)$와 실초곡면에 유도되는 構造方程式 사이의 관계를 기하학적으로 잘 해석하는 것이다.

예컨대, 실초곡면이 $A\xi = \alpha\xi$를 만족하면 구조벡터장 $\xi$를 主曲率벡터라 하고, $\alpha$를 $\xi$에 대응되는 主曲率이라고 한다. 이 때 실초곡면을 Hopf초곡면이라고 한다.

복소공간형의 주곡률이 일정한 실초곡면을 분류하는 문제는 微分幾何學에서 한 중요한 문제이다.

1973년 Takagi는 「複素射影空間의 주곡률이 일정한 Hopf실초곡면은 6가지 꼴 $A_1$, $A_2$, B, C, D 및 E로 분류된다」라는 정

리를 처음으로 증명하였다.

이보다 늦게 1989년 Berndt는 「複素雙曲空間의 주곡률이 일정한 Hopf실초곡면은 4가지 꼴 $A_0$, $A_1$, $A_2$ 및 B로 분류된다」라는 정리를 증명하였다.

복소사영공간의 실초곡면이 $A_1$ 또는 $A_2$꼴이고 복소쌍곡공간의 실초곡면이 $A_0$, $A_1$ 또는 $A_2$꼴일 때, 이와 같은 실초곡면을 통틀어 A꼴이라고 한다.

실초곡면의 연구에서 A꼴의 분류문제는 部分多樣体論에서 한 중요한 과제였다. 복소공간형의 실초곡면에서 제 2 기본텐서 $A$와 구조텐서장 $\phi$가 可換인 것은 Okumura(1975년)와 Montiel - Romero(1986년)에 의하여 A꼴로 완전히 분류되었다.

Jacobi作用素는 수학의 여러 분야에서 중요하게 사용되고 있는데, 실초곡면의 분류문제에 이것을 이용하는 것은 기하학적으로 큰 뜻이 있다.

복소공간형의 Jacobi構造作用素 $R_\xi$가 구조텐서장 $\phi$및 제 2 기본텐서 $A$와 각각 可換일 때, 실초곡면은 Cho-Ki(1998년)에 의하여 완전히 분류되었다.

이 논문에서는 복소공간형의 실초곡면에서 Jacobi구조작용소 $R_\xi$가 구조텐서장 $\phi$및 Ricci텐서장 $S$와 각각 可換이라는 전제 아래서 실초곡면을 분류하는 문제를 고찰하여 다음 정리 등을 얻었다.

**定理** 복소공간형의 平均曲率이 일정한 실초곡면이 $R_\xi\phi=\phi R_\xi$ 및 $R_\xi S=SR_\xi$를 만족하면 이 실초곡면은 Hopf초곡면이다. 이 때, 실초곡면은 A꼴로 분류된다.

## 0. Introduction

An $n$-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $c$.

As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let $M$ be a real hypersurfaces of $M_n(c)$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from the Kaehlerian metric and complex structure $J$ of $M_n(c)$. The structure vector $\xi$ is said to be *principal* if $A\xi = \alpha\xi$, where $A$ is the shape operator in the direction of the unit normal $N$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant ([10]) and that $M$ is called a *Hopf hypersurface* ([15]). We denote by $\nabla$, the Levi-Civita connection with respect to the Riemannian metric tensor $g$. Takagi ([18]) classified all homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be $\mathrm{A}_1$, $\mathrm{A}_2$, B, C, D and E, and Cecil-Ryan ([2]) and Kimura ([11]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Namely, he proved the following

**Theorem T([18]).** *Let $M$ be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then $M$ is a tube of radius $r$ over one of the following Kaehlerian submanifolds:*

(A$_1$) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(A$_2$) *a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a complex quadric $Q_{n-1}$, where $0 < r < \frac{\pi}{4}$,*

(C) *$P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,*

(D) *a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*

(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Also Berndt ([1]) showed that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n\mathbb{C}$ are realized as the tubes of constant radius over certain submanifolds when the structure vector $\xi$ is principal. Nowadays in $H_n\mathbb{C}$ they are said to be of type A$_0$, A$_1$, A$_2$, and B. He proved the following

**Theorem B([1]).** *Let $M$ be a real hypersurface of $H_n\mathbb{C}$. Then $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally congruent to one of the followings:*

($A_0$) *a self-tube, that is, a horosphere,*

($A_1$) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

($A_2$) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

Let $M$ be a real hypersurface of type $A_1$ or type $A_2$ in a complex projective space $P_n\mathbb{C}$ or that of type $A_0$, $A_1$, or $A_2$ in a complex hyperbolic space $H_n\mathbb{C}$. Then $M$ is said to be of *type A* for simplicity. By a theorem due to Okumura ([16]) and to Montiel - Romero ([14]) we have

**Theorem O-MR([16],[14]).** *If the shape operator $A$ and the structure tensor $\phi$ commute to each other, then a real hypersurface of a complex space form $M_n(c), c \neq 0$ is locally congruent to be of type* A.

Characterization problems for a real hypersurface of type A in a complex space form were studied by many authors (cf. [5],[6],[7],[9],[12] and [14] etc.).

The curvature tensor field $R$ on $M$ is defined by $R(X,Y) = [\nabla_X, \nabla_Y] -$

$\nabla_{[X,Y]}$, where $X$ and $Y$ are vector fields on $M$. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector filed $X$. Then we see that $R_X$ is a self-adjoint endomorphism of the tangent space. It is related with (the Jacobi vector equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic $\gamma$. It is well-known that the notion of Jacobi vector fields involve many important geometric properties. Some works have recently studied several conditions on the structure Jacobi operator $R_\xi$ and given some results on the classification of real hypersurfaces in a complex space form ([3], [4], [6], [8] and [15] etc.). One of them, Cho and Ki proved the following:

**Theorem CK([3]).** *Let $M$ be a connected real hypersurface of $P_n\mathbb{C}$. If $M$ satisfies $R_\xi\phi = \phi R_\xi$ and at the same time satisfies $R_\xi A = AR_\xi$. Then $M$ is a Hopf hypersurface. Further if $\eta(A\xi) \neq 0$, then $M$ is locally congruent to one of the following spaces:*

- ($A_1$) *a geodesic hypersphere (that is, a tube of radius $r$ over a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$),*
- ($A_2$) *a tube of radius $r$ over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$.*

In this paper we study a real hypersurface of a nonflat complex space form $M_n(c)$ which satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = S R_\xi$, where $S$ denotes the Ricci tensor of the hypersurface. The main purpose of the present paper is to improve Theorem CK.

All manifolds in the present paper are assume to be connected and of class $C^\infty$ and the real hypersurfaces supposed to be orientable.

## 1. Fundamental facts of real hypersurfaces

Let $M$ be a real hypersurface of $M_n(c)$ and $N$ be a unit normal vector field on $M$. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields $X$ and $Y$ on $M$, where $g$ denotes the Riemannian metric of $M$ induced from $\tilde{g}$ and $A$ is the shape operator of $M$ in $M_n(c)$. For any vector field $X$ tangent to $M$, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we may see that the structure $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector fields $X$ and $Y$ on $M$.

From the fact $\tilde{\nabla} J = 0$ and making use of the Gauss and Weingarten formulas, we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX. \tag{1.1}$$

Since the ambient space is of constant holomorphic sectional curvature $c$, we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ & - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{1.3}$$

where $R$ is the Riemann-Christoffel curvature tensor on $M$.

In what follows, to write our formulas in convention forms, we denote by $\alpha = g(A\xi, \xi), \beta = g(A^2\xi, \xi), \gamma = g(A^3\xi, \xi)$ and $h = \mathrm{Tr}A$, and for a function $f$ we denote by $\nabla f$ the gradient vector field of $f$.

If we put $U = \nabla_\xi \xi$, then $U$ is orthogonal to the structure vector $\xi$. We get

$$\phi U = -A\xi + \alpha\xi, \tag{1.4}$$

which shows that $g(U,U)=\beta-\alpha^2$. Thus we easily see that $\xi$ is a principal curvature vector, that is, $A\xi=\alpha\xi$ if and only if $\beta-\alpha^2=0$. From the Gauss structure equation (1.2), the Ricci tensor $S$ of $M$ is given by

$$S=\frac{c}{4}\{(2n+1)I-3\eta\otimes\xi\}+hA-A^2, \tag{1.5}$$

$I$ is an identity map, which implies

$$S\xi=\frac{c}{2}(n-1)\xi+hA\xi-A^2\xi. \tag{1.6}$$

We put

$$A\xi=\alpha\xi+\mu W, \tag{1.7}$$

where $W$ is a unit vector field orthogonal to $\xi$. Then we have $U=\mu\phi W$, and $W$ is also orthogonal to $U$. Further we have $\mu^2=\beta-\alpha^2$.

By the definition of $U$ and the second equation of (1.1) and (1.7), it is verified that

$$g(\nabla_X\xi,U)=\mu g(AW,X). \tag{1.8}$$

Now, differentiating (1.4) covariantly along $M$ and using (1.1), we find

$$\begin{aligned}\eta(X)g(AU+\nabla\alpha,Y)+g(\phi X,\nabla_Y U)\\ =g((\nabla_Y A)X,\xi)-g(A\phi AX,Y)+\alpha g(A\phi X,Y),\end{aligned} \tag{1.9}$$

which shows that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.10}$$

because of (1.3). From (1.9) we also have

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.11}$$

where we have used (1.1) and (1.8). Since $W$ is orthogonal to $U$, we see, using (1.1), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.12}$$

Because of (1.2), the structure Jacobi operator $R_\xi$ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{1.13}$$

for any vector field $X$ on $M$.

## 2. The Jacobi operator of real hypersurfaces

Let $M$ be a real hypersurface in a complex space form $M_n(c), c \neq 0$ satisfying $R_\xi \phi = \phi R_\xi$, which means that the eigenspace of $R_\xi$ is invariant by the structure operator $\phi$. Then by (1.13) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi. \tag{2.1}$$

We set $\Omega = \{p \in M : \mu(p) \neq 0\}$, and suppose that $\Omega$ is nonvoid, that is, $\xi$ is not a principal curvature vector on $M$. In the sequel we discuss our arguments on the open set $\Omega$ of $M$ unless otherwise stated. Then, it is, using (2.1), clear that $\alpha \neq 0$ on $\Omega$. So a function $\lambda$ given by $\beta = \alpha\lambda$ is defined. Thus, replacing $X$ by $U$ in (2.1) and using (1.4), we find

$$\phi AU = \lambda A\xi - A^2\xi. \tag{2.2}$$

In what follows we assume that

$$A^2\xi = \rho A\xi + \sigma\xi \tag{*}$$

for certain scalars $\rho$ and $\sigma$ on $M$. Then we have

$$\sigma = \alpha(\lambda - \rho). \tag{2.3}$$

Combining $(*)$ with (2.2), it is seen that

$$AU = (\rho - \lambda)U. \tag{2.4}$$

From $(*)$ and (1.7) we also have

$$AW = \mu\xi + (\rho - \alpha)W \tag{2.5}$$

and hence

$$A^2W = \rho AW + \alpha(\lambda - \rho)W \tag{2.6}$$

by virtue of $\mu \neq 0$.

Differentiating $(*)$ covariantly along $\Omega$ and taking account of (1.1), we find

(2.7)
$$((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y)$$
$$= g(\nabla\rho, X)g(A\xi, Y) + \rho g((\nabla_X A)\xi, Y) + g(\nabla\sigma, X)\eta(Y)$$
$$+ \alpha(\lambda - \rho)g(\phi AX, Y)$$

for any vector fields $X$ and $Y$ on $M$, which together with (1.3) and (1.10) implies that

$$(\nabla_\xi A)A\xi = \rho AU - \frac{c}{4}U + \frac{1}{2}\nabla\beta.$$

We put $X = \xi$ in (2.7) and use (1.10), (2.4) and the last equation. Then we obtain

(2.8)

$$\frac{1}{2}\nabla\beta = -A\nabla\alpha+\rho\nabla\alpha+g(\nabla\rho,\xi)A\xi+g(\nabla\sigma,\xi)\xi-\{(\rho-\lambda)(\rho+\alpha-3\lambda)-\frac{c}{4}\}U,$$

which connected to (2.4) and $(*)$ gives

(2.9)

$$\frac{1}{2}(A\nabla\beta-\rho\nabla\beta) = -A^2\nabla\alpha+2\rho A\nabla\alpha-\rho^2\nabla\alpha+g(\nabla\sigma,\xi)A\xi$$
$$+g(\sigma\nabla\rho-\rho\nabla\sigma,\xi)\xi+\lambda\{(\rho-\lambda)(\rho+\alpha-3\lambda)-\frac{c}{4}\}U.$$

Because of (2.3), we see, using (2.8), that

$$\frac{1}{2}d\beta(\xi) = \alpha d\alpha(\xi)+\mu d\alpha(W),$$

where $d$ denotes the exterior differential operator, which together with $\beta = \alpha\lambda$ implies that

$$\alpha d\lambda(\xi) = (2\alpha-\lambda)d\alpha(\xi)+2\mu d\alpha(W). \tag{2.10}$$

We verify also, making use of (2.5) and (2.8), that

$$\frac{1}{2}d\beta(W)-\alpha d\alpha(W) = \mu(d\rho(\xi)-d\alpha(\xi)),$$

which enables us to obtain

$$\alpha d\lambda(W) = (2\alpha - \lambda)d\alpha(W) + 2\mu(d\rho(\xi) - d\alpha(\xi)). \tag{2.11}$$

Now, define a 1-form $u$ by $u(X) = g(U, X)$ for any vector field $X$, it is, using (1.3) and (2.7), seen that

(2.12)

$$\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \frac{c}{2}(\rho - \alpha)g(\phi Y, X) - g(A^2\phi AX, Y)$$

$$+ g(A^2\phi AY, X) + 2\rho g(\phi AX, AY) - \alpha(\lambda - \rho)\{g(\phi AY, X) - g(\phi AX, AY)\}$$

$$= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + d\rho(Y)g(A\xi, X) - d\rho(X)g(A\xi, Y)$$

$$+ d(\beta - \rho\alpha)(Y)\eta(X) - d(\beta - \rho\alpha)(X)\eta(Y).$$

On the other hand, differentiating (2.5) covariantly along $\Omega$, we find

$$(\nabla_X A)W + A\nabla_X W = d\mu(X)\xi + \mu\nabla_X\xi + d(\rho - \alpha)(X)W + (\rho - \alpha)\nabla_X W.$$

By taking the inner product this with $W$, we get

$$g((\nabla_X A)W, W) = -2(\rho - \lambda)u(X) + d\rho(X) - d\alpha(X) \tag{2.13}$$

with the aid of (2.4) and the fact that $W$ is a unit orthogonal to $\xi$. We also have by applying $\xi$

$$\mu g((\nabla_X A)W, \xi) = (\rho - \lambda)(\rho - 2\alpha)u(X) + \frac{1}{2}d\beta(X) - \alpha d\alpha(X), \tag{2.14}$$

where we have used (1.12) and (2.4), which together with the Codazzi equation (1.3) gives

$$\mu(\nabla_W A)\xi = \{(\rho-\lambda)(\rho-2\alpha)-\frac{c}{2}\}U+\frac{1}{2}\nabla\beta-\alpha\nabla\alpha, \tag{2.15}$$

$$\mu(\nabla_\xi A)W = \{(\rho-\lambda)(\rho-2\alpha)-\frac{c}{4}\}U+\frac{1}{2}\nabla\beta-\alpha\nabla\alpha. \tag{2.16}$$

Replacing $X$ by $\mu W$ in (2.12) and making use of (1.7), (1.10), (2.4), (2.5), (2.6), (2.14) and (2.15), we find

$$\begin{aligned}&\alpha A\nabla\alpha-\frac{1}{2}A\nabla\beta+\frac{1}{2}(\rho-\alpha)\nabla\beta+\alpha(\lambda-\rho)\nabla\alpha-\alpha(\lambda-\alpha)\nabla\rho\\ &=\{(\rho-\lambda)(\alpha\lambda-2\rho\lambda+2\rho\alpha+\alpha^2)+\frac{c}{2}(\lambda-\alpha)\}U\\ &\quad-\mu d\rho(W)A\xi-\mu d(\beta-\rho\alpha)(W)\xi.\end{aligned} \tag{2.17}$$

If we replace $X$ by $A\xi$ in (2.7) and take account of (1.7), (1.10), (2.4), (2.13)∼ (2.16) and (∗), then we obtain

$$\begin{aligned}&\frac{1}{2}(A\nabla\beta-\rho\nabla\beta)+\alpha^2\nabla\lambda+\mu^2\nabla\rho\\ &=g(A\xi,\nabla\rho)A\xi+g(A\xi,\nabla\sigma)\xi\\ &+\{(\rho-\lambda)(2\rho\lambda-3\alpha\rho+2\alpha\lambda)+\frac{c}{4}(3\alpha-2\lambda)\}U.\end{aligned}$$

Substituting (2.9) into this, we find

$$
(2.18)\quad \begin{aligned}
&d\alpha^2\nabla\lambda + \mu^2\nabla\rho - A^2\nabla\alpha + 2\rho A\nabla\alpha - \rho^2\nabla\alpha \\
&= \{g(A\xi,\nabla\rho) - d\sigma(\xi)\}A\xi \\
&+ \{g(A\xi,\nabla\sigma) + \rho d\sigma(\xi) - (\beta-\rho\alpha)d\rho(\xi)\}\xi \\
&\quad + \{(\rho-\lambda)(\rho\lambda - 3\alpha\rho + \alpha\lambda + 3\lambda^2) + \frac{c}{4}(3\alpha-\lambda)\}U.
\end{aligned}
$$

Now, it is, using (2.1), verified that

$$
(2.19)\qquad \alpha\phi A\phi AX + \alpha A^2X = \rho g(A\xi,X)A\xi + \eta(X)A\xi - g(AU,X)U
$$

because of properties of almost contact metric structure.

On the other hand, we have from (1.9)

$$
\nabla_X U + g(A^2\xi,X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX,
$$

which together with (*) and (2.19) yields

$$
(2.20)\quad \begin{aligned}
\nabla_X U &+ \{\rho g(A\xi,X) + \alpha(\lambda-\rho)\eta(X)\}\xi = \phi(\nabla_X A)\xi + \alpha AX - A^2X \\
&+ \frac{1}{\alpha}\{\rho g(A\xi,X) + \alpha(\lambda-\rho)\eta(X)\}A\xi - \frac{\rho-\lambda}{\alpha}g(AU,X)U.
\end{aligned}
$$

If we put $X=U$ in (2.20) and take account of (2.4), then we get

$$
(2.21)\qquad \nabla_U U = \phi(\nabla_U A)\xi + (\rho-\lambda)(2\alpha-\rho)U.
$$

Using (1.7) and (2.4), we can write the equation (1.11) as

$$\nabla_\xi U = \mu(3\lambda - 3\rho + \alpha)W - \alpha(\lambda - \alpha)\xi + \phi\nabla\alpha. \tag{2.22}$$

Since the exterior derivative $du$ of a 1-form $u$ is given by

$$du(Y, X) = \frac{1}{2}\{Yu(X) - Xu(Y) - u([Y, X])\},$$

we verify, using (1.8), (2.22) and (*), that

$$du(\xi, X) = (3\lambda - 2\rho)\mu w(X) + g(\phi\nabla\alpha, X), \tag{2.23}$$

where a 1-form $w$ is defined by $w(X) = g(W, X)$, which shows that

$$du(\xi, U) = \mu d\alpha(W). \tag{2.24}$$

Now, differentiating (2.4) covariantly, we find

$$(\nabla_X A)U + A(\nabla_X U) = d(\rho - \lambda)(X)U + (\rho - \lambda)\nabla_X U. \tag{2.25}$$

If we take the inner product this with $\xi$ and make use of (1.3) and (2.22), then we obtain

$$\begin{aligned}(\nabla_U A)\xi =& \frac{c}{4}\mu W + \varepsilon U - \mu(3\lambda - 3\rho + \alpha)\{AW - (\rho - \lambda)W\}\\ &+ \alpha(\lambda - \alpha)\{A\xi - (\rho - \lambda)\xi\} - A\phi\nabla\alpha + (\rho - \lambda)\phi\nabla\alpha,\end{aligned} \tag{2.26}$$

where we have put $\varepsilon = d(\rho - \lambda)(\xi)$. Thus, it follows, using (2.1), that

$$\begin{aligned}\phi(\nabla_U A)\xi =& \{3(\lambda - \rho)(\lambda - \alpha) - \frac{c}{4} - \frac{1}{\alpha} d\alpha(U)\}U + \mu\varepsilon W \\ &+ (\rho - \lambda)(\nabla\alpha - d\alpha(\xi)\xi) - A\nabla\alpha + \frac{1}{\alpha} g(A\xi, \nabla\alpha)A\xi.\end{aligned} \tag{2.27}$$

Substituting this into (2.21), we get

(2.28)
$$\begin{aligned}\nabla_U U =& \{(\rho - \lambda)(3\lambda - \alpha - \rho) + \frac{c}{4} + \frac{1}{\alpha} d\alpha(U)\}U + A\nabla\alpha - (\rho - \lambda)\nabla\alpha \\ &+ \{(\rho - \lambda)d\alpha(\xi) - g(A\xi, \nabla\alpha)\}\xi - \mu\{\varepsilon + \frac{1}{\alpha} g(A\xi, \nabla\alpha)\}W,\end{aligned}$$

which tells us that

$$\begin{aligned}A(\nabla_U U) - (\rho - \lambda)\nabla_U U =& A^2\nabla\alpha - 2(\rho - \lambda)A\nabla\alpha + (\rho - \lambda)^2\nabla\alpha \\ &+ \{(\rho - \lambda)d\alpha(\xi) - g(A\xi, \nabla\alpha)\}\{A\xi - (\rho - \lambda)\xi\} \\ &- \mu(\varepsilon + \frac{1}{\alpha} g(A\xi, \nabla\alpha))\{AW - (\rho - \lambda)W\}.\end{aligned} \tag{2.29}$$

Because of (1.3) and (1.4), the relationship (2.25) implies that

$$\begin{aligned}&\frac{c}{4}\mu\{(\eta(Y)w(X) - \eta(X)w(Y)\} + g(AX, \nabla_Y U) - g(AY, \nabla_X U) \\ &= d(\rho - \lambda)(Y)u(X) - d(\rho - \lambda)(X)u(Y) \\ &\quad + (\rho - \lambda)\{(\nabla_Y u)(X) - (\nabla_X u)(Y)\}.\end{aligned} \tag{2.30}$$

If we replace $X$ by $U$ in (2.30) and make use of (2.4), then we obtain

$$A(\nabla_U U) - (\rho - \lambda)\nabla_U U = \mu^2(\nabla\lambda - \nabla\rho) + d(\rho - \lambda)(U)U,$$

which together with (2.29) gives

$$(2.31)\quad \begin{aligned} &A^2\nabla\alpha - 2\rho A\nabla\alpha + \rho^2\nabla\alpha + 2\lambda(A\nabla\alpha - \rho\nabla\alpha) + \lambda^2\nabla\alpha \\ &= \{g(A\xi, \nabla\alpha) - (\rho-\lambda)d\alpha(\xi)\}\{A\xi - (\rho-\lambda)\xi\} \\ &\quad + \mu\{\varepsilon + \frac{1}{\alpha}g(A\xi, \nabla\alpha)\}\{AW - (\rho-\lambda)W\} \\ &\quad + \mu^2(\nabla\lambda - \nabla\rho) + d(\rho-\lambda)(U)U. \end{aligned}$$

Substituting (2.18) into (2.31) and using (2.9), we find

$$(2.32)\quad \begin{aligned} &2\mu^2(\nabla\rho - \nabla\lambda) + d(\lambda-\rho)(U)U - 3(\lambda-\alpha)\{(\rho-\lambda)^2 - \frac{c}{4}\}U \\ &= \{g(A\xi, \nabla\alpha) - d\sigma(\xi) - 2\lambda d\rho(\xi)\}A\xi \\ &\quad + \{g(A\xi, \nabla\sigma) + (\rho - 2\lambda)d\sigma(\xi) - \sigma d\rho(\xi)\}\xi \\ &\quad + \{g(A\xi, \nabla\alpha) - (\rho-\lambda)d\alpha(\xi)\}\{A\xi - (\rho-\lambda)\xi\} \\ &\quad + \mu\{\varepsilon + \frac{1}{\alpha}g(A\xi, \nabla\alpha)\}\{AW - (\rho-\lambda)W\}. \end{aligned}$$

Since $A\xi$ and $AW$ are orthogonal to $U$, it follows that

$$(2.33)\quad d(\rho-\lambda)(U) = 3(\lambda-\alpha)\{(\rho-\lambda)^2 - \frac{c}{4}\}.$$

Using this, (1.7) and (2.5), the equation (2.32) can be written as

$$\mu^2(\nabla\rho - \nabla\lambda) = \mu^2(a\xi + bW) + 3(\lambda-\alpha)\{(\rho-\lambda)^2 - \frac{c}{4}\}U$$

for some functions $a$ and $b$, which shows that $a = \varepsilon$ and $b = d(\rho - \lambda)(W)$. Since $\lambda - \alpha$ does not vanish on $\Omega$, it follows that

$$(2.34) \qquad \alpha(\nabla\rho - \nabla\lambda) = \alpha(\varepsilon\xi + bW) + \{3(\rho - \lambda)^2 - \frac{3}{4}c\}U.$$

On the other hand, if we take the inner product (2.32) with $W$, and straitforward calculation, then we obtain

$$\alpha^2 d\rho(W) = 3\alpha\mu d\rho(\xi) + \alpha(4\alpha - 3\lambda)d\alpha(W) - \mu(4\alpha - \lambda)d\alpha(\xi),$$

where we have used (2.3), (2.10) and (2.11). Comparing this with (2.10) and (2.11), it is seen that

$$\alpha d(\rho - \lambda)(W) = \mu d(\rho - \lambda)(\xi),$$

that is, $b\alpha = \mu\varepsilon$. From this and (1.7), the equation (2.34) becomes

$$(2.35) \qquad \alpha(\nabla\rho - \nabla\lambda) = \varepsilon A\xi + 3\{(\rho - \lambda)^2 - \frac{c}{4}\}U.$$

*Remark 1.* We notice here, using (2.33), that $\rho - \lambda \neq 0$ on $\Omega$.

## 3. Lemmas

We will continue now, our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and $(*)$ as in section 2.

First of all, we prove

**Lemma 3.1.** *Let $M$ be a real hypersurface satisfying $R_\xi\phi = \phi R_\xi$ and $(*)$. Then we have*

$$\alpha(\nabla\rho - \nabla\lambda) = \theta U \tag{3.1}$$

*on $\Omega$, where $\theta$ is given by*

$$\theta = 3(\rho-\lambda)^2 - \frac{3}{4}c. \tag{3.2}$$

*Proof.* By differentiating (2.35) covariantly and taking the skew-symmetric parts obtained one, we find

$$\begin{aligned}
&d\alpha(Y)d(\rho-\lambda)(X) - d\alpha(X)d(\rho-\lambda)(Y)\\
&-6(\rho-\lambda)\{d(\rho-\lambda)(Y)u(X) - d(\rho-\lambda)(X)u(Y)\}\\
&= d\varepsilon(Y)g(A\xi, X) - d\varepsilon(X)g(A\xi, Y) - \frac{c}{2}\varepsilon g(\phi Y, X)\\
&-2\varepsilon g(A\phi AY, X) + \theta du(Y, X)
\end{aligned}$$

by virtue of (1.3), or using (2.35) again,

$$(3.3)\quad \begin{aligned}&\theta\{u(Y)d\alpha(X)-u(X)d\alpha(Y)\}-\frac{c}{2}\varepsilon\alpha g(\phi Y,X)\\&\qquad -2\varepsilon\alpha g(A\phi AY,X)+\theta\alpha du(Y,X)\\&\quad =\{\varepsilon d\alpha(Y)-\alpha d\varepsilon(Y)+6(\rho-\lambda)\varepsilon u(Y)\}g(A\xi,X)\\&\qquad -\{\varepsilon d\alpha(X)-\alpha d\varepsilon(X)+6(\rho-\lambda)\varepsilon u(X)\}g(A\xi,Y).\end{aligned}$$

Putting $Y=\xi$ in (3.3), we get

$$\begin{aligned}&\varepsilon\{d\alpha(X)+6(\rho-\lambda)u(X)\}-\alpha d\varepsilon(X)\\&\quad=\{\frac{\varepsilon}{\alpha}d\alpha(\xi)-d\varepsilon(\xi)\}g(AX,\xi)+\theta du(\xi,X)+\{\frac{\theta}{\alpha}d\alpha(\xi)-2\varepsilon(\rho-\lambda)\}u(X).\end{aligned}$$

Combining this with (3.3), we have

$$\begin{aligned}&\theta\{u(Y)d\alpha(X)-u(X)d\alpha(Y)\}\\&-\frac{c}{2}\varepsilon\alpha g(\phi Y,X)-2\varepsilon\alpha g(A\phi AY,X)+\theta\alpha du(Y,X)\\&=\{\theta du(\xi,Y)-2\varepsilon(\rho-\lambda)u(Y)+\frac{\theta}{\alpha}d\alpha(\xi)u(Y)\}g(A\xi,X)\\&\qquad-\{\theta du(\xi,X)-2\varepsilon(\rho-\lambda)u(X)+\frac{\theta}{\alpha}d\alpha(\xi)u(X)\}g(A\xi,Y).\end{aligned}$$

If we put $Y=U$ in this and take account of (1.4), (2.4) and (2.24), then we get

$$(3.4)\quad \begin{aligned}&\{-\mu\theta d\alpha(W)+\theta(\lambda-\alpha)d\alpha(\xi)-2\varepsilon(\rho-\lambda)\mu^2\}A\xi\\&\quad=\theta\{\mu^2\nabla\alpha-d\alpha(U)U\}+\frac{c}{2}\mu\alpha\varepsilon W-2\alpha\varepsilon(\rho-\lambda)\mu AW+\theta\alpha TU,\end{aligned}$$

where $g(TU, X) = du(U, X)$.

On the other hand, it is, using (2.20), verified that

$$TU =\{(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{4}+\frac{1}{\alpha}d\alpha(U)\}U + A\nabla\alpha + (\lambda+\alpha-\rho)\nabla\alpha$$
$$-\frac{1}{2}\nabla\beta - \mu\varepsilon W + (\rho-\lambda)d\alpha(\xi)\xi - \{d\alpha(\xi)+\frac{\mu}{\alpha}d\alpha(W)\}A\xi,$$

where we have used (2.14) and (2.27), or using (2.3), (2.8) and (2.10),

$$TU = \{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{2}+\frac{1}{\alpha}d\alpha(U)\}U + \alpha(\nabla\alpha - \nabla\lambda)$$
$$-\{d\alpha(\xi)+\frac{\mu}{\alpha}d\alpha(W) - d\lambda(\xi)\}A\xi.$$

Substituting this into (3.4), we find

$$\{\theta(\lambda d\alpha(\xi) - \alpha d\lambda(\xi)) - 2\varepsilon(\rho-\lambda)\mu^2\}A\xi \tag{3.5}$$
$$= \theta\alpha(\lambda\nabla\alpha - \alpha\nabla\lambda) + \frac{c}{2}\mu\alpha\varepsilon W - 2\varepsilon\alpha(\rho-\lambda)\mu AW$$
$$+\theta\alpha\{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\frac{c}{2}\}U.$$

If we take the inner product (3.5) with $W$ and use (2.5), we find

(3.6)

$$\theta\{\mu\lambda d\alpha(\xi) - \mu\alpha d\lambda(\xi) - \alpha\lambda d\alpha(W) + \alpha^2 d\lambda(W)\} + 2\{(\rho-\lambda)^2 - \frac{c}{4}\}\epsilon\mu\alpha = 0.$$

Since $\epsilon = d(\rho-\lambda)(\xi)$, we, using (2.11) and this, verify that

$$3\theta\{(\lambda-2\alpha)d\alpha(\xi) - 2\mu d\alpha(W)\} + \theta\alpha\{8d\rho(\xi) - 5d\lambda(\xi)\} = 0,$$

which together with (2.10) implies that $\theta d(\rho-\lambda)(\xi)=0$, that is, $\theta\varepsilon=0$ and hence $\varepsilon=0$ because of (3.2). Thus (3.1) is established by virtue of (2.35). □

**Lemma 3.2.** *Under the same hypotheses as those in Lemma 3.1, if $\theta\neq 0$ and $d(h-\rho)(\xi)=0$, then we have*

$$\nabla\alpha=(3\lambda-2\rho)U, \tag{3.7}$$

$$\alpha\nabla\lambda=\{2(\rho-\lambda)(3\lambda-\alpha-\rho)+\lambda(3\lambda-2\rho)+\frac{c}{2}\}U, \tag{3.8}$$

$$\alpha\nabla\rho=(\rho^2-2\rho\alpha+2\alpha\lambda-\frac{c}{4})U. \tag{3.9}$$

*Proof.* Using (3.1) and (3.2), it is clear that $\alpha\nabla\theta=6(\rho-\lambda)\theta U$. Thus, differentiating (3.1) covariantly and taking the skew-symmetric part obtained one, we find

$$d\alpha(Y)d(\rho-\lambda)(X)-d\alpha(X)d(\rho-\lambda)(Y)=\theta du(Y,X),$$

which together with (3.1) and $\theta\neq 0$ gives

$$d\alpha(Y)u(X)-d\alpha(X)u(Y)=\alpha du(Y,X). \tag{3.10}$$

Since $\epsilon = 0$ and $\theta \neq 0$, (3.6) can be written as

$$\lambda d\alpha(\xi) - \alpha d\lambda(\xi) = 2\mu d\alpha(W) - 2\alpha(d\rho(\xi) - d\alpha(\xi)), \tag{3.11}$$

where we have used (2.11).

On the other hand, if we take the trace of (1.9) and make use of (1.3), (1.4) and (3.10), then we obtain

$$\alpha d(h - \alpha)(\xi) = \mu d\alpha(W),$$

which together with (3.11) implies that

$$\lambda d\alpha(\xi) - \alpha d\lambda(\xi) = 2\alpha d(h - \rho)(\xi).$$

Thus, (3.5) turns out to be

$$\lambda \nabla \alpha - \alpha \nabla \lambda = 2\{(\rho - \lambda)^2 + (\rho - \lambda)(\alpha - 2\lambda) - \frac{c}{4}\}U \tag{3.12}$$

since we have $\epsilon = 0$ and $d(h - \rho)(\xi) = 0$ was assumed.

Using the same method as that used to derive (3.10) from (3.1), we can derive from (3.12) the following:

$$\begin{aligned} & d\lambda(Y)d\alpha(X) - d\lambda(X)d\alpha(Y) \\ & = (\rho - \lambda)\{d(\alpha - 2\lambda)(Y)u(X) - d(\alpha - 2\lambda)(X)u(Y)\} \\ & + \{(\rho - \lambda)^2 + (\rho - \lambda)(\alpha - 2\lambda) - \frac{c}{4}\}du(Y, X), \end{aligned}$$

where we have used (3.1). From (3.10), (3.12) and the last equation, we verify that $\theta d\alpha(W) = 0$ and hence $d\alpha(\xi) = 0$ by virtue of $\theta \neq 0$. Thus putting $Y = \xi$ in (3.10), we have $du(\xi, X) = 0$ for any vector field $X$. Therefore (2.23) turns out to be $\phi\nabla\alpha = \mu(2\rho - 3\lambda)W$, which shows that $\nabla\alpha = (3\lambda - 2\rho)U$. Thus (3.10) is reduced to $du = 0$. So (2.28) implies that

$$\frac{1}{2}\nabla g(U,U) = \{(\rho - \lambda)(3\lambda - \alpha - \rho) + (\lambda - \alpha)(3\lambda - 2\rho) + \frac{c}{4}\}U$$

withe the aid of (3.7). From this and (3.7), it follows, using $g(U,U) = \alpha(\lambda - \alpha)$, that (3.8) is accomplished. Because of (3.1) and (3.8), we see that (3.9) is established. Hence, required formulas are obtained. □

*Remark 2.* In the proof of above lemma, we verify that Lemma 3.2 is valid if we replace the assumption $\theta \neq 0$ and $d(h - \rho)(\xi) = 0$ by $du(\xi, X) = 0$ for any vector field $X$.

**Lemma 3.3.** *Let $M$ be a real hypersurface satisfying $R_\xi\phi = \phi R_\xi$ and $(*)$ in $M_n(c)$. If $\nabla\sigma = 0$, then $\Omega$ is void.*

*Remark 3.* This lemma was proved in [8]. But, we give a simple proof of it here.

*Proof.* Since $\nabla\sigma = 0$ is assumed, we see, making use of (2.3) and (3.1), that

$$(\rho - \lambda)\nabla\alpha + \theta U = 0, \tag{3.13}$$

which implies $d\alpha(\xi) = 0$ and $d\alpha(W) = 0$ by virtue of Remark 1. By differentiating (3.13) and using (3.1), we obtain (3.10) and hence $\theta du(\xi, X) = 0$.

Now, if we suppose that $du(\xi, X) \neq 0$. Then we have $\theta = 0$ and hence $(\rho - \lambda)^2 = \frac{c}{4}$. So (3.13) and Remark 1 tells us that $\nabla\alpha = 0$. Since we have $\nabla\rho = \nabla\lambda$, it is seen that $\nabla\beta = \alpha\nabla\rho$. From these and (2.10) we have $d\rho(\xi) = 0$. Thus, (2.8) turns out to be

$$\alpha\nabla\rho = 2(\rho - \lambda)(2\lambda - \alpha)U \tag{3.14}$$

by virtue of $(\rho - \lambda)^2 = \frac{c}{4}$. Using above arguments, it is, making use of (3.14), verified that $(\rho - \lambda)(2\lambda - \alpha)du(\xi, X) = 0$ and consequently $2\lambda - \alpha = 0$, that is, $2\mu^2 + \alpha^2 = 0$, which produces a contradiction. Thus, we have $du(\xi, X) = 0$ on $\Omega$. By Remark 2, we verify that Lemma 3.2 is valid. Combining (3.13) with (3.2) and (3.7), we have

$$\rho(\rho - \lambda) = \frac{3}{4}c. \tag{3.15}$$

Thus, (3.2) is reduced to

$$\theta = (\rho - \lambda)(2\rho - 3\lambda). \tag{3.16}$$

Differentiation (3.15) gives $(\rho - \lambda)\nabla\rho = -\rho\theta U$ because of (3.1), which together with (3.16) yields

$$\nabla\rho = \rho(3\lambda - 2\rho)U$$

and hence $\nabla\rho = (\rho^2 - \frac{9}{4}c)U$ with the aid of (3.15), or using (3.7)

$$\nabla\rho = \rho\nabla\alpha. \tag{3.17}$$

If we substitute $\nabla\rho = (\rho^2 - \frac{9}{4}c)U$ into (3.9), then we obtain

$$\alpha(\rho^3 - \frac{9}{4}c\rho + \frac{3}{2}c) = \rho^3 - \frac{c}{4}\rho. \tag{3.18}$$

Differentiating this and taking account of (3.17), we find

$$\{9\alpha(\rho^2 - \frac{c}{4}) - 3\rho^2 - \frac{c}{4}\}\nabla\alpha = 0,$$

which connected to (3.18) gives $\nabla\alpha = 0$. Therefore we see, using (3.7) and (3.16), that $\theta = 0$, which together with (3.2) and (3.15) implies that $\lambda = 0$, a contradiction. Hence, $\Omega$ is void. □

*Remark 4.* We notice here, using (1.6) and (1.13), that the condition (*) with $\sigma = \frac{c}{4}$, that is, $A^2\xi = \rho A\xi + \frac{c}{4}\xi$ if and only if $R_\xi A = AR_\xi$ on $\Omega$. In fact, from (1.13) we have

(3.19)
$$\begin{aligned} g(R_\xi Y, AX) - g(R_\xi X, AY) =& g(A^2\xi, Y)g(A\xi, X) - g(A^2\xi, X)g(A\xi, Y) \\ &+ \frac{c}{4}\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}. \end{aligned}$$

The if part is immediately true from above equation. So we are going to check the only if part. Since $R_\xi A - AR_\xi = 0$, by putting $X = \xi$ in (3.19), we find

$$\frac{c}{4}A\xi = \beta A\xi - \alpha A^2\xi + \frac{c}{4}\alpha\xi. \tag{3.20}$$

Substituting this into (3.19), we obtain

$$\mu\{w(Y)g(A^2\xi, X) - w(X)g(A^2\xi, Y)\} = \beta\{\eta(X)g(A\xi, Y) - \eta(Y)g(A\xi, X)\},$$

where we have used (1.7). Replacing $X$ by $A\xi$, we have

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi$$

by virtue of $\mu^2 = \beta - \alpha^2$. If we put $\mu^2\rho = \gamma - \beta\alpha$, then a function $\rho$ is defined on $\Omega$. Hence, it follows that

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi.$$

From this and (3.20) we see that $\beta = \rho\alpha + \frac{c}{4}$ because of $\mu \neq 0$.

## 4. Real hypersurfaces satisfying $R_\xi \phi = \phi R_\xi$ and $R_\xi S = SR_\xi$

Let $M$ be a real hypersurface of a complex space form $M_n(c), c \neq 0$. Suppose that the Ricci tensor $S$ of type (1,1) and the Jacobi operator $R_\xi$ with respect to the structure vector $\xi$ commute to each other, that is, $R_\xi S = SR_\xi$. Then we have

$$\begin{aligned} &g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ &= g(A^2\xi, Y)g(hA\xi - \frac{c}{4}\xi, X) - g(A^2\xi, X)g(hA\xi - \frac{c}{4}\xi, Y) \\ &\quad + \frac{c}{4}h\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}, \end{aligned}$$

where we have used (1.6) and (1.13), which shows that

$$\alpha A^3\xi = (\alpha h - \frac{c}{4})A^2\xi + (\gamma - \beta h + \frac{c}{4})A\xi + \frac{c}{4}(\beta - h\alpha)\xi. \tag{4.1}$$

Combining above two equations, we, using (1.7), that

$$\mu\{g(A^2\xi, Y)w(X) - g(A^2\xi, X)w(Y)\} = \beta\{\eta(Y)g(A\xi, X) - \eta(X)g(A\xi, Y)\}.$$

Putting $Y = A\xi$ in this, we find

$$\mu^2 g(A^2\xi, X) = \mu\gamma w(X) - \beta\alpha g(A\xi, X) + \beta^2\eta(X).$$

Thus, it follows that

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi$$

and consequently

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{4.2}$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$ and $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$ on $\Omega$. Thus the condition $(*)$ stated in section 2 is established.

From (4.2) we have

$$A^3\xi = (\rho^2 - \beta - \rho\alpha)A\xi + \rho(\beta - \rho\alpha)\xi.$$

Comparing this with (4.1), we find

$$\mu(h - \rho)(\beta - \rho\alpha - \frac{c}{4}) = 0. \tag{4.3}$$

Let $\Omega_0$ be a set of points such that $\mu(p)(h(p) - \rho(p)) \neq 0$ at $p \in M$. Then we have $\beta - \rho\alpha = \frac{c}{4}$ on $\Omega_0$. Thus, by Lemma 3.3 we see that $\xi$ is a principal curvature vector. Hence we have $h = \rho$ on $\Omega$. From this fact and (4.2), the equation (1.6) turns out to be

$$S\xi = g(S\xi, \xi)\xi, \tag{4.4}$$

where we have put $g(S\xi,\xi) = \frac{c}{2}(n-1) - (\beta - h\alpha)$.

If $g(S\xi,\xi)$ =const., then we conclude that $\xi$ is a principal curvature vector by virtue of Lemma 3.3. Hence (2.1) implies that $A\phi = \phi A$ if $\alpha \neq 0$.

Thus, by Theorem O-MR, we have

**Theorem 4.1.** *Let $M$ be a real hypersurface in a nonflat complex space form $M_n(c)$. If it satisfies $R_\xi\phi = \phi R_\xi$, $R_\xi S = SR_\xi$ and $g(S\xi,\xi) =$ const., then $M$ is locally congruent to be of type* A *provided that $\eta(A\xi) \neq 0$.*

*Remark 5.* It is proved in [8] that a real hypersurface satisfying $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = \tau\xi$ for some constant $\tau$ in a complex space form $M_n(c)$ is a Hopf hypersurface in $M_n(c)$.

According to Theorem 4.1 and Remark 4, we have

**Corollary 4.2.** *Let $M$ be a real hypersurface in a nonflat complex space form. If it satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi A = AR_\xi$, then $M$ is locally congruent to be of type* A *provided that $\eta(A\xi) \neq 0$.*

Now, we prove

**Theorem 4.3.** *Let $M$ be a real hypersurface with constant mean curva-*

*ture in a nonflat complex space form. If it satisfies $R_\xi \phi = \phi R_\xi$ and at the same time satisfies $R_\xi S = S R_\xi$, then $M$ is locally congruent to be of type* A *provided that $\eta(A\xi) \neq 0$.*

*Proof.* By Lemma 3.3 and (4.3), we may only discuss the case where $h = \rho$ on $\Omega$. The mean curvature of $M$ being constant, (3.1) becomes

$$\alpha \nabla \lambda = -\theta U. \tag{4.5}$$

This shows that $d\lambda(\xi) = 0$ and $d\lambda(W) = 0$. Thus, we verify, using (2.10) and (2.11), that $\{(2\lambda - \alpha)^2 + 4\mu^2\} d\alpha(\xi) = 0$ since we have $\nabla \rho = 0$. So we have $d\alpha(\xi) = 0$. Hence, the same method as that used in Lemma 3.2, it is, making use of (4.5), seen that $\theta du(\xi, X) = 0$ for any vector field $X$.

If we assume $du(\xi, X) \neq 0$ on $\Omega$, then we have $\theta = 0$ and hence $\nabla \lambda = 0$. So, by definition we have $\nabla \beta = \lambda \nabla \alpha$, which together with (2.3) and $d\alpha(\xi) = 0$ gives $d\sigma(\xi) = 0$. We also have $d\alpha(W) = 0$ because of (2.11). From these facts (2.8) and (2.17) are reduced respectively to

$$A \nabla \alpha = (\rho - \frac{1}{2}\lambda) \nabla \alpha + (\rho - \lambda)(2\lambda - \alpha) U, \tag{4.6}$$

$$(\alpha - \frac{1}{2}\lambda) A \nabla \alpha + (\frac{1}{2}\lambda\rho + \frac{1}{2}\alpha\lambda - \alpha\rho) \nabla \alpha = (\rho - \lambda)(3\alpha\lambda - 2\rho\alpha - 2\lambda^2) U,$$

where we have used $(\rho - \lambda)^2 = \frac{c}{4}$. Combing the last two equations, it follows that

$$\lambda^2 \nabla\alpha = 2(\rho - \lambda)(\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha)U.$$

In the same way as above, we have from this

$$(\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha)du(\xi, X) = 0,$$

which shows that $\alpha\lambda - 2\lambda^2 + 2\alpha^2 - 4\rho\alpha = 0$. From this we see that $\nabla\alpha = 0$ because of (3.2) and $\nabla\lambda = 0$. Therefore (4.6) implies that $2\lambda - \alpha = 0$, a contradiction. Thus, it follows that $du(\xi, X) = 0$ on $\Omega$. So we have $\nabla\alpha = (3\lambda - 2\rho)$ by virtue of (2.23). By Remark 2, we see that (3.8) and (3.9) are valid. Since the mean curvature of $M$ being constant, that is, $\nabla\rho = 0$, (3.9) means that $\rho^2 - 2\alpha\rho + 2\alpha\lambda - \frac{c}{4} = 0$, which implies that $(\rho - \lambda)\nabla\alpha = \alpha\nabla\lambda$. From this, (3.2), (3.7) and (4.5) we have $\rho(\rho - \lambda) = \frac{3}{4}c$, which enables us to obtain $\nabla\lambda = 0$ and hence $\theta = 0$ by virtue of (4.5). Thus, (3.8) gives $2(\rho - \lambda)(2\lambda - \alpha) + \lambda(3\lambda - 2\rho) = 0$, which tells us that $\nabla\alpha = 0$ because $\rho$ and $\lambda$ are both constant. By (3.7) we have $2\lambda - \alpha = 0$, a contradiction. Therefore we conclude that $\Omega$ is void. So by (2.2) we have $A\phi = \phi A$ if $\alpha \neq 0$. Owing to Theorem O-MR, we arrive at the conclusion.

□

Finally, we shall discuss real hypersurfaces satisfying $R_\xi\phi = \phi R_\xi$ and $R_\xi S = SR_\xi$ in a complex hyperbolic space $H_n\mathbb{C}$. By (3.2) we see that $\theta \neq 0$ because of $c < 0$. Futher, we may only consider the case where $\rho = h$. Therefore (3.12) is established. Thus, Lemma 3.2 is accomplished. Furthermore, it is, using (3.10), that $du = 0$ by virtue of (3.7). Thus, (2.30) is reduced to

$$\frac{c}{4}\mu(\eta(Y)w(X) - \eta(X)w(Y)) + g(AX, \nabla_Y U) - g(AY, \nabla_X U) = 0$$

withe the aid of (3.1). Putting $Y = \xi$ in this, we find

$$\frac{c}{4}\mu w(X) + g(AX, \nabla_\xi U) - g(A\xi, \nabla_X U) = 0,$$

or, using (2.22) and (3.7),

$$FU = \mu\{(\alpha - \rho)AW + \frac{c}{4}W - \mu A\xi\}, \tag{4.7}$$

where we have put $g(FU, X) = g(A\xi, \nabla_X U)$.

Because of (3.1) and (3.7), the equation (2.26) turns out to be

$$(\nabla_U A)\xi = \lambda\mu^2\xi + \mu\{\rho(\lambda - \alpha) + \frac{c}{4}\}W. \tag{4.8}$$

On the other hand, the divergence of the vector field $U$ is given by

$$divU = \alpha h + \lambda(\lambda - \alpha) + \frac{c}{2}(n-1) - h_{(2)}, \tag{4.9}$$

where we have used (1.3) and (2.20), and defined $h_{(2)} = Tr(tAA)$.

In the following, let $\{e_1, \cdots, e_{2n-1}\}$ be an orthonormal frame field of $M$. And the coefficients of tensors are defined by $\xi^i = g(\xi, e_i), U^i = g(U, e_i), A_i^j = g(Ae_i, e_j), \phi_i^j = g(\phi e_i, e_j)$, and so on. Furthermore, we use Einstein rule, that is, we take the summation for the same index in the equation.

Then (2.1) is equivalent to the following:

$$\sum_r \{\alpha(A_{jr}\phi_{ir} + A_{ir}\phi_{jr}) + U_i A_{jr}\xi_r + U_j A_{ir}\xi_r\} = 0$$

for any indices $i$ and $j$. If we operate $\nabla^i = \sum_j g^{ij}\nabla_j$ to this, and summing for $i$, we find

$$\mu(2\rho - 3\lambda)\{AW - (\rho - \lambda)W\} + \alpha\{(h_{(2)} - \frac{c}{2}(n-1))\xi - hA\xi - \phi\nabla h\}$$
$$+ (divU)A\xi + FU + (\nabla_U A)\xi - \mu(\rho - \lambda)AW = 0,$$

where we have used (1.1), (1.3), (2.4) and (3.7), which together with (4.7), (4.8) and (4.9) implies that

$$\alpha\phi\nabla h + \mu\{h_{(2)} - \frac{c}{2}n + 2(\rho - \lambda)^2\}W = 0. \tag{4.10}$$

As is already seen that $\Omega_0$ is void, we may only consider such that $h = \rho$ on $\Omega$. Therefore (3.9) shows that $dh(\xi) = 0$. From this fact, (4.10) implies that

$$\alpha\nabla h = \{2(h-\lambda)^2 + h_{(2)} - \frac{c}{2}n\}U,$$

where we have used properties of almost contact metric structure. Comparing this with (3.9), we have

$$h^2 + 2\lambda^2 + 2\alpha h - 2\alpha\lambda - 4h\lambda + h_{(2)} - \frac{c}{4}(2n-1) = 0.$$

On the other hand, it is, using (1.5), seen that the scalar curvature $r$ of $M$ is given by

$$r = c(n^2-1) + h^2 - h_{(2)}.$$

Further, we see, using (1.6) and (4.2) with $\rho = h$, that $g(S\xi,\xi) = \frac{c}{2}(n-1) + \alpha(h-\lambda)$.

From the last three equations, it follows that

$$r - 2g(S\xi,\xi) = 2(\rho-\lambda)^2 + \frac{c}{4}(4n^2-6n+1).$$

If we assume $r - 2g(S\xi,\xi) = \text{const.}$, then we have $\nabla\rho = \nabla\lambda$, which together with (3.1) gives $\theta = 0$, a contradiction. Hence $\Omega$ is void. Thus we have

**Theorem 4.4.** *Let $M$ be a real hypersurface in a complex hyperbolic space $H_n\mathbb{C}$. If it satisfies $R_\xi\phi = \phi R_\xi$, $R_\xi S = SR_\xi$ and $r - 2g(S\xi,\xi)$ =const., then $M$ is a Hopf hypersurface in $H_n\mathbb{C}$.*

## REFERENCE


1. J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Agnew. Math. **395** (1989), 132–141.
2. T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.
3. J.T. Cho and U-H. Ki, *Real hypersurfaces of a complex projective space in terms of the Jacobi operators*, Acta Math. Hungar. **80** (1998), 155–167.
4. J.T. Cho and U-H. Ki, *Jacobi operators on real hypersurfaces of a complex projective space*, Tsukuba J. Math. **22** (1998), 145–156.
5. U-H. Ki, *Cyclic-parallel real hypersurfaces of a complex space form*, Tsukuba J. Math. **12** (1988), 259–268.
6. U-H. Ki, H.-J. Kim and A.-A. Lee, *The Jacobi operator of real hypersurfaces in a complex space form*, Comm. Korean Math. Soc. **13** (1998), 545–560.
7. U-H. Ki, S.-J. Kim and S.-B. Lee, *Some characterizations of a real hypersurfaces of type A*, Kyungpook Math. J. **31** (1991), 73–82.
8. U-H. Ki, A.A. Lee and S.-B. Lee, *On real hypersurfaces of a complex*

*space form in terms of Jacobi operators*, Comm. Korean Math. Soc. **13** (1998), 317–336.

9. U-H. Ki and H. Song, *Jacobi operators on a semi-invariant submanifold of codimension 3 in a complex projective space*, Nihonkai Math. J. **14** (2003), 1–16.

10. U-H. Ki and Y.J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32** (1990), 207–221.

11. M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.

12. M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. **45** (1995), 135–148.

13. S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Japan **37** (1985), 515–535.

14. S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedicata **20** (1986), 245–261.

15. R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms, in Tight and Taut submanifolds*, Cambridge Univ. Press (1998(T.E. Cecil and S.S. Chern, eds.)), 233–305.

16. M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.

17. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

18. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures I, II*, J. Math. Soc. Japan **27** (1975), 43–53, 507–516.

19. K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser (1983).

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