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On the construction of 4-polytopes with certain flag vector pairs

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지도교수 김 진 홍

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목 차

국문초록

특별한 플래그벡터 순서쌍을 만족하는 4차원 다면체의 구성에 관한 연구

김 혜 미
지도교수 : 김 진 홍
조선대학교 교육대학원 수학교육전공

Sjöberg와 Ziegler는 4차원 다면체의 플래그벡터 순서쌍 (f_0,f_{03})를 완벽하게 결정하는 연구 결과를 발표하였다. 이 발표를 토대로 Kim과 Park은 4차원 다면체의 플래그벡터의 순서쌍 (f_0,f_{02})의 범위에 관해 새로운 결과를 제시하였다. 본 논문에는 Kim과 Park의 연구 결과에서 제시한 범위를 만족하는 4차원 다면체의 구체적인 예를 꼭짓점의 개수가 7개이거나 또는 8개인 경우에서 찾았다. 이를 위해 먼저 4차원 다면체 $P_i(i=1,...,15)$에 대하여 P_i의 경계를 구성하고 있는 면(facet)들의 정확한 구조를 규명하였다. 그런 다음 $m=3f_0(f_0-3)-f_{02}$라 할 때, P_i의 m 값이 각각

9, 12, 15, 18, 21, 24, 27, 33, 36, 39, 42, 43, 45, 48, 49

로 주어짐을 구체적인 계산에 통해 확인하여 Kim과 Park이 증명한 필요조건이 충분조건이 될 가능성을 보여주는 몇 가지 구체적인 예를 제시했다.

Ⅰ. Introduction

Our main concern of this thesis is the convex polytope P of dimension d in the Euclidean space $\mathbb{R}^{d+1}$ equipped with the Euclidean metric $< \cdot , \cdot >$ which is one of the fundamental geometric objects in geometry. We say that a linear inequality $<c,x> \le x_0$ is valid if it is satisfied for all point $x \in P$. A face of P is the set of the form

$$F = P \cap \{x \in \mathbb{R}^{d+1} \mid <c,x> \le c_0\},$$

where $<c,x> \le c_0$ is a valid inequality for P. The dimension of a face F is defined to be the dimension of its affine hull aff(F). By definition, for a valid inequality $<0,x> \le 0$ for P, we can obtain a face P itself. All other faces F of P such that $F \subset P$, F is called a proper face of P. It is clear that by definition the inequality $<0,x> \le -1$ for P, we have the empty face $\varnothing$ of P. The faces of dimension $0, 1, \dim P$ are called vertices, edges, and facets, respectively.

Let P be a convex polytope of dimension d, in short called a d-dimensional polytope. Then we say that P is simplicial if every facet of P is a simplex. This is equivalent to saying that every face of P is a simplex. Thus every facet of a simplicial polytope has exactly d vertices. Conversely, if every facet of a polytope P has exactly d vertices, then P is simplicial.

Now, let $f_i = f_i(P)$ denote the number of i-dimensional faces of P for $0 \le i \le d-1$. Then the f-vector of P is defined to

$$(f_0(P), f_1(P), \ldots, f_{d-1}(P)).$$

It is well-known from the Euler-Poincare formula that we have

$$f_0(P) - f_1(P) + \cdots + (-1)^d f_{d-1} = 1 - (-1)^d.$$

More generally, the so-called Dehn-Sommerville equations hold for a d-dimensional polytope P (see Chapter 2 for more details). In order to explain them, we need to define the flag vectors of P. To be more precise, let S be a subset of $\{0,1,2,\ ...,d-1\}$, and let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1,\ ...,\dim F_r\} = S.$$

It is more convenient to make use of the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1,i_2,\ldots,i_k\}}(P)$ for any subset $\{i_1,i_2,\ldots,i_k\}$ of $\{0,1,2,\ldots d-1\}$. For example, $f_{02}(P)$ then will mean $f_{\{0,2\}}(P)$. With these understood, the flag vector of P is defined to be

$$(f_S)_{S \subseteq \{0,\ \ldots,\ d-1\}}.$$

It is clear that the f-vector $f(P)$ is just a vector which is formed by some part of components of the whole flag vector $(f_S)_{S \subseteq \{0,\ \ldots,\ d-1\}}$. Namely, for example, if we take $S = \{i\}$ for each $0 \le i \le d-1$, then we have

$$f_S(P) = f_i(P).$$

We remark that the notion of the flag vector as well as the f-vector is one of the fundamental combinatorial invariants for convex polytopes and that the notion of the f-vector is more well-known than that of the flag-vector.

In [11], Sjöberg and Ziegler has recently proved very remarkable results that completely characterize the flag vector pair (f_0, f_{03}) of any 4-dimensional polytopes. It is worth mentioning that Altshuler and Steinberg's results of a 4-dimensional polytopes with up to 8 vertices and geometric methods such as stacking, general stacking on cyclic polytopes, facet splitting, and truncating played important roles in finding out the structure of specific 4-dimensional polytopes.

Right after the results of Sjöberg and Ziegler, in [11] Kim and Park proved some necessary conditions for the ranges of flag vector pairs such as (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), (f_1, f_{03}) of 4-dimensional polytopes. However, currently their results are far from complete in that it is not obvious whether or not their results give rise to necessary and sufficient conditions for flag vector pairs (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), (f_1, f_{03}) to be satisfied by 4-dimensional polytopes.

One of the aims of this thesis is to explicitly construct various and concrete examples of 4-dimensional polytopes which satisfy necessary conditions for the ranges of flag vector pair (f_0, f_{02}) proved by Kim and Park in [7]. This will provide some evidence that their results might be a necessary and sufficient condition for the range of flag vector pair (f_0, f_{02}) as well as the validity for the results given in [11].

In order to construct such examples, we make use of the examples of 4-dimensional polytopes $P_1, P_2, \ldots, P_{15}$ with the number of vertices equal to 7 or 8 given in the paper [11] of Sjöberg and Ziegler (see [Table 1.1]). The polytopes in [Table 1.1] are listed by their facet list. More precisely, Fukuda, Miyata, and Moriyama provide a complete list of all 31 polytopes with 7 vertices and 1294 polytopes with 8 vertices [4]. The third column in [Table 1.1] such as $7.x$ means that the polytope can be found as the x-th polytope listed in the classification of 4-polytopes with 7 vertices.

polytope	facet list	row
P_1	[654321] [65430] [6520] [6420] [5310] [5210] [4310] [4210]	7.3
P_2	[65432] [65431] [65210] [64210] [5320][5310] [4320] [4310]	7.21
P_3	[65432] [65431] [65210] [6421] [5320] [5310] [4320] [4310] [4210]	7.22
P_4	[65432] [65410] [6531] [6431] [5420] [5321] [5210] [4320] [4310] [3210]	7.11
P_5	[65432] [6541] [6531] [6431] [5421] [5320] [5310] [5210] [4320] [4310] [4210]	7.16
P_6	[65432] [65431] [6521] [6420] [6410] [6210] [5320] [5310] [5210] [4320] [4310]	7.24
P_7	[65432] [6541] [6531] [6430] [6410] [6310] [5421] [5320] [5310] [5210] [4320] [4210]	7.13

P_8	[765432] [765410] [76321] [75310] [64210] [5430] [4320] [3210]	8.186
P_9	[765432] [76541] [76310] [75310] [64210] [6320] [5420] [5410] [5320]	8.285
P_{10}	[76543] [76542] [76321] [75310] [75210] [64310] [64210] [5430] [5420]	8.1145
P_{11}	[765432] [76541] [76310] [54310] [7531] [6421] [6320] [6210][4320] [4210]	8.241
P_{12}	[765432] [76541] [76320] [75310] [54310] [7610] [6421] [6210] [4320] [4210]	8.353
P_{13}	[765432] [76541] [73210] [63210] [7631] [7520] [7510] [6420] [6410] [5420] [5410]	8.201
P_{14}	[765432] [76541] [76310] [7531] [6430] [6410] [5420] [5410] [5321] [5210] [4320] [3210]	8.306
P_{15}	[765432] [76510] [7641] [7541] [6530] [6421] [6321] [6310] [5420] [5410] [5320] [4210] [3210]	8.117

[Table 1.1] 4-polytopes P_i with $f_0 =$ 7 or 8

To be more precise, our main result goes as follows.

Theorem 1.1

For each $1 \le i \le 15$, let P_i be a 4-dimensional polytope as in [Table 1.1], and let

$$m = 3f_0(P_i)(f_0(P_i) - 3) - f_{02}(P_i).$$

Then each m has the following values:

9, 12, 15, 18, 21, 24, 27, 33, 36, 39, 42, 43, 45, 48, 49.

Furthermore, the values of f_0, f_1, f_{02}, f_{03}, and m_i for each polytope P_i are given by the following [Table 1.2]:

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_1	7	21	57	35	27
P_2	7	19	60	36	24
P_3	7	19	63	39	21
P_4	7	19	66	42	18
P_5	7	19	69	45	15
P_6	7	20	72	46	12
P_7	7	20	75	49	9

P_8	8	24	71	39	49
P_9	8	23	72	42	48
P_{10}	8	25	77	43	43
P_{11}	8	23	75	45	45
P_{12}	8	24	78	46	42
P_{13}	8	24	81	49	39
P_{14}	8	24	84	52	36
P_{15}	8	24	87	55	33

[Table 1.2] Values of f_0, f_1, f_{02}, f_{03}, and m

The proof of Theorem 1.1 will be given in Chapter 3. In fact, it should be remarked that in a similar context Y. Seol also investigated the properties of polytopes $P_{16}, P_{17}, \ldots, P_{27}$ in the [6, Table 3] of Sjöberg and Ziegler in her thesis [11], which actually needs some corrections.

The thesis is organized as follows.

In Chapter 2, we first summarize some basic definitions, notation, and useful facts which are necessary for explaining our main results given in Chapter 3. We refer the reader to [1], [3], [5], [6], [7], [8], [9]. [10], [12] and [13] more details. Moreover, in this chapter we summarize some important results previously obtained by Kim and Park in [3] which are our main concern of this thesis.

Finally, Chapter 3 is devoted to giving a proof of Theorem 1.1 and completely determining the values of m for the polytopes P_i $(1 \leq i \leq 15)$ in [Table 1.1].

Ⅱ. Certain flag vector pairs of 4-polytopes

This chapter summarizes the definitions and notation used in the thesis. In addition, in this chapter we will explain the important facts which are essential in understanding this thesis.

To do so, we begin with recalling that a simple polytope means that faces meet at one vertex as many as the number of dimensions of polytope. As in Chapter 1, let S be a subset of $\{0,1,2,\ ...,d-1\}$, and let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1,\ ...,\ \dim F_r\} = S.$$

Let $\mathscr{F}^4$ denote the set of all 4-polytopes, up to the combinatorial equivalence. Our main concern of this thesis is to characterize the set

$$\Pi_{0,02}(\mathscr{F}^4) = \{(f_0(P), f_{02}(P)) \in \mathbb{Z}^2 \mid P : \text{4-polytope}\},$$

so the following generalized Euler-Poincare equations play an important role.

Theorem 2.1 (Dehn-Sommerville equations, Bayer and Billera [2]).
Let P be a d-polytope and $S \subseteq \{0,1,\ \cdots,d-1\}$. Let $\{i,k\} \subseteq S \cup \{-1,d\}$ such that $i < k-1$ and such that there is no $j \in S$ for which is $i < j < k$. the following identity holds:

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1-(-1)^{k-i-1}).$$

The following lemma holds:

Lemma 2.2 The flag vector of every 4-polytope P satisfies the following identity:

$$2f_0(P) - 2f_1(P) + f_{02}(P) - f_{03}(P) = 0.$$

Proof. For the proof, we apply the generalized Dehn-Sommerville equation (Theorem 2.1) with $S=0$, $i=0$, $k=4$. Then it is easy to obtain

$$\sum_{j=1}^{3}(-1)^{j-1}f_{0j}=f_0(1-(-1)^{4-0-1}).$$

This implies that we have

$$f_{01}-f_{02}+f_{03}=2f_0.$$

By using the identity $f_{01}=2f_1$, it is now obvious to show

$$2f_0-2f_1+f_{02}-f_{03}=0,$$

as desired. □

As a consequence of Lemma 2.2, we can show the following

Lemma 2.3 The flag vector of every 4-polytope P satisfies the inequalities:

$$30\le 6f_0(P)\le f_{02}(P)\le 3f_0(P)(f_0(P)-3).$$

Proof. For the proof, we make use of a result of Sjöberg and Ziegler in [11]. That is, we have

$$20\le 4f_0\le f_{03}\le 2f_0(f_0-3).$$

Thus it follows from Lemma 2.2 that we have

$$2f_0(f_0-3)\ge f_{03}=2f_0-2f_1+f_{02}.$$

This implies that we have

$$\begin{aligned} f_{02} &\le -2f_0+2f_1+2f_0(f_0-3) \\ &\le -2f_0+f_0(f_0-1)+2f_0(f_0-3) \\ &= 3f_0(f_0-3). \end{aligned}$$

(2.1)

Since we have $f_{03}\ge 4f_0$ and $f_1\ge 2f_0$, it follows from the identity $f_{03}=2f_0-2f_1+f_{02}$ that

$$f_{02}\ge 2f_0+2f_1\ge 6f_0\ge 30.$$

(2.2)

Finally, using (2.1) and (2.2), it is easy to obtain

$$30\le 6f_0\le f_{02}\le 3f_0(f_0-3).$$ □

In fact, it turns out that in [7] the following result, essentially due to Kim and Park, holds:

Theorem 2.4 The flag vector pair $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ of a 4-polytope P satisfies the following two conditions:
(1) $30 \le 6f_0 \le f_{02} \le 3f_0(f_0 - 3)$.
(2) $f_0 \ge 6$ and for $m \in \{1,2,3,4,5,6,7,8,10,11\}$, $f_{02} \ne 3f_0(f_0 - 3) - m$.

Note that Theorem 2.4 is one of our key motivations for our concrete enumeration of flag vector pairs (f_0, f_{02}) for certain 4-polytopes. Indeed, our main Theorem 1.1 provides some affirmative evidence for the validity of Theorem 2.4. Hopefully, we expect that Theorem 2.4 is very closely related to a necessary and sufficient condition for a complete characterization of flag vector pairs $(f_0(P), f_{02}(P))$ of 4-polytopes.

Note also that the bipyramid P over the tetrahedron contains a unique non-edge so that P satisfies

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (6, 14, 48, 32)$$

and $f_{02} = 3f_0(f_0 - 3) - 6$. Thus, there exists a 4-polytope where $m = 6$ in Theorem 2.4 is actually achieved.

Next, we list some examples of polytopes with small polytopal pairs (f_0, f_{03}) for $f_{03} \le 80$ with simplex facet or simple vertices, following the paper of Sjöberg and Ziegler in [11]. Actually these examples play an important role in finding some concrete examples that satisfy the results given in Theorem 2.1. To do so, we first explain some well-known 4-polytopes listed in [Table 2.1].

In order to define the cyclic polytope, we first need to define the moment curve at $\mathbb{R}^d$, as follows:

$$\alpha : \mathbb{R} \rightarrow \mathbb{R}^d,\ t \mapsto (t, t^2,\ \ldots, t^d) \in \mathbb{R}^d.$$

For any $n > d$, the standard d-th cyclic polytope with n vertices, denoted by $C_d(t_1, t_2, \ldots, t_n)$, is defined as the convex hull in $\mathbb{R}^d$ of n different points $\alpha(t_1), \ldots, \alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$. The set of all sides of the (convex) polytope P is a partially ordered set (or poset) when partially aligned by inclusion. The two polytopes are said to be equivalent in combination of the same combination type. The cyclic polytope $C_d(n)$ are exactly an equivalent combination to the standard cyclic polytope $C_d(t_1, t_2, \ldots t_n)$.

We next explain how to construct a stacking from a given polytope. Indeed, let P be a 4-polytope with face F, and let v be a point beyond face F below the other side. Let Q be the convex hull of P and v, i.e., Q=conv $(\{v\} \cup P)$. In this case, Q is said to be a 4-polytope obtained by stacking. Thus, by stacking, for example, to a square cone P, we can obtain a new 4-polytope Q, a convex shell of P, and a new vertex v.

On the other hand, a pyramid over triangular bipyramid just means the polytope obtained by taking the pyramid over a 3-dimensional triangular bipyramid.

For the facet splitting, consider plane F of 4-polytope P and hyperplane H

intersecting the relative interior of F in polygon X. If the only vertex of P is a simple vertex on one side of H, we can obtain a new polytope Q by separating facet F into two new sides by polygon X. In this case, Q is said to be obtained from P by splitting a facet. For example, splitting the bipyramid means that we obtain a new polytope by dividing one side of the bipyramid. We refer the reader to [11] more details.

For any convex polytope $P \subset \mathbb{R}^n$, one can defines its dual polytope P^* in

$(\mathbb{R}^n)^*$ by

$$P^* = \{ y \in (\mathbb{R}^n)^* \mid < y, x > \geq -1,\ x \in P \}.$$

It can be shown that the dual polytope P^* is convex in the dual space $(\mathbb{R}^n)^*$ and the origin 0 is always contained in the interior of P^*. If, in addition, P contains the origin 0 in its interior, then P^* is also a convex polytope which is bounded and $(P^*)^* = P$. In this paper, we always assume that P contains the origin in its interior, unless stated otherwise. Note that there is a one-to-one order-reversing correspondence between the face poset of P and that of P^*. Moreover, if P is simple, its dual polytope P^* is simplicial, and the converse is also true. It is easy to see that the dual of a simplex is again a simplex itself and the dual of a cube in $\mathbb{R}^3$ is a cross-polytope, that is, an octahedron.

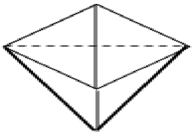
With these understood, our table [Table 2.1] goes as follows:

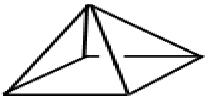
(f_0, f_{03})	Description	(f_0, f_{03})	Description
Polytopes with $\triangle_3$-facet and simple vertex		$(11, 45)$	P_5^*
$(5, 20)$	4-simplex	$(11, 49)$	P_{13}^*
$(6, 26)$	2-fold pyramid over quadrangle	$(11, 52)$	dual of $(9, 52)$
$(6, 29)$	pyramid over triangular bipyramid	$(11, 55)$	dual of $(10, 55)$
$(7, 29)$	pyramid over triangular prism	$(12, 52)$	P_{14}^*
$(7, 32)$	2-fold pyramid over pentagon	$(13, 55)$	P_{15}^*
$(7, 35)$	P_1	polytopes with $\triangle_3$-facet	
$(7, 36)$	P_2	$(6, 36)$	cyclic polytope $C_4(6)$
$(7, 39)$	P_3	$(7, 42)$	P_4
$(7, 45)$	P_5	$(7, 46)$	P_6
$(8, 35)$	P_1^*	$(7, 49)$	P_7

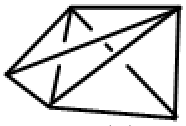
$(8,36)$	P_2^*	$(7,52)$	$R_2(6)$
$(8,38)$	2-fold pyramid over hexagon	$(7,56)$	cyclic polytope $C_4(7)$
$(8,39)$	P_8	$(8,43)$	P_{10}
$(8,42)$	P_9	$(8,60)$	P_{17}
$(8,45)$	P_{11}	$(8,63)$	P_{19}
$(8,46)$	P_{12}	$(8,65)$	P_{20}
$(8,49)$	P_{13}	$(8,66)$	P_{21}
$(8,52)$	P_{14}	$(8,68)$	P_{22}
$(8,55)$	P_{15}	$(8,69)$	P_{23}
$(8,59)$	P_{16}	$(8,70)$	P_{24}
$(8,62)$	P_{18}	$(8,72)$	P_{25}
$(9,39)$	P_3^*	$(8,73)$	P_{26}
$(9,42)$	P_9^*	$(8,76)$	P_{27}
$(9,45)$	split bipyramid in $(9,42)$	$(8,80)$	cyclic polytope $C_4(8)$
$(9,46)$	split bipyramid in $(9,43)$	$(9,79)$	stack onto square pyramid in $(8,63)$
$(9,49)$	split bipyramid in $(9,46)$	Polytopes with simple vertex	
$(9,52)$	stack onto square pyramid in $(8,36)$	$(9,36)$	dual of cyclic polytope $C_4(6)$
$(10,45)$	P_{11}^*	$(9,43)$	P_{10}^*
$(10,46)$	P_{12}^*	$(10,42)$	P_4^*
$(10,49)$	dual of $(9,49)$	$(11,46)$	P_6^*
$(10,52)$	split bipyramid in $(10,49)$	$(12,49)$	P_7^*
$(10,55)$	stack onto square pyramid in $(9,39)$	$(13,52)$	$R_2(6)^*$

[Table 2.1] some polytopal pairs

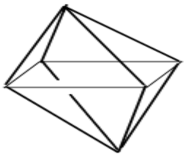
Finally, we list 3-polytopes with five and six vertices. They will play an important role in completely determining the facet structures of a given 4-polytope in Chapter 3. In fact, we have the following list (see [Table 2.2] and [Table 2.3]):


bipyramids over a triangle


square pyramid


$C_3(5)$

[Table 2.2] 3-polytopes with five vertices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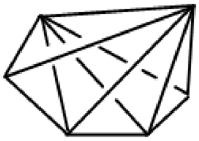
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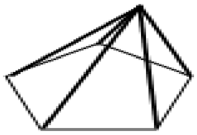
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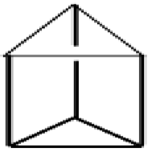
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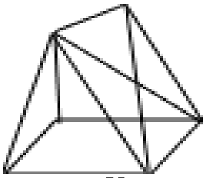
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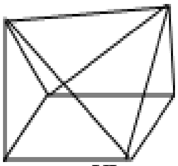
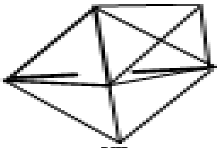

Ⅵ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[Table 2.3] 3-polytopes with six vertices

III. Examples of 4-polytopes with (f_0, f_{02}): P_k $(1 \le k \le 15)$

In order to find a 4-polytope satisfying the conditions of Theorem 2.4, we start with a 4-polytope given in [Table 1.1] of Chapter 1. Note that a 3-polytope consisting of four vertices only is a tetrahedron. Furthermore, as mentioned in [Chapter 2, Table 2.2], a 3-polytope consisting of only five vertices is either a bipyramid over a triangle, $C_3(5)$, or a square pyramid. Finally, note that, 3-polytopes consisting of only six vertices are I, Ⅱ, Ⅲ, Ⅳ, Ⅴ, Ⅵ, and Ⅶ, as in [Table 2.3] given in Chapter 2.

Now, we begin with the case of P_1, as in [Table 1.1] listed by Fukuta, Miyata, and Moriyama in [4].

4.1 P_1 case

In this section, we deal with P_1 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_1 is a 4-polytope with the following facet list:

[654321] [65430] [6520] [6420] [5310] [5210] [4310]
[4210]

(see Table 1.1 for more details). Thus it has 5 tetrahedra and 1 bipyramids over a triangle and one square biypramid. It turns out that there are one possibilities for P_1 with such a facet list.

Below, we list all possibilities for P_1 with the facet list, and by explicitly calculating the value

$$m = 3f_0(f_0 - 3) - f_{02}$$

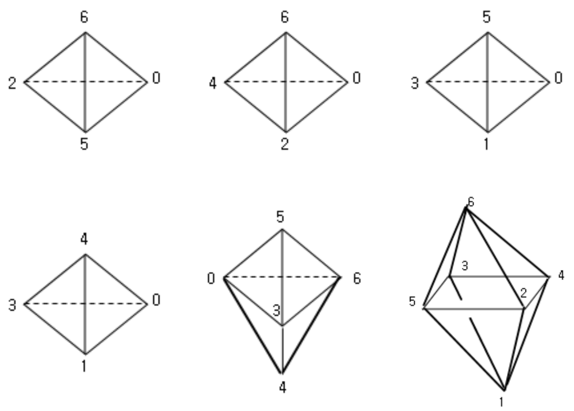
we show that each case fits well with Theorem 2.4 of Kim and Park and thus supports Theorem 2.4 positively. We will explain how to obtain the value m only for the first case. in detail, and leave the details of other cases to a reader.

(1) $f_1(P_1) = 18$ with the followings edges (here, all non-edge are 61,54,32):

$$65, 64, 63, 62, 60, 53, 52, 51, 50, 43, 42, 41, 40, 31, 30, 21, 20, 10$$

All 2-faces are:

~~[025],[026],[056],[256],[024],[026],[046],[246],[013],[015],[135],[035]~~
~~[012],[015],[025],[125],[013],[014],[034],[134],[012],[014],[024],[124]~~
~~[035],[056],[356],[034],[046],[346],[124],[134],[135],[125],[246],[256]~~
~~[346],[356]~~.



[Figure 4.1] The first case of all facets of P_1

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_1) = 7,\ f_{03}(P_1) = 35,\ f_1(P_1) = 18,$$

$$f_{02}(P_1) = -2f_0(P_1) + 2f_1(P_1) + f_{03}(P_1),$$

Thus it is easy to obtain

$$f_{02}(P_1) = -2\times 7 + 2\times 18 + 35 = 57.$$

It follows from the equation $f_{02}(P_1) = 3f_0(P_1)(f_0(P_1)-3) - m$ that we have $m = 27$. Consequently, this case provides an example P_1 of a 4-polytope which satisfies

$$f_{02}(P_1) = 3f_0(P_1)(f_0(P_1)-3) - 27.$$

Other cases for P_1 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_1 has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_1	7	18	57	35	$m=27$

[Table 4.1] P_1

4.2 P_2 case

In this section, we deal with P_2 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_2 is a 4-polytope with the following facet list:

[65432] [65431] [65210] [64210] [5320] [5310] [4320]
[4310]

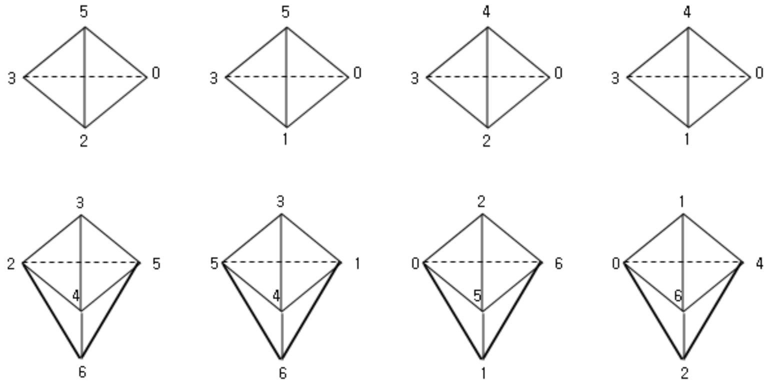
(see Table 1.1 for more details). Thus it has 4 tetrahedra and 4 bipyramids over a triangle. It turns out that there are four possibilities for P_2 with such a facet list. as follow.

(1) $f_1(P_2)=19$ with the followings edges (here, all non-edges are:63,21):

$$65,64,62,61,60,54,53,52,51,50,43,42,41,40,32,31,30,20,10$$

All 2-faces are:

~~[532],[530],[520],[320],[531],[501],[530],[301],[432],[402],[430],[320]~~
~~[413],[401],[430],[301],[532],[324],[534],[256],[246],[456],[354],[351]~~
~~[341],[546],[516],[461],[205],[256],[206],[061],[051],[561],[106],[104]~~
~~[164],[204],[062],[246]~~



[Figure 4.2] The first case of all facets of P_2

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_2) = 7,\ f_{03}(P_2) = 36,\ f_1(P_2) = 19,$$

$$f_{02}(P_2) = -2f_0(P_2) + 2f_1(P_2) + f_{03}(P_2),$$

Thus it is easy to obtain

$$f_{02}(P_2) = -2\times 7 + 2\times 19 + 36 = 60.$$

It follows from the equation $f_{02}(P_2) = 3f_0(P_2)(f_0(P_2) - 3) - m$ that we have $m = 24$. Consequently, this case provides an example P_2 of a 4-polytope which satisfies

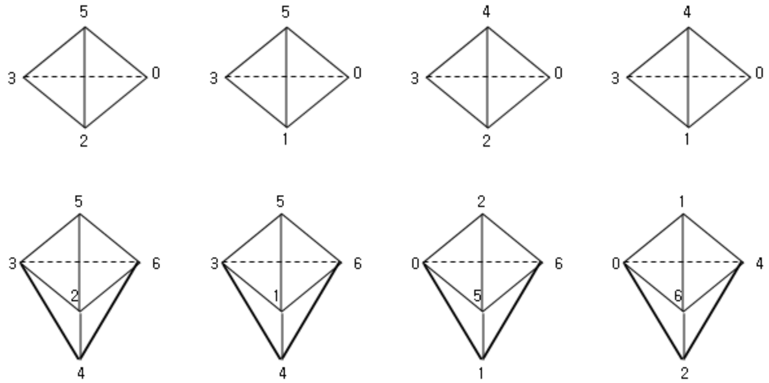
$$f_{02}(P_2) = 3f_0(P_2)(f_0(P_2) - 3) - 24.$$

(2) $f_1(P_2) = 19$ with the followings edges: (here, all non-edges are:54,21)

65,64,63,62,61,60,53,52,51,50,43,42,41,40,32,31,30,20,10

All 2-faces are:

~~[532],[530],[520],[320],[531],[501],[530],[301],[432],[402],[430],[320]~~
~~[413],[401],[430],[301],[532],[324],[526],[346],[536],[264],[351],[314]~~
~~[356],[364],[516],[164],[205],[015],[256],[561],[206],[061],[106],[042]~~
~~[164],[062],[014],[624]~~



[Figure 4.3] The second case of all facets of P_2

In this case, we have

$$f_0(P_2) = 7,\ f_{03}(P_2) = 36,\ f_1(P_2) = 19$$

Thus, $f_{02}(P_2) = -2\times 7 + 2\times 19 + 36 = 60$, and $m = 24$

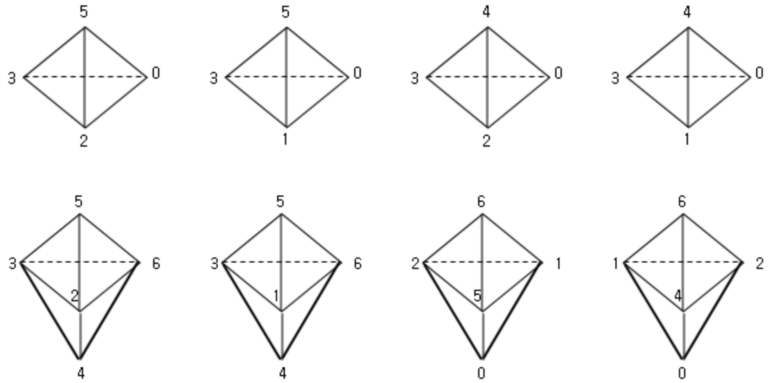
$$= 3f_0(P_2)(f_0(P_2)-3) - 24.$$

(3) $f_1(P_2) = 19$ with the followings edges: (here, all non-edges are:54,60)

$$65,64,63,62,61,53,52,51,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[532],[530],[520],[320],[531],[501],[530],[301],[432],[402],[430],[320]~~
~~[413],[401],[430],[301],[532],[324],[526],[346],[536],[264],[351],[314]~~
~~[356],[364],[516],[164],[205],[015],[256],[561],[621],[210],[614],[042]~~
~~[162],[642],[014],[120]~~



[Figure 4.4] The third case of all facets of P_2

In this case, we have

$$f_0(P_2) = 7,\ f_{03}(P_2) = 36,\ f_1(P_2) = 19$$

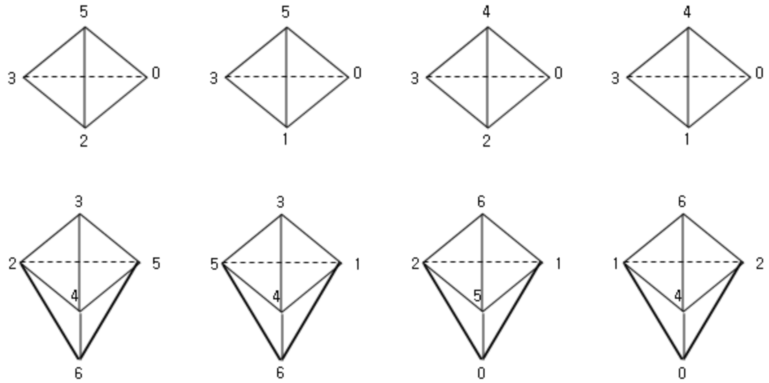
Thus, $f_{02}(P_2) = -2\times 7 + 2\times 19 + 36 = 60$, and $m = 24$

$$= 3f_0(P_2)(f_0(P_2) - 3) - 24.$$

(4) $f_1(P_2) = 19$ with the followings edges: (here, all non-edges are:60,63)

$$65, 64, 62, 61, 54, 53, 52, 51, 50, 43, 42, 41, 40, 32, 31, 30, 21, 20, 10$$

All 2-faces are:
[532],[530],[520],[320],[531],[501],[530],[301],[432],[402],[430],[320]
[413],[401],[430],[301],[532],[256],[324],[246],[534],[546],[534],[561]
[531],[564],[341],[164],[265],[501],[261],[250],[651],[210],[642],[140]
[641],[420],[612],[120]



[Figure 4.5] The fourth case of all facets of P_2

In this case, we have

$$f_0(P_2) = 7,\ f_{03}(P_2) = 36,\ f_1(P_2) = 19$$

Thus, $f_{02}(P_2) = -2\times 7 + 2\times 19 + 36 = 60$, and $m = 24$

$$= 3f_0(P_2)(f_0(P_2) - 3) - 24.$$

Other cases for P_2 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together. Therefore, P_2 has four case.

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_2	7	19	60	36	$m = 24$

[Table 4.2] P_2

4.3 P_3 case

In this section, we deal with P_3 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_3 is a 4-polytope with the following facet list:

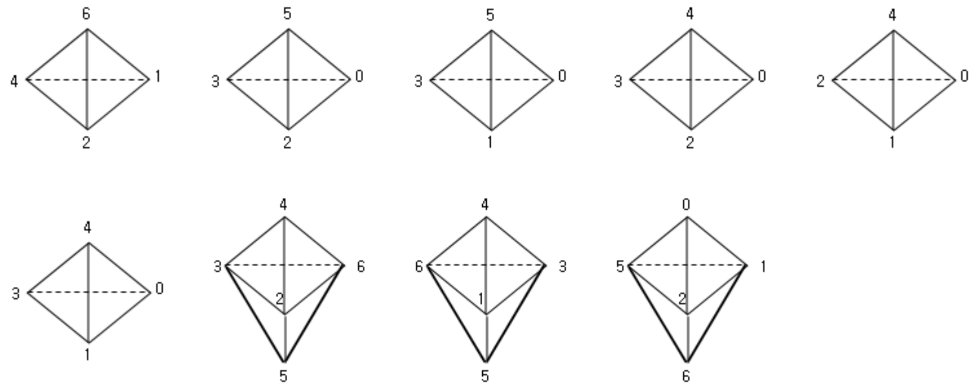
[65432] [65431] [65210] [6421] [5320] [5310] [4320]
[4310] [4210]

(see Table 1.1 for more details). Thus it has 6 tetrahedra and 3 bipyramids over a triangle. It turns out that there are two possibilities for P_3 with such a facet list. as follow.

(1) $f_1(P_3) = 19$ with the followings edges (here, all non-edges are:60,54):

$$65, 64, 63, 62, 61, 53, 52, 51, 50, 43, 42, 41, 40, 32, 31, 30, 21, 20, 10$$

All 2-faces are:
[641],[612],[642],[421],[530],[502],[531],[302],[530],[510],[531],[310]
[432],[402],[430],[302],[431],[410],[430],[301],[421],[410],[402],[210]
[642],[532],[563],[432],[436],[256],[641],[463],[531],[561],[431],[635]
[520],[526],[012],[612],[051],[516]



[Figure 4.6] The first case of all facets of P_3

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_3) = 7,\ f_{03}(P_3) = 39,\ f_1(P_3) = 19,$$

$$f_{02}(P_3) = -2f_0(P_3) + 2f_1(P_3) + f_{03}(P_3),$$

Thus it is easy to obtain

$$f_{02}(P_3) = -2\times 7 + 2\times 19 + 39 = 63.$$

It follows from the equation $f_{02}(P_2) = 3f_0(P_2)(f_0(P_2) - 3) - m$ that we have

$m=21$. Consequently, this case provides an example P_3 of a 4-polytope which satisfies

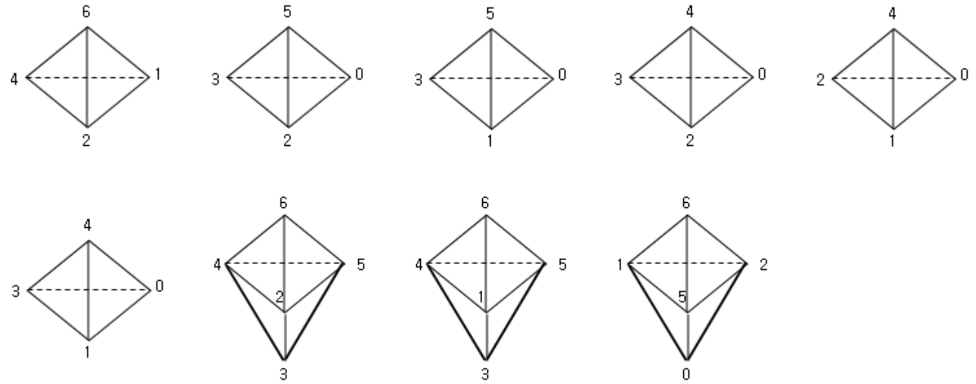
$$f_{02}(P_3) = 3f_0(P_3)(f_0(P_3)-3)-21.$$

(2) $f_1(P_3) = 19$ with the followings edges: (here, all non-edges are:60,63)

$$65,64,62,61,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[641],[612],[642],[421],[530],[502],[531],[302],[530],[510],[531],[310]~~
~~[432],[402],[430],[302],[431],[410],[430],[301],[421],[410],[402],[210]~~
~~[642],[532],[645],[432],[256],[453],[641],[615],[531],[645],[431],[435]~~
~~[520],[612],[012],[615],[051],[652]~~



[Figure 4.7] The second case of all facets of P_3

In this case, we have

$$f_0(P_3) = 7,\ f_{03}(P_3) = 39,\ f_1(P_3) = 19$$

Thus, $f_{02}(P_3) = -2\times 7 + 2\times 19 + 39 = 63$, and $m=21$

$$= 3f_0(P_3)(f_0(P_3)-3)-21.$$

Other cases for P_3 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_3 has two case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_3	7	19	63	39	$m=21$

[Table 4.3] P_3

4.4 P_4 case

In this section, we deal with P_4 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_4 is a 4-polytope with the following facet list:

[65432] [65410] [6531] [6431] [5420] [5312] [5210]
[4320] [4310] [3210]

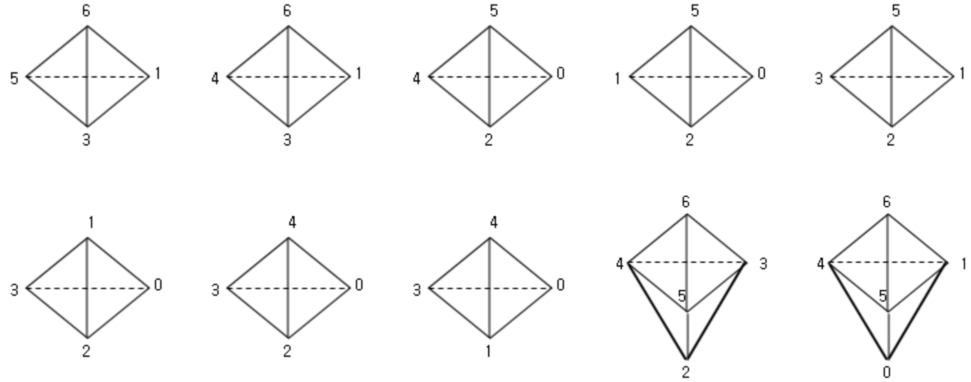
(see Table 1.1 for more details). Thus it has 8 tetrahedra and 2 bipyramids over a triangle. It turns out that there are one possibilities for P_4 with such a facet list. as follow.

(1) $f_1(P_4)=19$ with the followings edges: (here, all non-edges are:60,62)

$$65,64,63,61,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[653],[613],[615],[513],[643],[613],[641],[143],[542],[520],[540],[240]~~
~~[532],[321],[531],[512],[520],[521],[210],[510],[430],[432],[320],[420]~~
~~[431],[430],[410],[310],[321],[310],[320],[210],[653],[432],[643],[542]~~
~~[643],[532],[645],[410],[615],[540],[641],[510]~~



[Figure 4.8] The first case of all facets of P_4

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_4) = 7,\ f_{03}(P_4) = 42,\ f_1(P_4) = 19,$$

$$f_{02}(P_4) = -2f_0(P_4) + 2f_1(P_4) + f_{03}(P_4),$$

Thus it is easy to obtain

$$f_{02}(P_4) = -2\times 7 + 2\times 19 + 42 = 66.$$

It follows from the equation $f_{02}(P_4) = 3f_0(P_4)(f_0(P_4)-3) - m$ that we have $m = 18$. Consequently, this case provides an example P_4 of a 4-polytope which satisfies

$$f_{02}(P_4) = 3f_0(P_4)(f_0(P_4)-3) - 18.$$

Other cases for P_4 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_4 has only one case.

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0-3) - f_{02}$
P_4	7	19	66	42	$m = 18$

[Table 4.4] P_4

4.5 P_5 case

In this section, we deal with P_5 case. For 7 vertices labeled with 0,1,2

,⋯,6 P_5 is a 4-polytope with the following facet list:

[65432] [6541] [6531] [6431] [5421] [5320] [5310]
[5210] [4320] [4310] [4210]

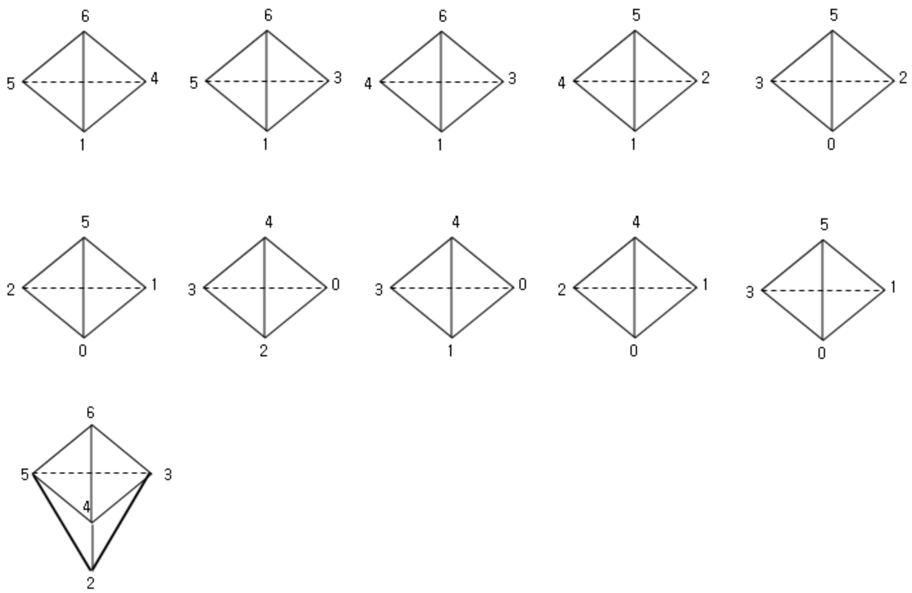
(see Table 1.1 for more details). Thus it has 10 tetrahedra and 1 bipyramids over a triangle. It turns out that there are one possibilities for P_5 with such a facet list. as follow.

(1) $f_1(P_5) = 19$ with the followings edges (here, all non-edges are:60,62):

$$65,64,63,61,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[651]~~,~~[654]~~,~~[614]~~.~~[514]~~,~~[651]~~,~~[653]~~,~~[631]~~,~~[513]~~,~~[641]~~,~~[643]~~,~~[613]~~,~~[413]~~
~~[541]~~,~~[421]~~,~~[542]~~,~~[421]~~,~~[542]~~,~~[512]~~,~~[532]~~,~~[530]~~,~~[520]~~,~~[320]~~,~~[530]~~,~~[531]~~
~~[301]~~,~~[510]~~,~~[521]~~,~~[520]~~,~~[210]~~,~~[510]~~,~~[432]~~,~~[430]~~,~~[420]~~,~~[320]~~,~~[430]~~,~~[413]~~
~~[410]~~,~~[310]~~,~~[420]~~,~~[210]~~,~~[421]~~,~~[410]~~,~~[654]~~,~~[653]~~,~~[643]~~,~~[542]~~,~~[532]~~,~~[432]~~



[Figure 4.9] The first case of all facets of P_5

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_5)=7,\ f_{03}(P_5)=45,\ f_1(P_5)=19,$$
$$f_{02}(P_5)=-2f_0(P_5)+2f_1(P_5)+f_{03}(P_5),$$

Thus it is easy to obtain

$$f_{02}(P_5)=-2\times 7+2\times 19+45=69.$$

It follows from the equation $f_{02}(P_5)=3f_0(P_5)(f_0(P_5)-3)-m$ that we have $m=15$. Consequently, this case provides an example P_5 of a 4-polytope which satisfies

$$f_{02}(P_5)=3f_0(P_5)(f_0(P_5)-3)-15.$$

Other cases for P_5 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_5 has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_5	7	19	69	45	$m=15$

[Table 4.5] P_5

4.6 P_6 case

In this section, we deal with P_6 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_6 is a 4-polytope with the following facet list:

[65432] [65431] [6521] [6420] [6410] [6210] [5320]
[5310] [5210] [4320] [4310]

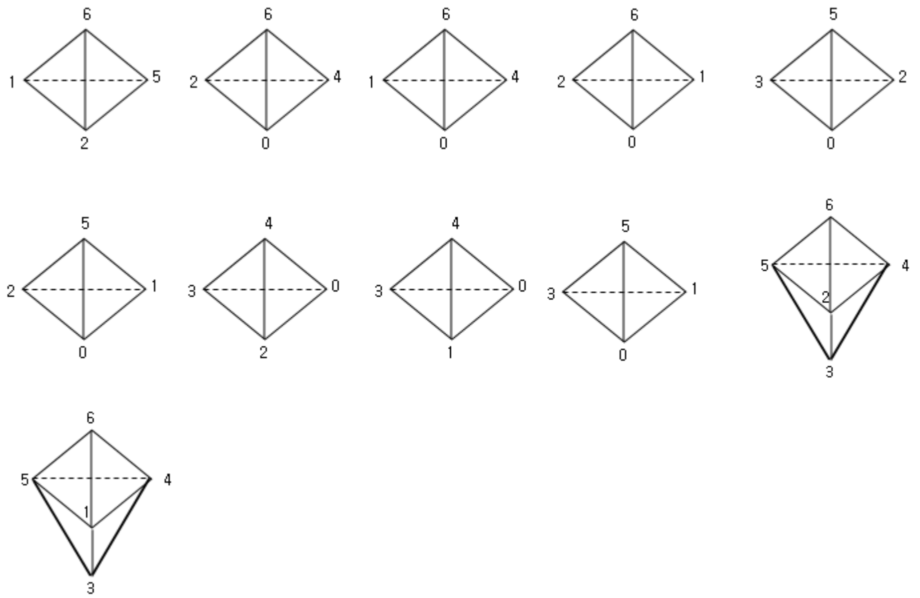
(see Table 1.1 for more details). Thus it has 9 tetrahedra and 2 bipyramids over a triangle. It turns out that there are one possibilities for P_6 with such a facet list. as follow.

(1) $f_1(P_6) = 20$ with the followings edges (here, all non-edges are:63):

$$65,64,62,61,60,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

[562],[612],[516].[512],[642],[602],[640],[420],[641],[640],[610],[410]
[621],[021],[620],[610],[532],[530],[520],[320],[531],[310],[530],[501]
[520],[521],[021],[510],[430],[320],[432],[420],[431],[401],[430],[301]
[562],[642],[654],[543],[523],[324],[651],[641],[531],[654],[453],[134]



[Figure 4.10] The first case of all facets of P_6

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_6) = 7,\ f_{03}(P_6) = 46,\ f_1(P_6) = 20,$$
$$f_{02}(P_6) = -2f_0(P_6)+2f_1(P_6)+f_{03}(P_6),$$

Thus it is easy to obtain

$$f_{02}(P_6) = -2\times 7+2\times 20+46=72.$$

It follows from the equation $f_{02}(P_6) = 3f_0(P_6)(f_0(P_6)-3)-m$ that we have

$m = 12$. Consequently, this case provides an example P_6 of a 4-polytope which satisfies

$$f_{02}(P_6) = 3f_0(P_6)(f_0(P_6) - 3) - 12.$$

(2) $f_1(P_6) = 20$ with the followings edges: (here, all non-edges are:64)

$$65, 63, 62, 61, 60, 54, 53, 52, 51, 50, 43, 42, 41, 40, 32, 31, 30, 21, 20, 10$$

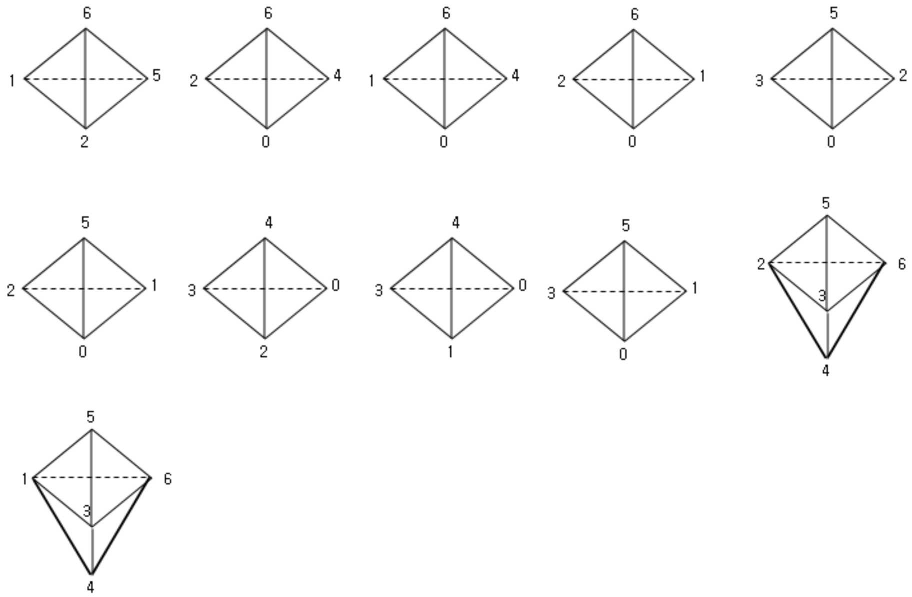
All 2-faces are:

~~[562]~~,~~[612]~~,~~[516]~~.~~[512]~~,~~[642]~~,~~[602]~~,~~[640]~~,~~[420]~~,~~[641]~~,~~[640]~~,~~[610]~~,~~[410]~~

~~[621]~~,~~[021]~~,~~[620]~~,~~[610]~~,~~[532]~~,~~[530]~~,~~[520]~~,~~[320]~~,~~[531]~~,~~[310]~~,~~[530]~~,~~[501]~~

~~[520]~~,~~[521]~~,~~[021]~~,~~[510]~~,~~[430]~~,~~[320]~~,~~[432]~~,~~[420]~~,~~[431]~~,~~[401]~~,~~[430]~~,~~[301]~~

~~[562]~~,~~[642]~~,~~[532]~~,~~[432]~~,~~[563]~~,~~[364]~~,~~[651]~~,~~[641]~~,~~[531]~~,~~[431]~~,~~[436]~~,~~[536]~~



[Figure 4.11] The second case of all facets of P_6

In this case, we have

$$f_0(P_6) = 7,\ f_{03}(P_6) = 46,\ f_1(P_6) = 20,$$

Thus, $f_{02}(P_6) = -2\times 7 + 2\times 20 + 46 = 72$, and $m = 12$

$$= 3f_0(P_6)(f_0(P_6) - 3) - 12.$$

Other cases for P_6 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_6 has two case.

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_6	7	20	72	46	$m = 12$

[Table 4.6] P_6

4.7 P_7 case

In this section, we deal with P_7 case. For 7 vertices labeled with 0,1,2 ,⋯,6 P_7 is a 4-polytope with the following facet list:

[65432] [6541] [6531] [6430] [6410] [6310][5421] [5320]
[5310] [5210] [4320] [4210]

(see Table 1.1 for more details). Thus it has 11 tetrahedra and 1 bipyramids over a triangle. It turns out that there are one possibilities for P_7 with such a facet list. as follow.

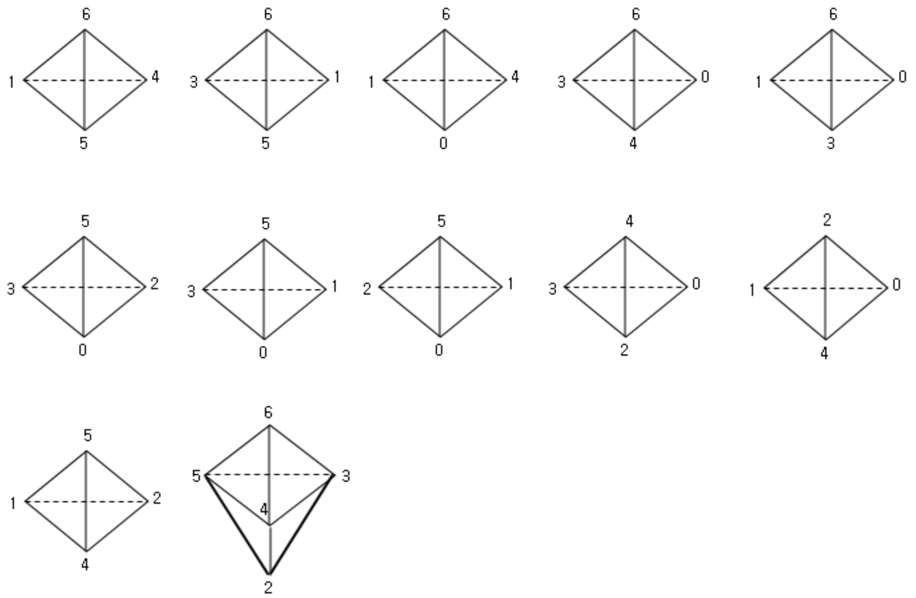
(1) $f_1(P_7) = 20$ with the followings edges (here, all non-edges are:62):

65,64,63,61,60,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10

All 2-faces are:

~~[654],[615],[514].[614],[653],[651],[531],[613],[640],[643],[630],[430]~~
~~[641],[640],[410],[610],[631],[610],[310],[360],[541],[542],[512],[412]~~
~~[523],[320],[053],[520],[530],[351],[510],[310],[520],[210],[521],[510]~~

~~[430],[320],[432],[420],[654],[653],[542],[432],[436],[432]~~



[Figure 4.12] The first case of all facets of P_7

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_7) = 7,\ f_{03}(P_7) = 49,\ f_1(P_7) = 20,$$

$$f_{02}(P_7) = -2f_0(P_7) + 2f_1(P_7) + f_{03}(P_7),$$

Thus it is easy to obtain

$$f_{02}(P_7) = -2\times 7 + 2\times 20 + 49 = 75.$$

It follows from the equation $f_{02}(P_7) = 3f_0(P_7)(f_0(P_7)-3) - m$ that we have $m=9$. Consequently, this case provides an example P_7 of a 4-polytope which satisfies

$$f_{02}(P_7) = 3f_0(P_7)(f_0(P_7)-3) - 9.$$

Other cases for P_7 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_7 has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_7	7	20	75	49	$m=9$

[Table 4.7] P_7

4.8 P_8 case

In this section, we deal with P_8 case. For 8 vertices labeled with 0,1,2 ,⋯,7 P_8 is a 4-polytope with the following facet list:

[765432] [765410] [76321] [6420] [75310] [64210] [5430]
[4320] [3210]

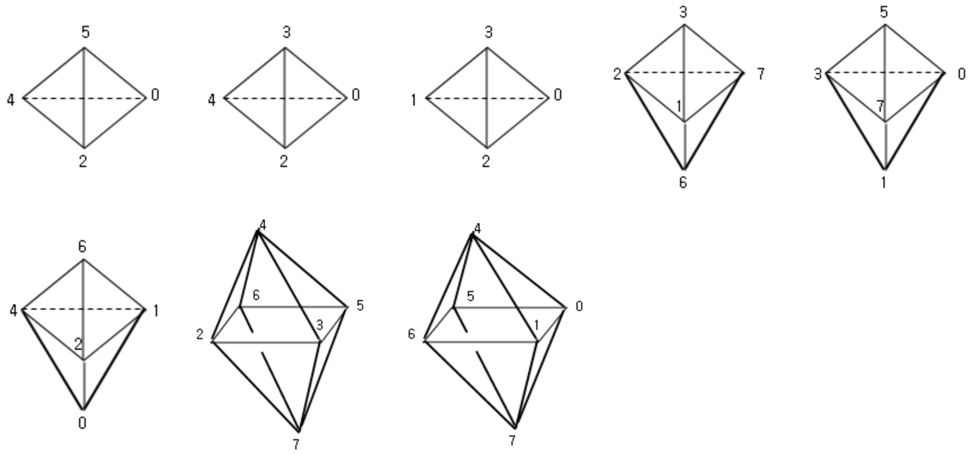
(see Table 1.1 for more details). Thus it has 3 tetrahedra and 3 bipyramids over a triangle and 2 simplicial. It turns out that there are one possibilities for P_8 with such a facet list.

(1) $f_1(P_8)=24$ with the followings edges (here, all non-edge are:74,76,52,51):

$$76,75,73,72,71,70,65,64,62,60,54,53,50,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[543]~~,~~[530]~~,~~[540]~~,~~[430]~~,~~[432]~~,~~[420]~~,~~[430]~~,~~[320]~~,~~[321]~~,~~[320]~~,~~[310]~~,~~[210]~~
~~[321]~~,~~[276]~~,~~[273]~~,~~[617]~~,~~[317]~~,~~[216]~~,~~[530]~~,~~[310]~~,~~[357]~~,~~[570]~~,~~[371]~~,~~[710]~~
~~[642]~~,~~[641]~~,~~[216]~~,~~[042]~~,~~[041]~~,~~[210]~~,~~[426]~~,~~[276]~~,~~[423]~~,~~[765]~~,~~[456]~~,~~[273]~~
~~[345]~~,~~[357]~~,~~[456]~~,~~[657]~~,~~[461]~~,~~[617]~~,~~[450]~~,~~[570]~~,~~[410]~~,~~[710]~~



[Figure 4.13] The first case of all facets of P_8

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_8) = 8,\ f_{03}(P_8) = 39,\ f_1(P_8) = 24,$$

$$f_{02}(P_8) = -2f_0(P_8) + 2f_1(P_8) + f_{03}(P_8),$$

Thus it is easy to obtain

$$f_{02}(P_8) = -2\times 8 + 2\times 24 + 39 = 71.$$

It follows from the equation $f_{02}(P_8) = 3f_0(P_8)(f_0(P_8)-3) - m$ that we have $m=49$. Consequently, this case provides an example P_8 of a 4-polytope which satisfies

$$f_{02}(P_8) = 3f_0(P_8)(f_0(P_8)-3) - 49.$$

Other cases for P_8 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_8 has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_8	8	24	71	39	$m=49$

[Table 4.8] P_8

4.9 P_9 case

In this section, we deal with P_9 case. For 8 vertices labeled with 0,1,2 ,⋯,7 P_9 is a 4-polytope with the following facet list:

$$[765432]\ [765410]\ [76321]\ [6420]\ [75310]\ [64210]\ [5430]$$
$$[4320]\ [3210]$$

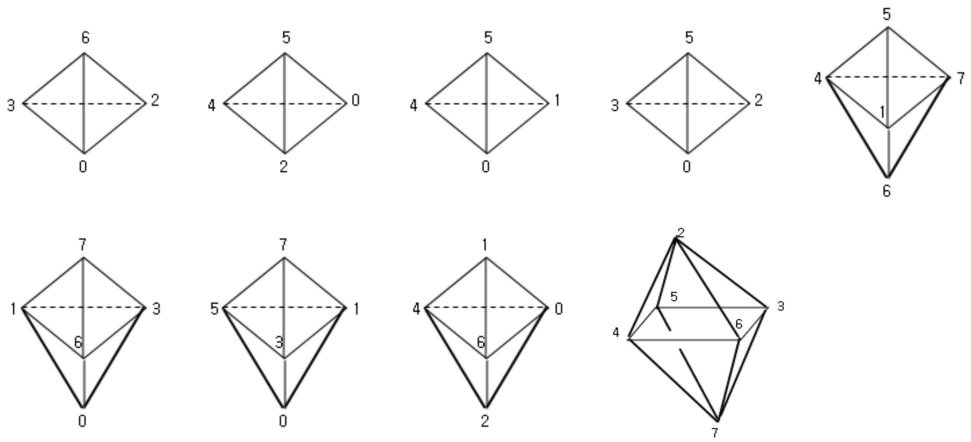
(see Table 1.1 for more details). Thus it has 4 tetrahedra and 4 bipyramids over a triangle and one simplicial or one $C_7(3)$. It turns out that there are one possibilities for P_9 with such a facet list.

(1)$f_1(P_9) = 23$ with the followings edges (here, all non-edge are:27,21,43,70,65):

$$76,75,74,73,71,64,63,62,61,60,54,53,52,51,50,42,41,40,32,31,30,20,10$$

All 2-faces are:

~~[632],[620],[630],[320],[540],[520],[542],[240],[541],[410],[540],[510]~~
~~[532],[530],[320],[520],[540],[547],[517],[416],[476],[167],[176],[173]~~
~~[763],[610],[630],[130],[753].[751],[731],[530],[510],[130],[146],[140]~~
~~[160],[264],[240],[620],[542],[632],[246],[253],[547],[476],[537],[367]~~



[Figure 4.14] The first case of all facets of P_9

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_9) = 8,\ f_{03}(P_9) = 42,\ f_1(P_9) = 23,$$
$$f_{02}(P_9) = -2f_0(P_9) + 2f_1(P_9) + f_{03}(P_9),$$

Thus it is easy to obtain

$$f_{02}(P_9) = -2\times 8 + 2\times 23 + 42 = 72.$$

It follows from the equation $f_{02}(P_9) = 3f_0(P_9)(f_0(P_9)-3)-m$ that we have $m=48$. Consequently, this case provides an example P_9 of a 4-polytope which satisfies

$$f_{02}(P_9) = 3f_0(P_9)(f_0(P_9)-3)-48.$$

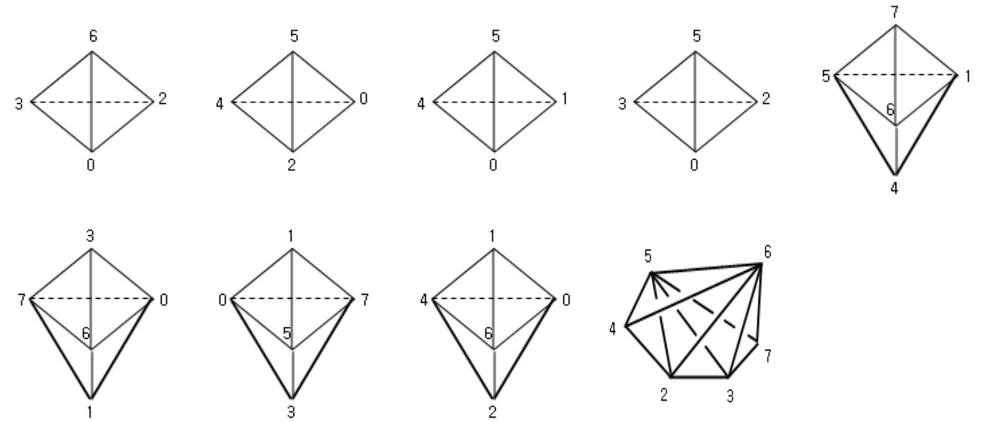
(2)$f_1(P_9) = 23$ with the followings edges:

(here, all non-edges are:74,72,21,43,31)

65,64,62,61,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10

All 2-faces are:

~~[632],[620],[630],[320],[540],[520],[542],[240],[541],[410],[540],[510]~~
~~[532],[530],[320],[520],[756],[751],[761],[546],[514],[614],[673],[730]~~
~~[630],[761],[710],[610],[510],[053],[107],[073],[157],[537],[146],[160]~~
~~[148],[246],[240],[206],[542],[523],[537],[567],[642],[623],[637],[654]~~



[Figure 4.15] The second case of all facets of P_9

In this case, we have

$$f_0(P_9) = 8,\ f_{03}(P_9) = 42,\ f_1(P_9) = 23,$$

Thus, $f_{02}(P_9) = -2\times 8 + 2\times 23 + 42 = 72$, and $m=48$

$$= 3f_0(P_9)(f_0(P_9)-3)-48.$$

Other cases for P_9 are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_9 has two case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_9	8	23	72	42	$m=48$

[Table 4.9] P_9

4.10 P_{10} case

In this section, we deal with P_{10} case. For 8 vertices labeled with 0,1,2 ,⋯,7 P_{10} is a 4-polytope with the following facet list:

[76543] [76542] [76321][75310][75210] [64310] [64210]
[5430][5420]

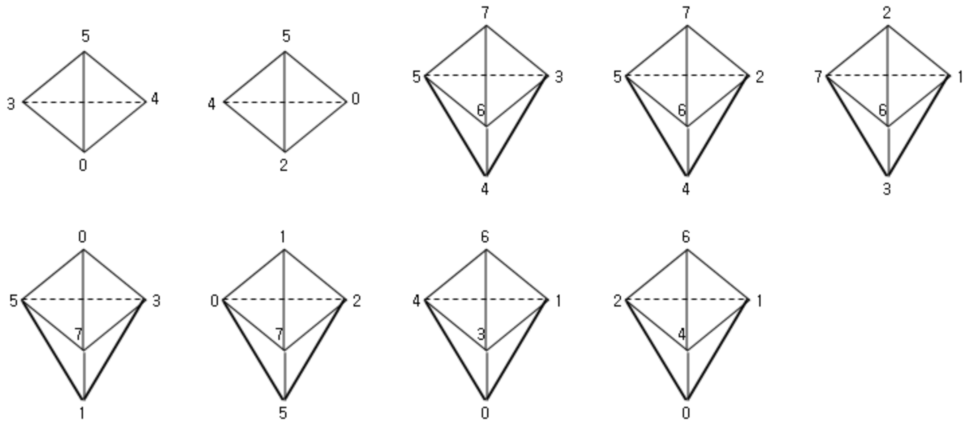
(see Table 1.1 for more details). Thus it has 2 tetrahedra and 7 bipyramids over a triangle. It turns out that there are one possibilities for P_{10} with such a facet list.

(1) $f_1(P_{10}) = 25$ with the followings edges (here, all non-edge are:74,60,51):

$76,75,73,72,71,70,65,64,63,62,61,54,53,52,50,43,42,41,40,32,31,30,21,20,10$

All 2-faces are:

~~[540]~~,~~[530]~~,~~[543]~~,~~[430]~~,~~[540]~~,~~[520]~~,~~[542]~~,~~[420]~~,~~[765]~~,~~[564]~~,~~[753]~~,~~[643]~~
~~[763]~~,~~[534]~~,~~[765]~~,~~[752]~~,~~[762]~~,~~[564]~~,~~[264]~~,~~[524]~~,~~[762]~~,~~[721]~~,~~[261]~~,~~[763]~~
~~[713]~~,~~[613]~~,~~[530]~~,~~[570]~~,~~[537]~~,~~[710]~~,~~[130]~~,~~[713]~~,~~[107]~~,~~[120]~~,~~[172]~~,~~[705]~~
~~[752]~~,~~[025]~~,~~[643]~~,~~[631]~~,~~[641]~~,~~[430]~~,~~[310]~~,~~[410]~~,~~[420]~~,~~[264]~~,~~[261]~~,~~[120]~~
~~[614]~~,~~[410]~~



[Figure 4.16] The first case of all facets of P_{10}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{10}) = 8,\ f_{03}(P_{10}) = 43,\ f_1(P_{10}) = 25,$$

$$f_{02}(P_{10}) = -2f_0(P_{10}) + 2f_1(P_{10}) + f_{03}(P_{10}),$$

Thus it is easy to obtain

$$f_{02}(P_{10}) = -2\times 8 + 2\times 25 + 43 = 77.$$

It follows from the equation $f_{02}(P_{10}) = 3f_0(P_{10})(f_0(P_{10})-3)-m$ that we have $m=43$. Consequently, this case provides an example P_{10} of a 4-polytope which satisfies

$$f_{02}(P_{10}) = 3f_0(P_{10})(f_0(P_{10})-3)-43.$$

Other cases for P_{10} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{10} has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{10}	8	25	77	43	$m=43$

[Table 4.10] P_{10}

4.11 P_{11} case

In this section, we deal with P_{11} case. For 8 vertices labeled with 0,1,2 ,⋯,7 P_{11} is a 4-polytope with the following facet list:

$$[765432]\ [76541]\ [76310][54310][7531]\ [6421][6320]\ [6210]\ [4320][4210]$$

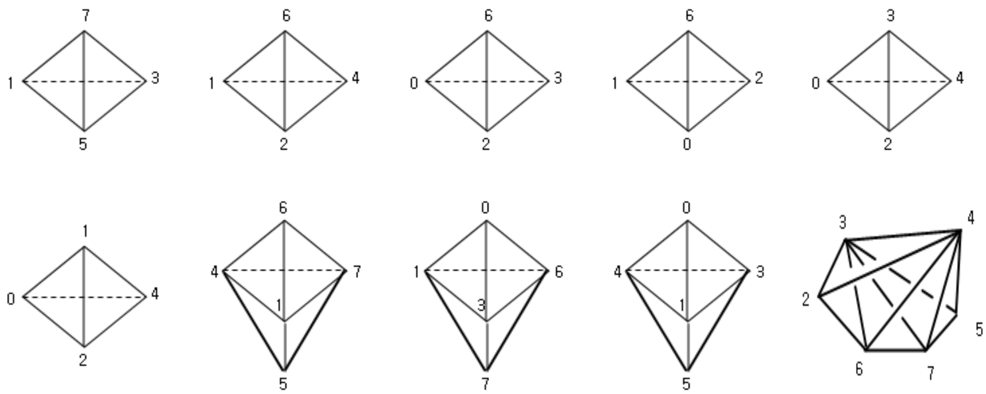
(see Table 1.1 for more details). Thus it has 6 tetrahedra and 3 bipyramids over a triangle and one $C_7(3)$. It turns out that there are one possibilities for P_{11} with such a facet list.

(1) $f_1(P_{11}) = 23$ with the followings edges (here, all non-edge are:72,70,65,52,50)

$$: 76,75,74,73,71,64,63,62,61,60,54,53,51,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[753]~~,~~[713]~~,~~[751]~~,~~[153]~~,~~[642]~~,~~[641]~~,~~[612]~~,~~[412]~~,~~[261]~~,~~[260]~~,~~[210]~~,~~[610]~~
~~[432]~~,~~[430]~~,~~[320]~~,~~[420]~~,~~[421]~~,~~[420]~~,~~[210]~~,~~[410]~~,~~[632]~~,~~[630]~~,~~[320]~~,~~[620]~~
~~[641]~~,~~[647]~~,~~[617]~~,~~[145]~~,~~[175]~~,~~[475]~~,~~[103]~~,~~[106]~~,~~[306]~~,~~[137]~~,~~[167]~~,~~[367]~~
~~[041]~~,~~[043]~~,~~[013]~~,~~[415]~~,~~[435]~~,~~[153]~~,~~[326]~~,~~[367]~~,~~[354]~~,~~[234]~~,~~[426]~~,~~[467]~~
~~[475]~~,~~[375]~~



[Figure 4.17] The first case of all facets of P_{11}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{11}) = 8,\ f_{03}(P_{11}) = 45,\ f_1(P_{11}) = 23,$$

$$f_{02}(P_{11}) = -2f_0(P_{11}) + 2f_1(P_{11}) + f_{03}(P_{11}),$$

Thus it is easy to obtain

$$f_{02}(P_{11}) = -2\times 8 + 2\times 23 + 45 = 75.$$

It follows from the equation $f_{02}(P_{11}) = 3f_0(P_{11})(f_0(P_{11})-3)-m$ that we have $m=45$. Consequently, this case provides an example P_{11} of a 4-polytope which satisfies

$$f_{02}(P_{11}) = 3f_0(P_{11})(f_0(P_{11})-3)-45.$$

Other cases for P_{11} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{11} has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{11}	8	23	75	45	$m=45$

[Table 4.11] P_{11}

4.12 P_{12} case

In this section, we deal with P_{12} case. For 8 vertices labeled with 0,1,2,⋯,7 P_{12} is a 4-polytope with the following facet list:

[765432] [76541] [76320][75310][54310][7610][6421][6210][4320][4210]

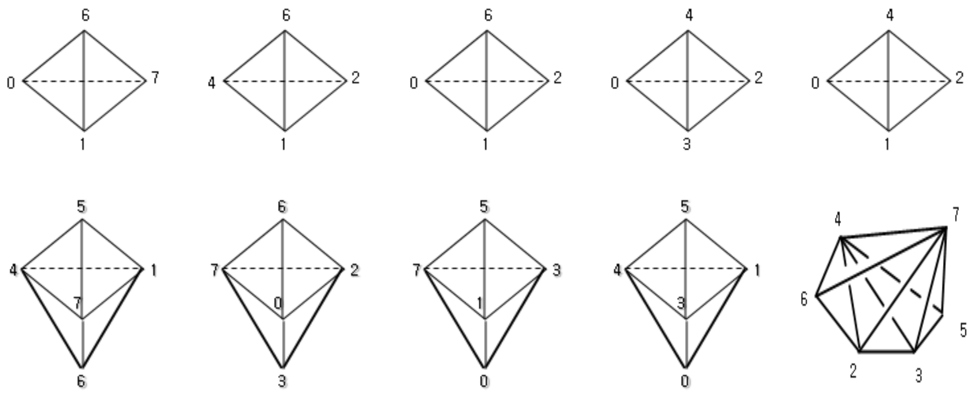
(see Table 1.1 for more details). Thus it has 5 tetrahedra and 4 bipyramids over a triangle and one simplicial or $C_7(3)$. It turns out that there are one possibilities for P_{12} with such a facet list.

(1)$f_1(P_{12}) = 24$ with the followings edges (here, all non-edge are:65,63,52,50):

$$76,75,74,73,72,71,64,62,61,60,54,53,51,43,42,41,40,32,31,30,21,20,10$$

All 2-faces are:

~~[761]~~,~~[760]~~,~~[710]~~,~~[610]~~,~~[642]~~,~~[641]~~,~~[421]~~,~~[612]~~,~~[621]~~,~~[610]~~,~~[620]~~,~~[210]~~
~~[432]~~,~~[430]~~,~~[402]~~,~~[302]~~,~~[412]~~,~~[420]~~,~~[210]~~,~~[410]~~,~~[475]~~,~~[571]~~,~~[451]~~,~~[476]~~
~~[671]~~,~~[461]~~,~~[760]~~,~~[762]~~,~~[260]~~,~~[703]~~,~~[723]~~,~~[023]~~,~~[735]~~,~~[715]~~,~~[513]~~,~~[710]~~
~~[730]~~,~~[130]~~,~~[435]~~,~~[451]~~,~~[351]~~,~~[430]~~,~~[410]~~,~~[310]~~,~~[642]~~,~~[423]~~,~~[435]~~,~~[475]~~
~~[764]~~,~~[762]~~,~~[723]~~,~~[735]~~



[Figure 4.18] The first case of all facets of P_{12}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{12}) = 8,\ f_{03}(P_{12}) = 46,\ f_1(P_{12}) = 24,$$
$$f_{02}(P_{12}) = -2f_0(P_{12}) + 2f_1(P_{12}) + f_{03}(P_{12}),$$

Thus it is easy to obtain

$$f_{02}(P_9) = -2\times 8 + 2\times 24 + 46 = 78.$$

It follows from the equation $f_{02}(P_{12}) = 3f_0(P_{12})(f_0(P_{12})-3)-m$ that we have $m=42$. Consequently, this case provides an example P_{12} of a 4-polytope which satisfies

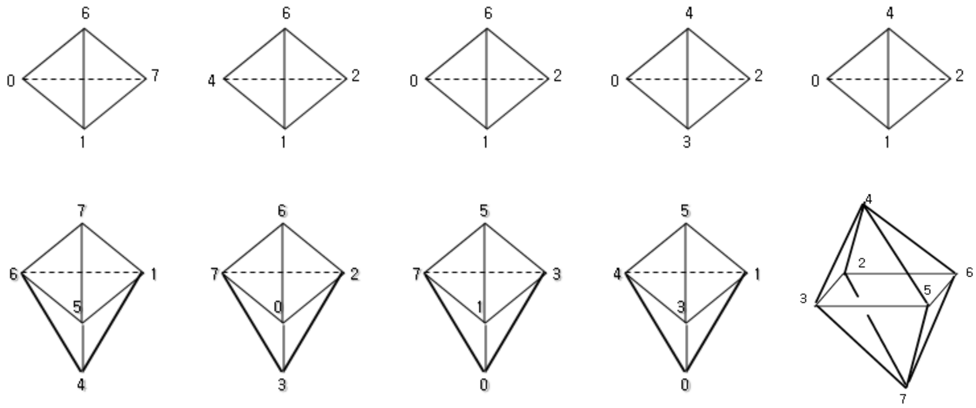
$$f_{02}(P_{12}) = 3f_0(P_{12})(f_0(P_{12})-3)-42.$$

(2)$f_1(P_{12}) = 24$ with the followings edges:(here, all non-edge are:74,63,52,50)

76,75,73,72,71,70,65,64,62,61,60,54,53,51,43,42,41,40,32,31,30,21,20,10

All 2-faces are:

~~[761],[760],[710],[610],[642],[641],[421],[612],[621],[610],[620],[210]~~
~~[432],[430],[402],[302],[412],[420],[210],[410],[765],[715],[761],[645]~~
~~[614],[451],[762],[760],[620],[723],[703],[320],[751],[735],[513],[730]~~
~~[310],[710],[453],[451],[153],[430],[310],[410],[432],[436],[435],[465]~~
~~[375],[372],[267],[756]~~



[Figure 4.19] The second case of all facets of P_{12}

In this case, we have

$$f_0(P_{12}) = 8,\ f_{03}(P_{12}) = 46,\ f_1(P_{12}) = 24,$$

Thus, $f_{02}(P_{12}) = -2\times 8 + 2\times 24 + 46 = 78$, and $m = 42$

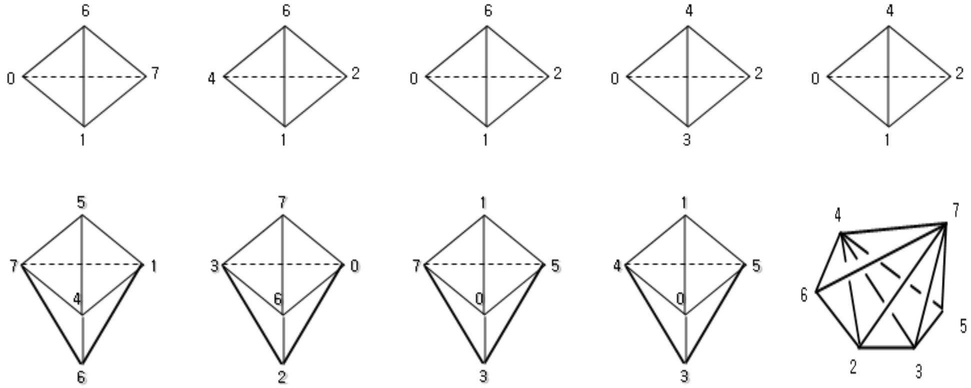
$$= 3f_0(P_{12})(f_0(P_{12}) - 3) - 42.$$

(3)$f_1(P_{12}) = 24$ with the followings edges:(here, all non-edge are:72,65,52,31)

76,75,74,73,71,70,64,63,62,61,60,54,53,51,50,43,42,41,40,32,30,21,20,10

All 2-faces are:

~~[761],[760],[710],[610],[642],[641],[421],[612],[621],[610],[620],[210]~~
~~[432],[430],[402],[302],[412],[420],[210],[410],[754],[715],[541],[476]~~
~~[716],[416],[736],[730],[760],[362],[302],[620],[710],[715],[105],[703]~~
~~[735],[035],[140],[105],[145],[403],[305],[453],[457],[476],[462],[432]~~
~~[345],[357],[376],[362]~~



[Figure 4.20] The third case of all facets of P_{12}

In this case, we have

$$f_0(P_{12}) = 8,\ f_{03}(P_{12}) = 46,\ f_1(P_{12}) = 24,$$

Thus, $f_{02}(P_{12}) = -2\times 8 + 2\times 24 + 46 = 78$, and $m = 42$

$$= 3f_0(P_{12})(f_0(P_{12}) - 3) - 42.$$

Other cases for P_{12} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{12} has three case.

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_{12}	8	24	78	46	$m = 42$

[Table 4.12] P_{12}

4.13 P_{13} case

In this section, we deal with P_{13} case. For 8 vertices labeled with 0,1,2 ,⋯,7 P_{13} is a 4-polytope with the following facet list:

[765432][76541][73210][63210][7631][7520][7510][6420][6410][5420][5410]

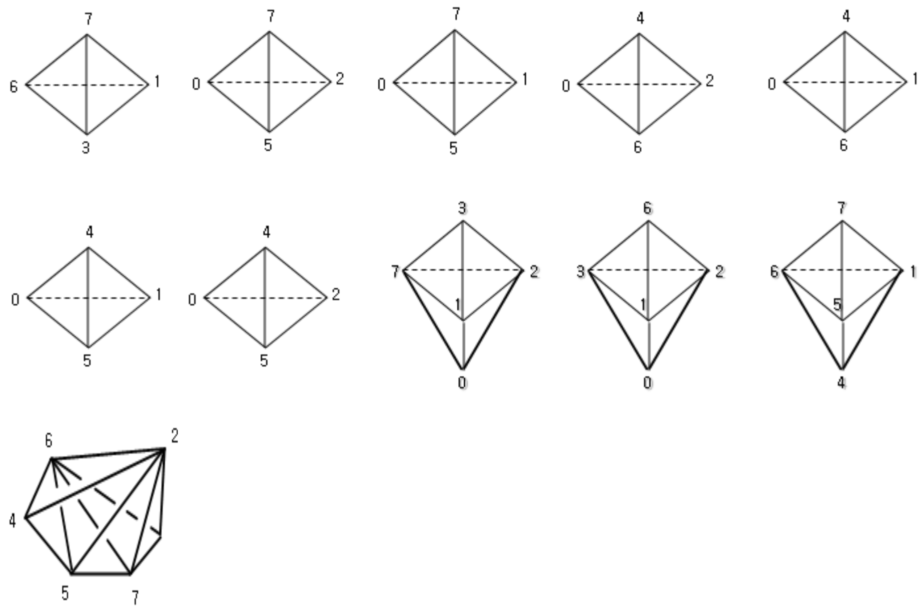
(see Table 1.1 for more details). Thus it has 7 tetrahedra and 3 bipyramids over a triangle and one $C_7(3)$. It turns out that there are one possibilities for P_{13} with such a facet list.

(1)$f_1(P_{13}) = 24$ with the followings edges (here, all non-edge are:74,53,43,30):

76,75,73,72,71,70,65,64,63,62,61,60,54,52,51,50,42,41,40,32,31,21,20,10

All 2-faces are:

~~[761],[631],[761],[713],[752],[720],[750],[520],[751],[750],[701],[510]~~
~~[642],[640],[620],[420],[641],[610],[640],[410],[512],[402],[540],[520]~~
~~[541],[540],[410],[510],[765],[654],[761],[614],[751],[541],[731],[732]~~
~~[710],[720],[312],[120],[613],[632],[312],[610],[620],[120],[645],[657]~~
~~[673],[623],[246],[245],[257],[273]~~



[Figure 4.21] The first case of all facets of P_{13}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{13}) = 8,\ f_{03}(P_{13}) = 49,\ f_1(P_{13}) = 24,$$

$$f_{02}(P_{13}) = -2f_0(P_{13}) + 2f_1(P_{13}) + f_{03}(P_{13}),$$

Thus it is easy to obtain

$$f_{02}(P_{13}) = -2\times 8 + 2\times 24 + 49 = 81.$$

It follows from the equation $f_{02}(P_{13}) = 3f_0(P_{13})(f_0(P_{13})-3)-m$ that we have $m=39$. Consequently, this case provides an example P_{13} of a 4-polytope which satisfies

$$f_{02}(P_{13}) = 3f_0(P_{13})(f_0(P_{13})-3)-39.$$

Other cases for P_{13} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{13} has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{13}	8	24	81	49	$m=39$

[Table 4.13] P_{13}

4.14 P_{14} case

In this section, we deal with P_{14} case. For 8 vertices labeled with 0,1,2,⋯,7 P_{14} is a 4-polytope with the following facet list:

[765432][76541][76310][7531][6430][6410][5420][5410][5321][5210][4320][3210]

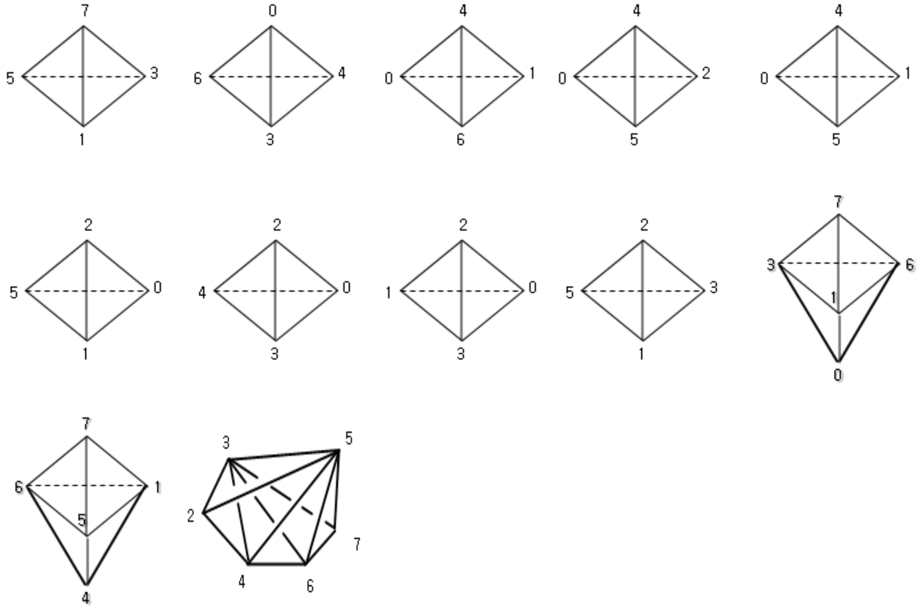
(see Table 1.1 for more details). Thus it has 9 tetrahedra and 2 bipyramids over a triangle and one $C_7(3)$. It turns out that there are one possibilities for P_{14} with such a facet list.

(1) $f_1(P_{14}) = 24$ with the followings edges (here, all non-edge are:74,72,70,62):

76,75,73,71,65,64,63,61,60,54,53,52,51,50,43,42,41,40,32,31,30,21,20,10

All 2-faces are:

~~[640]~~,~~[603]~~,~~[643]~~,~~[403]~~,~~[641]~~,~~[610]~~,~~[640]~~,~~[410]~~,~~[542]~~,~~[520]~~,~~[540]~~,~~[420]~~
~~[541]~~,~~[401]~~,~~[540]~~,~~[510]~~,~~[532]~~,~~[132]~~,~~[531]~~,~~[512]~~,~~[521]~~,~~[510]~~,~~[520]~~,~~[210]~~
~~[753]~~,~~[135]~~,~~[751]~~,~~[713]~~,~~[432]~~,~~[430]~~,~~[302]~~,~~[420]~~,~~[321]~~,~~[310]~~,~~[210]~~,~~[320]~~
~~[765]~~,~~[654]~~,~~[761]~~,~~[614]~~,~~[751]~~,~~[514]~~,~~[371]~~,~~[376]~~,~~[761]~~,~~[310]~~,~~[360]~~,~~[160]~~
~~[325]~~,~~[324]~~,~~[346]~~,~~[367]~~,~~[357]~~,~~[524]~~,~~[546]~~,~~[576]~~



[Figure 4.22] The first case of all facets of P_{14}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{14}) = 8,\ f_{03}(P_{14}) = 52,\ f_1(P_{14}) = 24,$$

$$f_{02}(P_{14}) = -2f_0(P_{14}) + 2f_1(P_{14}) + f_{03}(P_{14}),$$

Thus it is easy to obtain

$$f_{02}(P_{14}) = -2\times 8 + 2\times 24 + 52 = 84.$$

It follows from the equation $f_{02}(P_{14}) = 3f_0(P_{14})(f_0(P_{14})-3)-m$ that we have $m=36$. Consequently, this case provides an example P_{14} of a 4-polytope which satisfies

$$f_{02}(P_{14}) = 3f_0(P_{14})(f_0(P_{14})-3)-36.$$

Other cases for P_{14} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{14} has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{14}	8	24	84	52	$m=36$

[Table 4.14] P_{14}

4.15 P_{15} case

In this section, we deal with P_{15} case. For 8 vertices labeled with 0,1,2 ,⋯,7 $P_{15}$is a 4-polytope with the following facet list:

[765432][76510][7641][7541][6530][6421][6321][6310]5420][5410][5320][4210][3210]

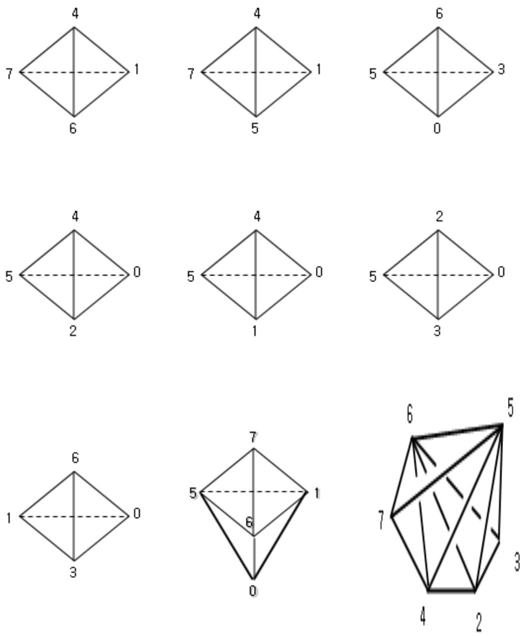
(see Table 1.1 for more details). Thus it has 11 tetrahedra and 1 bipyramids over a triangle and one $C_7(3)$. It turns out that there are one possibilities for P_{15} with such a facet list.

(1)$f_1(P_{15})=24$ with the followings edges (here, all non-edge are:73,72,70,43):

76,75,74,71,65,64,63,62,61,60,54,53,52,51,50,42,41,40,32,31,30,21,20,10

All 2-faces are:

~~[764],[614],[761],[714],[574],[451],[751],[741],[653],[630],[650],[530]~~
~~[642],[641],[612],[412],[631],[612],[632],[123],[631],[630],[130],[610]~~
~~[542],[540],[520],[420],[541],[540],[501],[410],[530],[502],[532],[320]~~
~~[421],[420],[401],[120],[320],[210],[321],[103],[576],[761],[571],[510]~~
~~[560],[610],[674],[642],[623],[635],[576],[542],[574],[532]~~



[Figure 4.23] The first case of all facets of P_{15}

The values of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{15}) = 8,\ f_{03}(P_{15}) = 55,\ f_1(P_{15}) = 24,$$

$$f_{02}(P_{15}) = -2f_0(P_{15}) + 2f_1(P_{15}) + f_{03}(P_{15}),$$

Thus it is easy to obtain

$$f_{02}(P_{15}) = -2\times 8 + 2\times 24 + 55 = 87.$$

It follows from the equation $f_{02}(P_{15}) = 3f_0(P_{15})(f_0(P_{15})-3)-m$ that we have $m=33$. Consequently, this case provides an example P_{15} of a 4-polytope which satisfies

$$f_{02}(P_{15}) = 3f_0(P_{15})(f_0(P_{15})-3)-33.$$

Other cases for P_{15} are not possible, since it can be shown that by inspection 2-faces coming from possible facets do not fit well together.

Therefore, P_{15} has only one case.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{15}	8	24	87	55	$m=33$

[Table 4.15] P_{15}

4.16 Our final results

Our goal is to find flag vector pairs (f_0,f_{02}) of 4-polytopes that satisfy the conditions which are proposed by by Kim and Park's thesis [8]. To do so, we have investigated some specific examples listed by Fukuta, Miyata, and Moriyama in [4] (see [Table 1.1] for more details).

As a result, we have found specific cases of 4-polytopes satisfying $m=3f_0(f_0-3)-f_{02}$. That is, the values of m for the 4-polytopes $P_1 \sim P_{15}$ we have found are

$$[9,12,15,18,21,24,27,33,36,39,42,43,45,48,49].$$

More specifically, we can summarize our main results by using the following table.

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_1	7	21	63	35	27
P_2	7	19	60	36	24
P_3	7	19	63	39	21
P_4	7	19	66	42	18
P_5	7	19	69	45	15
P_6	7	20	72	46	12
P_7	7	20	75	49	9
P_8	8	24	71	39	49
P_9	8	23	72	42	48

P_{10}	8	25	77	43	43
P_{11}	8	23	75	45	45
P_{12}	8	24	78	46	42
P_{13}	8	24	81	49	39
P_{14}	8	24	84	52	36
P_{15}	8	24	87	55	33

[Table 4.1] Summary of our main results

References

[1] A. Alsthuler and L. Shemer, *Consrtuction theorems for polytopes*, Israel Journal of Mathematics **47** (1984), 99-110

[2] M. Bayer and L. Billera, *Deneralized Dehn-Sommervill relations for polytopes, spheres and Eulerian partially ordered sets*, Inventiones Math. 79 (1985), 143-157.

[3] H. B. Cho and J. H. Kim, *On some obstructions of flag vector pairs* (f_1, f_{05}) *of 5-polytopes*, Involve: A Journal of Mathematics **12** (7) (2019), 1183-1192.

[4] K. Fukuda, H. Miyata, and S. Moriyama, *Classification of oriented matroids*, URL www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/, accessed September 2018.

[5] B. Grünbaum, *Convex polytopes,* Second Edition, New York and London, Springer-Verlag, 2003.

[6] H. K. Jeong, *On some optimal bounds for certain flag vector pairs of polytopes*, Master's Thesis, Chosun University, 2019.

[7] J. W. Kim, *On the constructions of 4-polytopes for flag vector pair* (f_0, f_{02}), Master's Thesis, Chosun University, 2019.

[8] N. R. Park, *Characterizing certain flag vector pairs of 4-polytopes*, Master's Thesis, Chosun University, 2019.

[9] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *Lower bound theorems for general polytopes*, preprint (2015): arxiv:1509.08218

[10] G. Pineda-Villavicencio, J. Ugon, and D. Yost, *The excess degree of a*

polytope, preprint (2017): arxiv:1703.10702.

[11] H. Sjöberg and G. M. Ziegler, *Characterizing face and flag vector pairs for polytopes*, Discrete and Computational Geometry **64** (2020), 174-199.

[12] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics 152, Revised edition, New York, Springer-Verlag, 2007.

[13] Y. M. Seol, *Some concrete enumeration of flag vector pairs* (f_0, f_{02}) *for certain 4-polytopes,* Master's Thesis, Chosun University, 2021.