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Some bounds for the fatness of 4-polytopes

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목　　　차

국문초록

4차원 다면체의 비만도의 범위에 관한 연구

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d차원 다면체의 f-벡터는 $i(0 \le i \le d-1)$ 차원의 면의 개수를 f_i라 할 때, $f=(f_0, f_1, \ldots f_{d-1})$로 정의되며, f-벡터보다 좀 더 자세한 정보를 얻을 수 있는 개념으로 플래그벡터를 정의할 수도 있다. 본 논문에서는 4차원 다면체 P의 비만도인 $\phi_4(P)=(f_1+f_2)/(f_0+f_3)$의 범위에 관한 상계와 하계에 관한 연구를 했다. 그 결과, P가 정확하게 두 개 또는 세 개의 비모서리(non-edges)를 가진 4차원 다면체일 때, 다음과 같은 새로운 부등식이 성립함을 보였다.

$$1 < \phi_4(P) < 3$$

Ⅰ. Introduction and Main Results

For a d-dimensional polytope P, let $f_i = f_i(P)$ denote the number of $i-$dimensional faces of P. Then the f-**vector** of P is defined to be

$$(f_0(P), f_1(P), \ldots, f_{d-1}(P)),$$

and it satisfies the well-known Euler-Poincaré equation given by

$$\sum_{i=0}^{d-1} (-1)^i f_i(P) = (1-(-1)^d).$$

In particular, the f-vector of 3-polytope P satisfies

$$f_0(P) - f_1(P) + f_2(P) = 2.$$

Due to Steinitz, the characterization for the set $\mathbb{F}_3$ of f-vectors of all 3-polytope is complete and well-known (see [11]). More precisely, an integer vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if it satisfies the following:

(1) $f_0 - f_1 + f_2 = 2$.

(2) $f_2 \leq 2f_0 - 4$ and the equality holds only for simplicial 3-polytopes.

(3) $f_0 \leq 2f_2 - 4$ and the equality holds only for simple 3-polytopes.

Contrary to the complete characterization of 3-polytopes, it is true that our understanding of the set $\mathbb{F}_4$ of the f-vectors of all 4-polytopes is very insufficient and incomplete. But the 2-dimensional coordinate projections $\Pi_{i,j}(\mathbb{F}_4)$ of the set $\mathbb{F}_4$ of 4-polytopes to the coordinate planes are complete determined by Grünbaum, and Barnette and Reay in

[2] and [6]. To be more precise, it follows from a result of Grünbaum in [2] that the set of f-vector pairs (f_0, f_3) of 4-polytopes satisfies

$$5 \le f_0 \le \frac{1}{2} f_3(f_3 - 3), \quad 5 \le f_3 \le \frac{1}{2} f_0(f_0 - 3).$$

Moreover, the set of f-vector pairs (f_0, f_1) of 4-polytopes satisfies

$$10 \le 2f_0 \le f_1 \le \frac{1}{2} f_0(f_0 - 1)$$

except for the cases of $(6,12),(7,14),(8,17),(10,20)$. The case for the set of f-vector pairs (f_1, f_2) of 4-polytopes is more complicated but its characterization is complete (see [2] for more details).

Another useful combinatorial invariant for convex polytopes, called the **flag vector**, is a less well-known generalization of the concept of f-vectors. That is, for $S \subseteq \{0, \ldots, d-1\}$, let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \dim F_2, \ldots, \dim F_r\} = S.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0,1,2,\ldots d-1\}$. For instance, $f_{01}(P)$ will mean $f_{\{0,1\}}(P)$. The f-vector of P is then $(f_0, f_1, \ldots, f_{d-1})$, and the flag vector of P is $(f_S)_{S \subseteq 0, \cdots, d-1}$.

For any two subsets S_1 and S_2 of $\{0,1,2,\ldots,d-1\}$, a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a **flag vector pair**.

More generally, for any k, not necessarily mutually disjoint, subsets $S_1, S_2, \ldots, S_k$ of $\{0,1,2,\ldots,d-1\}$, a k-tuple

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P))$$

or simply $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$, of flag numbers of P will be called a **flag vector k-tuple.**

As in the f-vectors, let us denote by $\Pi_{S_1, S_2, \ldots, S_k}$ the projection of the flag vector $(f_S)_{S \subseteq \{0, \ldots, d-1\}}$ onto its coordinates $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$. We call $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ a **polytopal flag vector k-tuple** if $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ belongs to the image of the set of all flag vectors of d-dimensional polytopes under the projection map $\Pi_{S_1, S_2, \ldots, S_k}$, that is, if there is a d-polytope P such that

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P)) = (f_{S_1}, f_{S_2}, \ldots, f_{S_k}).$$

Our main concern of this thesis is 4-dimensional polytopes, and, in particular, we want to answer the following important question:

Question 1.1 Is there a constant c, independently of 4-polytopes, so that all 4-polytopes P satisfy inequality

$$f_1(P) + f_2(P) \leq c(f_0(P) + f_3(P))?$$

For this question, we first define the **fatness function** $\phi_4 : \mathbb{F}_4 \to \mathbb{R}$ given by

$$\phi_4(P)=\frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)}$$

of a 4-polytope P. Similarly, for 3-polytopes P we define the **fatness function** $\phi_3(P)$ by

$$\phi_3(P)=\frac{f_1(P)}{f_0(P)+f_2(P)}.$$

There are some known results for the values of the fatness function ϕ_4 or ϕ_3. For example, the 4-simplex has fatness 2, while the 4-cube and the 4-cross polytope have fatness $\frac{56}{24}=\frac{7}{3}$. More generally, if P is simple, then by using the Dehn-Sommerville relations

$$f_2(P)=f_1(P)+f_3(P)-f_0(P),$$

$$f_1(P)=2f_0(P)$$

we can obtain the formula for fatness, as follows.

$$\phi_4(P)=\frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)}=\frac{3f_0(P)+f_3(P)}{f_0(P)+f_3(P)}<3$$

Since every 4-polytope and its dual have the same fatness, the same upper bound holds for simplicial 4-polytopes.

On the other hand, it is known that the neighborly cubical 4-polytopes defined by Jöswig and Ziegler in [8] have f-vectors

$$(4,2n,3n-6,n-2)\times 2^{n-2}.$$

Hence we can obtain the fatness

$$\phi_4=\frac{5n-6}{n+2},$$

which converges to 5 as n goes to ∞. In fact, by a result of Eppstein, Kuperberg, and Ziegler in [5], it is known that there is convex 4-polytope P whose $\phi_4(P)$ is greater than 5.048.

Our main aim of this thesis is to review and also prove the upper and lower bounds for the fatness function ϕ_4 as well as ϕ_3. More precisely, we first review the proofs of the following results (Theorems 1.1 and 1,2) in [9].

Theorem 1.1 Let P be a convex 3-polytope with $f_1(P) \geq 6$, Then we have

$$\frac{1}{2} < \phi_3(P) < 2.$$

Theorem 1.2 Let P be a convex 4-polytope with $f_1(P) \geq 10$. Then we have the following inequalities:

$$\phi_4(P) \geq \frac{(3f_1 + 3 + \sqrt{13 + 4\sqrt{1 + 8f_1}})}{(3f_1 - 1 - \sqrt{1 + 8f_1})} > 1.$$

As before, let P be a d-dimensional convex polytope. Then, in general, f_1 and f_0 satisfy the inequality

$$f_1 \leq \binom{f_0}{2} = \frac{f_0(f_0 - 1)}{2}.$$

So, if f_1 happens to be less than $\binom{f_0}{2}$, then there should be at least one pair of vertices v_1, v_2 of P whose does not form an edge. We call such a pair of vertices v_1, v_2 a **non-edge**. In particular, any facet of 4-polytope P which is not a simplex should contain at least one non-edge. This is because the only 3-polytope in which every two vertices form an edge is the 3-simplex.

Next, we prove new upper and lower bounds for the fatness function ϕ_4, as follows.

Theorem 1.3 Let P be a convex 4-polytope with only two non-edges. Then we have the following inequalities.

$$1 < \phi_4(P) < 3.$$

Theorem 1.4 Let P be a 4-polytope with exactly three non-edges. Then we have he following inequalities.

$$1 < \phi_4(P) < 3.$$

This thesis is structured, as follows.

In Chapter II, we review some important facts and theorems necessary for the proofs of our main results given in Chapter 3.

In Chapter III, we provide the proofs of Theorems 1.1, 1.2, 1.3, and 1.4.

Ⅱ. Preliminaries

In this chapter, we outline some of the important theorems needed to prove the main results in the next three chapters, and setsup the notation and definitions used later.

The convex hull of a set of finite points is a convex polytope and a **convex polyhedron** P is an intersection of finitely many half-spaces in $\mathbb{R}^n$:

$$P=\{x\in\mathbb{R}^n : \langle l_i,x\rangle\geq -a_i, i=1,\ldots,m\}$$

where $l_i\in(\mathbb{R}^n)^*$, dual space of $\mathbb{R}^n$, are some linear functions and $a_i\in\mathbb{R}, i=1,\ldots m$. A **(convex) polytope** is a bounded convex polyhedron, complete determined by [3] and [2, Chapter 1]. Let P be a convex d-polytope, and let $f(P)$ denote the f-vector of P defined by

$$f(P)=(f_0(P), f_1(P),\cdots, f_{d-1}(P)),$$

where $f_i(P)$ means the number of all i-dimensional faces of P. In other words, the f-vector of a cellulation X, denoted $f(P)=(f_0, f_1, \ldots)$, counts the number of cells in each dimension: $f_0(P)$ is the number of vertices, $f_1(P)$ is the number of edges, and so on.

Next, we collect a few well-known facts for convex d-polytopes which are necessary for the proofs of our main results given in Chapter 3. First, we recall the following theorem of Sjöberg and Ziegler in [10].

Theorem 2.1 Flag vector pair (f_0, f_{03}) for 4-polytopes satisfies the following relation:

$$\Pi_{0,03}(F^4) = \{(f_0(P), f_{03}(P)) \in \mathbb{Z}\times\mathbb{Z} \mid P \text{ is a 4-polytopes}\}.$$

Note that Theorem 2.1 describes the number of possible vertex-facet occurrences of a 4-polytope with a fixed number of vertices, which gives the average number of possible facets $\frac{f_{03}}{f_0}$ of the vertex plot for a given number of vertices f_0 (see the paper Sjöberg and Ziegler, [10]).

Theorem 2.2 (Generalized Dehn-Sommerville equation, [4, Theorem 2.1]) Let P be a d-polytope, and let $S \subseteq \{0,1,2,\cdots,d-1\}$. Let $\{i,k\} \subseteq S \cup \{-1,d\}$ such that $i < k-1$ and there is no $j \in S$ such that $i < j < k$. Then, we have

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}}(P) = f_S(P)(1-(-1)^{k-i-1}).$$

This theorem is a generalization of the well-known Euler's equation for convex polytopes.

Theorem 2.3 (Bayer, [3, Theorem 1.3 and 1.4]). The flag vector of every 4-polytope satisfies the inequalities

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \text{ and } -6f_0 + 6f_1 - f_{02} \geq 0.$$

This theorem, in particular, plays important roles in the proofs of the main results given in [9], among which [9, Theorem 1.5] is used in Chapter 3.

Ⅲ. Proofs of Main Results

The purpose of Chapter III is to review and also give the proofs of our main Theorems 1.1, 1.2, 1.3, and 1.4.

To do so, we start with the following theorem for convex 3-polytopes.

Theorem 3.1 Let P be a convex 3-polytope with $f_1(P) \geq 6$, The we have

$$\frac{1}{2} < \phi_3(P) < 2.$$

Proof. For the proof, note first that

$$f_1 = f_0 + f_2 - 2 \text{ (Euler's equation)},$$

$$\frac{3}{2} f_0 \leq f_1 \leq 3f_0 - 6 \text{ (Steinitz)},$$

$$f_2 \leq 2f_0 - 4, \text{ and } f_0 \leq 2f_2 - 4 \text{ (or } f_2 \geq \frac{1}{2} f_0 + 2).$$

By using these facts above, we have

$$\phi_3(P) = \frac{f_1}{f_0+f_2} \geq \frac{\frac{3}{2}f_0}{f_0+f_2}$$

$$\geq \frac{\frac{3}{2}f_0}{3f_0-4} = \frac{3f_0}{6f_0-8} = \frac{\frac{1}{2}(6f_0-8)+4}{6f_0-8}$$

$$= \frac{1}{2} + \frac{4}{6f_0-8}$$

$$= \frac{1}{2} + \frac{2}{3f_0-4} > \frac{1}{2}. \qquad -①$$

On the other hand, we also have

$$\phi_3(P) = \frac{f_1}{f_0+f_2} \leq \frac{3f_0-6}{f_0+\frac{1}{2}f_0+2} = \frac{6f_0-12}{3f_0+4}$$

$$= \frac{2(3f_0+4)-20}{3f_0+4}$$

$$= 2 - \frac{20}{3f_0+4} < 2. \qquad -②$$

Therefore, by ① and ② we have

$$\frac{1}{2} < \phi_3(P) < 2,$$

which was to be demonstrated. □

If P is a convex 4-polytope, recall that we have defined its fatness as

$$\phi_4(P) = \frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)}.$$

In case of simple 4-polytope P, we have

$$f_2(P) = f_1(P) + f_3(P) - f_0(P),$$

$$f_1(P) = 2f_0(P).$$

Thus, we have

$$\phi_4(P) = \frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)} = \frac{3f_0(P)+f_3(P)}{f_0(P)+f_3(P)} < 3.$$

On the other hand, it is known in the paper of Jöswig and Ziegler that there are convex 4-polytopes P_n with f-vectors

$$f(P) = (4, 2n, 3n-6, n-2)\cdot 2^{n-2}.$$

Thus,

$$\phi_4(P_n) = \frac{f_1(P_n)+f_2(P_n)}{f_0(P_n)+f_3(P_n)}$$

$$= \frac{5n-6}{n+2} \to 5, \quad \text{as } n \to \infty.$$

In fact, it turns out that there are convex 4-polytopes P with $\phi_4(P) > 5.048$ (see the paper by Eppstein, Kuperberg, and Ziegler in [5]).

Recall that our concern of this paper is to answer whether or not there is a constant C, independent of all convex 4-polytopes, such that any convex 4-polytopes P satisfy the inequality

$$\phi_4(P) \le c, \text{ i.e., } (f_1(P)+f_2(P)) \le c(f_0(P)) + f_3(P)).$$

Now, we provide the proof of a lower bound for ϕ_4 given in [9], as follows.

Theorem 3.2 Let P be a convex 4-polytope with $f_1(P) \geq 10$ Then, we have the following inequalities:

$$\phi_4(P) \geq \frac{(3f_1+3+\sqrt{13+4\sqrt{1+8f_1}})}{(3f_1-1-\sqrt{1+8f_1})} > 1.$$

Proof. Recall

$$f_0 - f_1 + f_2 - f_3 = 1-(-1)^4 = 0 \text{ (Euler's equation).}$$

Thus, we have

$$\begin{aligned}\phi_4(P) &= \frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)} \\ &= \frac{f_1+(-f_0+f_1+f_3)}{f_0+f_3} \\ &= \frac{-f_0+2f_1+f_3}{f_0+f_3}.\end{aligned}$$

Since $f_{03} \geq 4f_3$, we have

$$\phi_4(P) = \frac{-f_0+2f_1+f_3}{f_0+f_3} \geq \frac{-f_0+2f_1+f_3}{f_0+\frac{1}{4}f_{03}}.$$

Now, recall that the flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of non-neighborly 4-polytope P satisfies the following inequalities [9, Theorem 1.5]

$$f_{03} \leq 4f_1 - 2(1+\sqrt{1+8f_1}).$$

Thus,

$$f_0 + \frac{1}{4}f_{03} \leq f_0 + f_1 - \frac{1}{2}(1+\sqrt{1+8f_1}).$$

Note also that $f_1 \geq 2f_0$ (refer to [6, Theorem. 10.4.2] or [10, Theorem. 2.2]). Hence

$$\phi_4(P) \geq \frac{\frac{3}{2}f_1 + f_3}{\frac{3}{2}f_1 - \frac{1}{2}(1+2\sqrt{1+8f_1})}. \qquad \text{- ③}$$

By [10, Theorem. 2.1] or [6, Theorem. 10.4.1] we have

$$f_0 \leq \frac{1}{2}f_3(f_3-3).$$

That is,

$$f_3^{\,2} - 3f_3 - 2f_0 \geq 0.$$

Thus,

$$f_3 \geq \frac{3+\sqrt{9+8f_0}}{2}.$$

It is also true as in [10, Theorem. 2.2] or [6, Theorem 10.4.2] that

$$f_1 \leq \frac{1}{2}f_0(f_0-1).$$

That is,

$$f_0^{\,2} - f_0 - 2f_1 \geq 0, \text{ i.e., } f_0 \geq \frac{1+\sqrt{1+8f_1}}{2}.$$

Consequently, we have

$$\begin{aligned} f_3 &\geq \frac{3+\sqrt{9+8f_0}}{2} \\ &\geq \frac{3+\sqrt{9+4+4\sqrt{1+8f_1}}}{2} \\ &= \frac{3+\sqrt{13+4\sqrt{1+8f_1}}}{2}. \end{aligned}$$

By ③, we now have

$$\phi_4(P) \geq \frac{(\frac{3}{2}f_1 + \frac{3}{2} + \frac{1}{2}\sqrt{13+4\sqrt{1+8f_1}})}{\frac{3}{2}f_1 - \frac{1}{2} - \frac{1}{2}\sqrt{1+8f_1}}$$

$$= \frac{(3f_1 + 3 + \sqrt{13+4\sqrt{1+8f_1}})}{3f_1 - 1 - \sqrt{1+8f_1}} > 1,$$

which was to be demonstrated. □

Remark:

$$f(x) = \frac{(3x+3+\sqrt{13+4\sqrt{1+8x}})}{(3x-1-\sqrt{1+8x})}, \quad x \geq 10.$$

Then f is a decreasing function, and satisfies

$$1 < f(x) \leq 2, \ x \geq 10.$$

Let P be a convex d-polytope. As mentioned above, if there is a pair of vertices of P which does not form an edge, then such an edge is called a non-edge.

Let P be a convex 4-polytope with only one non-edge. Then we have $1 < \phi_4(P) < 3$ (see [9, Theorem 7.5]).

Next, we deal with 4-polytopes with exactly two non-edges.

Lemma 3.3 Let P be a 4-polytope with exactly two non-edges, and let F be a 3-dimensional facet of P. Then we have $f_0(F) = 4$ or 5.

Proof. By assumption, F has at most two non-edges. Thus, in this case we have

$$\binom{f_0(F)}{2} - 2 \le f_1(F) \le 3f_0(F) - 6.$$

This implies that $4 \le f_0(F) \le 5$. Namely, we have $f_0(F)$ equal to 4 or 5. □

Corollary 3.4 Let P be a 4-polytope with exactly two non-edges. Assume that P is not simplicial. Then either there are exactly two bipyramids over a triangle as facets such that each bipyramid contains exactly one non-edge, or there are exactly two square pyramids as facets such that two apices are connected by an edge.

Proof. For the proof, it suffices to note that the only 3-dimensional polytope with five vertices is either a bipyramid over a triangle or a square pyramid. □

Now we ready to state and prove one of our main results, as follows.

Theorem 3.3 Let P be a 4-polytope with only two non-edges. Then we have the following inequalities:

$$1 < \phi_4(P) < 3.$$

Proof. 1) Assume first that P is simplicial. Let t denote the number of all tetrahedral facets of P. Then it is easy to obtain

$$f_3(P) = t, f_2(P) = 2t, f_{03}(P) = 4t, f_1(P) = f_0(P) + t.$$

Indeed, it follows from the identity

$$f_{02}(P) = -2f_0(P) + 2f_1(P) + f_{03}(P) \text{ (Theorem 2.2) and}$$

$$f_{02}(P) = 3f_2(P) = 6t$$

that we have

$$6t = -2f_0(P) + 2f_1(P) + 4t. \text{ thus, } f_1(P) = f_0(P) + t.$$

Note that $f_0(P) \geq 5$ and $t \geq 5$.

Hence, we have

$$1 < \phi_4(P) = \frac{f_1(P) + f_2(P)}{f_0(P) + f_3(P)} = \frac{f_0(P) + 3t}{f_0(P) + t}$$

$$= 1 + \frac{2t}{f_0(P) + t} \leq 1 + \frac{2t}{t+5} < 3.$$

2) Assume next that P is not simplicial. Then, since P is assumed to have only two non-edges, exactly one of the following two cases holds:

(i) there are exactly two bi-pyramids as facets such that each bi-pyramid contains exactly one non-edge.

(ii) there are exactly two square pyramids as facets such that two apices ate connected by one edge.

For the case of (i), P is a 4-polytope with exactly two bi-pyramid facets and other remaining tetrahedral facets. Hence, we have

$$f_3(P) = t + 2, f_2(P) = 2t + 6, f_{03}(P) = 4t + 10.$$

By the formula

$$f_{02}(P) = -2f_0(P) + 2f_1(P) + f_{03}(P)$$

we have

$$f_{02}(P) = 6t + 18 = -2f_0(P) + 2f_1(P) + 4t + 10.$$

Thus,

$$f_1(P) = f_0(P) + t + 4.$$

Hence, it is easy to obtain

$$\begin{aligned} 1 < \phi_4(P) &= \frac{f_1 + f_2}{f_0 + f_3} = \frac{f_0 + 3t + 10}{f_0 + t + 2} \\ &= 1 + \frac{2t+8}{f_0 + t + 2} \leq 1 + \frac{2t+8}{t+7} \\ &= 3 - \frac{6}{t+7} < 3. \end{aligned}$$

For the case of (ii), P is a 4-polytope with exactly two square pyramid facets and other remaining tetrahedral facets. Note also that we have

$$f_3(P) = t + 2, f_2(P) = 2t + 4, f_{03}(P) = 4t + 10.$$

Thus,

$$f_{02}(P) = 6t + 12 = -2f_0(P) + 2f_1(P) + f_{03}(P) \text{ and so } f_1(P) = f_0(P) + t + 1.$$

Therefore, it is easy to show

$$\begin{aligned} 1 < \phi_4(P) &= \frac{f_1 + f_2}{f_0 + f_3} = \frac{f_0 + 3t + 5}{f_0 + t + 2} \\ &\leq 1 + \frac{2t+3}{t+7} = 3 - \frac{11}{t+7} < 3. \end{aligned}$$

This completes the proof of Theorem 3.3. $\square$

Finally, we deal with 4-polytopes with exactly three non-edges.

Theorem 3.4 Let P be a 4-polytope with exactly three non-edges. Then we have

$$1 < \phi_4(P) < 3.$$

For the proof, we first begin with the following lemma.

Lemma 3.7 Let P be a 4-polytope with exactly three non-edges, and let F be a 3-dimensional facet of P. Then we have $f_0(F)=4,\ 5,$ or 6.

Proof. By assumption, F has at most three non-edges. Thus we should have

$$\binom{f_0(F)}{2} - 3 \le f_1(F) \le 3f_0(F) - 6.$$

This implies that $4 \le f_0(F) \le 6$. That is, $f_0(F) = 4,\ 5,$ or 6. □

Corollary 3.8 Let P be a 4-polytope with exactly three non-edges. Assume that P is not simplicial. Then, one of the following statements holds;

(1) There is a 3-dimensional facet F with $f_0(F) = 6$ and $f_1(F) = 12$ such that F contains three non-edges. In this case, there are only two combinatorially different 3-polytopes F which are both simplicial, as in Figure 3.1.

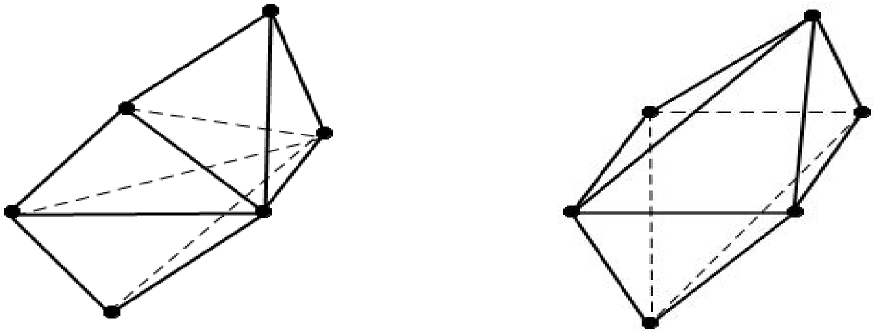

Figure 3.1.

(2) There are exactly three bi-pyramids over a triangle such that each bi-pyramid contains exactly one non-edge.

(3) There are exactly two square pyramids and one bi-pyramid over a triangle such that two apices of two square pyramids are connected by an edge and such that two squares of two square pyramids meet together (see Figure 3.2).

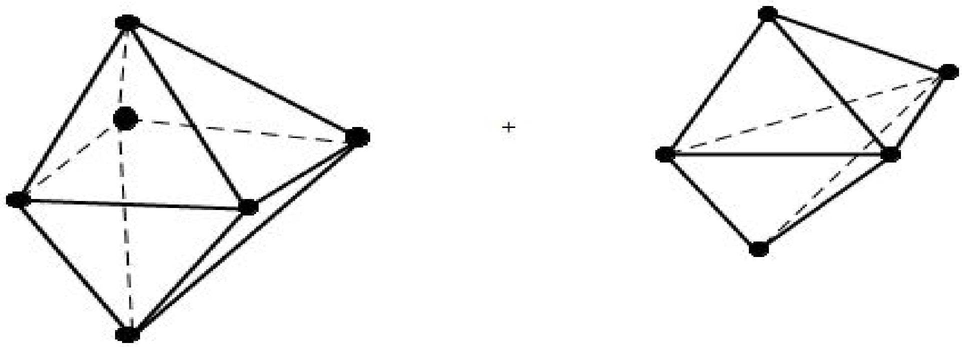

Figure 3.2

Proof of Theorem 3.4. Let F be a 3-dimensional facet of P. Then it follows from Lemma 3.7 that $f_0(F) = 4$, 5, or 6. The case (1) corresponds to the case that a non-tetrahedral facet F with $f_0(F) = 6$ exists, while two cases (2) and (3) correspond to those that three non-tetrahedral facets with $f_0(F) = 5$ exist.

Let t denote the number of tetrahedral facets of P, and let

$$f_3 := f_3(P),\ f_2 := f_2(P),\ f_1 := f_1(P),\ f_0 := f_0(P),$$

as above.

Case (1):

$$f_3 = t + 1, f_2 = 2t + 4, f_{03} = 4t + 6.$$

Thus,

$$f_{02} = -2f_0 + 2f_1 + f_{03},$$

$$3(2t+4) = f_{02} = -2f_0 + 2f_1 + 4t + 6,$$

$$\therefore f_1 = f_0 + t + 3.$$

Hence, we have

$$1 < \phi_4(P) = \frac{f_1 + f_2}{f_0 + f_3} = \frac{f_0 + 3t + 7}{f_0 + t + 1}$$

$$= 1 + \frac{2t+6}{f_0 + t + 1} \leq 1 + \frac{2t+6}{t+6}$$

$$= 3 - \frac{6}{t+6} < 3.$$

Case (2):

$$f_3 = t+3, f_2 = 2t+9, f_{03} = 4t+15.$$

Thus,

$$3(2t+9) = 3f_2 = f_{02} = -2f_0 + 2f_1 + 4t + 15, \text{ i.e.},$$

$$2f_1 = 2f_0 + 2t + 12 \quad \therefore f_1 = f_0 + t + 6.$$

Hence, we have

$$\begin{aligned} 1 < \phi_4(P) = \frac{f_1 + f_2}{f_0 + f_3} &= \frac{f_0 + 3t + 15}{f_0 + t + 3} \\ &= 1 + \frac{2t+12}{f_0 + t + 3} \\ &\le 1 + \frac{2t+12}{t+8} \\ &= 3 - \frac{4}{t+8} < 3. \end{aligned}$$

Case (3):

$$f_3 = t+3, f_2 = 2t+7, f_{03} = 4t+15.$$

Thus,

$$\begin{aligned} 3(2t+7) = 3f_2 &= f_{02} \\ &= -2f_0 + 2f_1 + f_{03} \\ &= -2f_0 + 2f_1 + 4t + 15, \text{ i.e}, \end{aligned}$$

$$f_1 = f_0 + t + 8.$$

Hence, we have

$$
\begin{aligned}
1 < \phi_4(P) = \frac{f_1 + f_2}{f_0 + f_3} &= \frac{f_0 + 3t + 15}{f_0 + t + 3} \\
&= 1 + \frac{2t + 12}{f_0 + t + 3} \\
&\leq 1 + \frac{2t + 12}{t + 8} = 3 - \frac{4}{t + 8} < 3.
\end{aligned}
$$

This completes the proof of Theorem 3.4. $\square$

References

[1] A. Altshuler and L. Steinberg, *Enumeration of the quasisimplicial 3-spheres and 4-polytopes with eight vertices*, Pacific Journal of Mathematics **113** (1984), 269-288.

[2] D. W. Barnette, *The projection of the* f*-vectors of 4-polytopes onto the* (E,S)*-plane*, Discrete Math. **10** (1974), 201-216.

[3] M. M. Bayer, *The extended f-vectors of 4-polytopes*, J. Combinatorial Theory Ser. A **44** (1987), 141-151.

[4] M. M. Bayer and L. J. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. **79** (1985), 143-157.

[5] Eppstein, D., Kuperberg, G., Ziegler, G.M.: *Fat 4-polytopes and fatter 3-spheres.* Pure and Appl. Math., Marcel Dekker Inc. **253**, 239–265 (2003).

[6] B. Grünbaum, *Convex polytopes, Springer,* 1967.

[7] Andrea Höppner and Günter M. Ziegler, *A census of flag-vectors of 4-polytopes,* Polytopes — combinatorics and com-putation (G. Kalai and G.M. Ziegler, eds.), DMV Seminars, vol. 29, Birkhäuser-Verlag, Basel, 2000, pp. 105–110.

[8] Michael Jöswig and Günter M. Ziegler, *Neighborly cubical polytopes,* Discrete Comput. Geom. **24** 2-3 (2000), 325–344, arXiv:math.CO/9812033.

[9] J. H. Kim and N. R. Park, Flag vector pairs, fatness, and their bounds of 4-polytopes, to appear in Contributions in Discrete Mathematics.

[10] H. Sjöberg and G. M. Ziegler, *Characterizing face and flag vector pairs for polytopes*, Discrete Comput. Geom. **64** (2020) 174-199.

[11] Ernst Steinitz, *Uber die Eulerschen Polyederrelationen*, Archivfür Mathematik und Physik **11** (1906), 86–88.