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New bounds for the fatness of 4-dimensional polytopes

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국문초록

4차원 다면체의 새로운 비만도에 관한 연구

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Sjöberg와 Ziegler는 2018년에 4차원 다면체의 플래그벡터 순서쌍 (f_0, f_{03})의 범위를 완벽하게 결정하는 연구결과를 발표하였다. 그 후, Kim과 Park은 2019년에 4차원 다면체의 플래그벡터 순서쌍 (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), (f_1, f_{03})의 새로운 범위를 증명하였다. 또한 Kim과 Park은 이 새로운 범위의 응용으로 4차원 다면체의 비만도 ϕ_4라는 개념을 정의하고 ϕ_4의 범위를 각각 일반적인 4차원 다면체, non-edge가 한 개인 4차원 다면체, 그리고 non-neighborly 조건을 만족하는 4차원 다면체의 경우에 증명하였다. 이에 본 논문은 Kim과 Park 그리고 Shin의 결과를 바탕으로 새로운 비만도 $\widetilde{\phi_4}$를 $\frac{f_2+f_3}{f_0+f_1}$으로 정의하고, 새롭게 정의된 비만도 $\widetilde{\phi_4}$의 범위를 각각 일반적인 4차원 다면체, non-edge가 한 개 또는 두 개인 4차원 다면체, 그리고 non-neighborly 조건에 만족하는 4차원 다면체의 경우에 증명하였다.

Ⅰ. Introduction

A d-**dimensional polytope** (or simply d-**polytope**) P is the convex hull of finitely many points in the Euclidean space $\mathbb{R}^d$. For a d-dimensional polytope P, let $f_i = f_i(P)$ denote the number of i-dimensional faces of P for $0 \le i \le d-1$. Faces of dimension 0, 1, and $d-1$ are called **vertices**, **edges**, and **facets**, respectively. The f-**vector** $f(P)$ of P is defined to be

$$f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P)).$$

In fact, we can generalize the concept of f-vectors in various ways. For example, for $S \subset \{0,1,2,\ldots,d-2\}$ let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \ \ldots \ \dim F_r\} = S.$$

For the sake of simplicity, we shall use the notation $f_{i_1 i_2 \ldots, i_{k-1}, i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_{k-1}, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0,1,2,\ldots d-2, d-1\}$. For example, $f_{01}(P)$ will mean $f_{\{0,1\}}(P)$. The **flag vector** (or **extended f-vector**) of P is defined to be

$$(f_S)_{S \subseteq \{0,1,\ldots,d-2,d-1\}}.$$

Let P^* be the dual d-polytope of P. The f-vector of P^* is given by

$$f(P^*) = (f_{d-1}(P), \ \ldots, f_1(P), f_0(P)).$$

Similarly, the flag vector component $f_S(P^*)$ of P^* is given by

$$f_S(P^*) = f_T(P),$$

where $T = \{d-1-s \mid s \in S\}$.

We shall denote by F^d the set of all f-vectors of d-dimensional polytopes. Clearly F^d will be a subset of $\mathbb{Z}^d$. Let $\Pi_{i,j}(F^d)$ denote the projection of f-vectors of $P \in F^d$ onto the coordinates f_i and f_j. Then $(n,m) \in \Pi_{i,j}(F^d)$ is called a **polytopal pair** such that there is a d-polytope P with $f_i(P) = n$ and $f_j(P) = m$.

For the moment curve in $\mathbb{R}^d$ defined by

$$\alpha : \mathbb{R} \to \mathbb{R}^d, \quad t \mapsto (t, t^2, \cdots, t^{d-1}, t^d)$$

and for any $n > d$, the **standard d-th cyclic polytope with n vertices**, denoted by $C_d(t_1, t_2, t_3, \cdots, t_n)$, is defined as the convex hull in $\mathbb{R}^d$ of n different points $\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_{n-1}), \alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$. **Cyclic polytopes** $C_d(n)$ are defined to be d-polytopes which are combinatorially equivalent to the standard cyclic polytopes (see [3] for more details). For $(n,m) \in \Pi_{0,d-1}(F^d)$, it is well-known that these pairs satisfy the following upper bound theorem saying

$$m \le f_{d-1}(C_d(n)), \quad n \le f_{d-1}(C_d(m)),$$

([4], [6] and [9, section 8.4]).

As in the case of the f-vectors of polytopes, for any two subsets S_1 and S_2 of $\{0, 1, \cdots, d-2, d-1\}$, a pair $(f_{S_1}(P), f_{S_2}(P))$ of flag numbers of P is called a **flag vector pair**. More generally, for any k subsets $S_1, S_2, \cdots, S_{k-1}, S_k$ of $\{0, 1, \cdots, d-2, d-1\}$, a k-tuple $(f_{S_1}(P), \ldots, f_{S_{k-1}}(P), f_{S_k}(P))$ of flag numbers of P is called a **flag vector k-tuple**.

One of the important problems in convex geometry is to completely characterize the f-vectors or flag vectors of polytopes. This problem has been solved completely only up to dimension 3. In particular, in [3] Steinitz showed that the set of f-vectors of 3-polytopes is given by

$$\{(f_0, f_0 + f_2 - 2, f_2) \mid 4 \le f_0 \le 2f_2 - 4,\ 3 \le f_2 \le 2f_0 - 4\}.$$

In [1], Bayer and Billera showed that the flag numbers of 3-polytopes satisfy the following restrictions

$$f_{01}=f_{02}=f_{12}=2f_1=2f_0+2f_2-4,\ f_{012}=4f_1$$

Further, Grünbaum, Barnette, and Reay determined the 2-dimensional coordinate projections $\Pi_{i,j}(F^4)$ of the set of f-vectors of 4-polytopes (see [8] for more details).

Recently, in [8] Sjöberg and Ziegler has given a complete characterization of the flag vector pair (f_0,f_{03}) of any 4-dimensional polytopes. To be precise, they proved the following result.

Theorem 1.1 [8, Theorem 2.5] There exists a 4-polytope P with $f_0(P)=f_0$ and $f_{03}(P)=f_{03}$ if and only if the following two conditions hold:

(1) f_0 and f_{03} are integers satisfying

$$20\le 4f_0\le f_{03}\le 2f_0(f_0-3),$$

and $f_{03}\ne 2f_0(f_0-3)-k,\ k\in\{1,2,3,5,6,9,13\}$.

(2) (f_0,f_{03}) is not one of the following 18 exceptional pairs

(6,24), (6,25), (6,28),
(7,28), (7,30), (7,31), (7,33), (7,34), (7,37), (7,40),
(8,33), (8,34), (8,37), (8, 40)
(9,37), (9,40), (10,40), (10,43).

In [2], Kim an Park has proved some bounds for the flag vector pairs (f_0,f_{02}), (f_{02},f_{03}), (f_1,f_{02}), (f_1,f_{03}) of any 4-dimensional polytopes. In particular, they have shown some bounds for the fatness functions ϕ_3 and ϕ_4 of 3-polytopes and 4-polytopes defined by

$$\phi_3=\frac{f_1}{f_0+f_2},\quad \phi_4=\frac{f_1+f_2}{f_0+f_3},$$

respectively. It turns out that for any 3-polytopes and 4-polytopes they satisfy the following inequalities

$$\frac{3}{4}\le\phi_3<2,$$

and

$$\phi_4 \geq \frac{2\left(3f_1+3+\sqrt{13+4\sqrt{1+8f_1}}\right)}{7f_1-3(1+\sqrt{1+8f_1})} > \frac{6}{7}.$$

Recall that a convex polytope P is called **neighborly** (or **2-neighborly**) if any pair of vertices of P is connected by an edge, forming a complete graph. So any non-neighborly polytope P should have at least one pair of vertices of P which do not form an edge. We call such a pair of vert ices a **non-edge**. In particular, any facet of a 4-polytope which is not a simplex should contain at least one non-edge. This is because the only 3-polytope in which every two vertices form an edge is the 3-simplex.

In [2], Kim and Park also proved some inequalities for any non-neighborly 4-polytopes and 4-polytopes with exactly one non-edge. After the paper [2] of Kim and Park, in [7] Shin extended the results of Kim and Park to any 4-polytopes with exactly two non-edges.

Following the papers of Sjöberg and Ziegler, Kim and Park, and Shin, we consider a new fatness function $\widetilde{\phi_4}$ of 4-dimension polytopes P, as follows.

Definition 1.1 The new fatness function $\widetilde{\phi_4}(P)$ of a 4-dimensional polytope P is defined by the following relation:

$$\widetilde{\phi_4}(P) = \frac{f_2(P)+f_3(P)}{f_0(P)+f_1(P)}.$$

Our main results of this thesis can be summarized as follows.

Theorem 1.2 Let P be a convex 4-polytope. Then the following inequalities hold:

$$\frac{1}{3} < \widetilde{\phi_4}(P) < \frac{7}{2}.$$

Theorem 1.3 Let P be a convex non-neighborly 4-polytope. Then the following inequality holds:

$$\frac{1}{3} < \widetilde{\phi_4}(P) < 3.$$

Theorem 1.4 Let be P a 4-polytope with a unique non-edge. Then the following inequalities hold:

$$0 < \widetilde{\phi_4}(P) < 3.$$

Theorem 1.5 Let P be a 4-polytope with exactly two non-edges. Then the following inequalities hold:

$$0 < \widetilde{\phi_4}(P) < 3.$$

The detailed proofs of the above results will be given in Chapter 3.

This thesis is organized as follows.

In Chapter 2, we first summarize some basic definitions, notation, and useful facts necessary for later chapters.

In Chapter 3, we state and prove some new bounds for the fatness of 4-dimension polytopes for the cases of any 4-polytopes, non-neighborly 4-polytopes, and 4-polytopes with a unique non-edge or exactly two non-edges.

Ⅱ. Preliminaries

This chapter reviews the important theorems needed to demonstrate our main results given in Chapter 3. In addition, in this chapter we set up some notation and definitions for later use.

The convex polytopes are the simplest kind of polytopes, and form the basis for several different generalizations of the concept of polytopes. As mentioned before, a convex polytope is the convex hull of a finite set of points in d-dimensional affine Euclidean space $\mathbb{R}^d$ (see [4, Definition 1.1]). For example, 2-polytope are polygons, while 3-polytopes are polyhedra. More precisely, for $i=1,2,\ldots,m$ let l_i describe a linear functional in $\mathbb{R}^d$ and let $a_i \in \mathbb{R}$. Then a convex polyhedron P is an intersection of finitely many half-spaces in $\mathbb{R}^d$ given by

$$P=\{x\in\mathbb{R}^d \mid \langle l_i, x\rangle \geq -a_i,\ i=1,2,\ \ldots, m-1, m\}.$$

In this paper, we also need the notion of a non-edge for a given polytope P. In order explain it, recall first that $f_0(P)$ is the number of vertices, $f_1(P)$ is the number of edges, $f_{02}(P)$ is the number of faces that make up the vertex, and $f_{03}(P)$ is the number of facets that make up the vertex. Then $f_0(P)$ and $f_1(P)$ satisfy the inequality

$$f_1(P) \leq \binom{f_0(P)}{2}.$$

If $f_1(P) \leq \binom{f_0(P)}{2}-1$, then there must have at least one vertex pair of P that does not form an edge. We shall call such a pair of vertices a **non-edge**. Similarly, if $f_1(P) \leq \binom{f_0(P)}{2}-2$, then there must be at least two non-edges. Some simple examples which show the above properties can be seen in Figure 2.1.

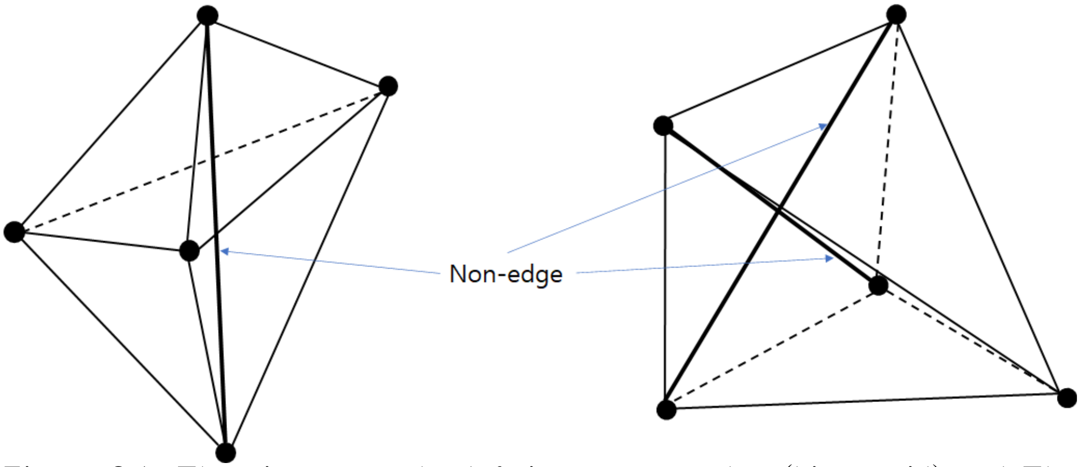

Figure 2.1. The picture on the left is one non-edge (bipyramid) and The picture on the left is two non-edge (square pyramid).

Next we review some previous results for some flag vector pairs of 4-polytopes which are needed in the proofs of our main results in Chapter 3.

Theorem 2.1 [3, Theorem 10.4.1] The set of f-vector pairs (f_0, f_3) of 4-polytopes is equal to

$$\Pi_{0,3}(F^4) = \left\{(f_0, f_3) \in \mathbb{Z}^2 \mid 5 \le f_0 \le \frac{1}{2} f_3(f_3 - 3),\ 5 \le f_3 \le \frac{1}{2} f_0(f_0 - 3)\right\}.$$

Theorem 2.2 [3, Theorem 10.4.2] The set of f-vector pairs (f_0, f_1) of 4-polytopes is equal to

$$\Pi_{0,1}(F^4) = \left\{(f_0, f_1) \in \mathbb{Z}^2 \mid 10 \le 2f_0 \le f_1 \le \frac{1}{2} f_0(f_0 - 1)\right\}$$
$$- \{(6,12),\ (7,14),\ (8,17),\ (10,20)\}.$$

Theorem 2.3 [8, Theorem 2.5] There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if and only if the following two conditions hold:

(1) f_0 and f_{03} are integers satisfying

$$20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0-3),$$

and $f_{03} \neq 2f_0(f_0-3)-k,\ k \in \{1,2,3,5,6,9,13\}$

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(8,33), (8,34), (8,37), (8, 40)
(9,37), (9,40), (10,40), (10,43).

2.1 Flag vector pairs (f_0, f_2)

The following well-known Dehn-Sommerville equation holds.

Theorem 2.4 [1, Theorem 2.1] Let P be a d-polytope, and let S be a subset of $\{0,1,\ldots,d-1\}$. Let $\{i,k\}$ be a subset of $S \cup \{-1,d\}$ such that $i < k-1$ and such that there does not exist an integer $j \in S$ with $i < j < k$. Then the following equation holds.

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1-(-1)^{k-i-1}).$$

The following lemma follows immediately from Theorem 2.4.

Lemma 2.5 The flag vector of a 4-polytope P satisfies the following identity

$$2f_0(P) - 2f_1(P) + f_{02}(P) - f_{03}(P) = 0.$$

Proof. For the proof, if we set $S = \{0\}$, $i = 0$, and $k = 4$ in Theorem 2.4, then it is easy to obtain the equality, as desired. □

We also need the following result (see [2, Lemma 3.5]).

Lemma 2.6 Let P be a 4-polytope with a unique non-edge and let t be the number of all tetrahedral facets of P. Then the following statements hold:
(1) If the polytope P is not simplicial, then P is a polytope with one bipyramid facet and remaining tetrahedral facets, and f_{02} satisfies

$$f_{02} = 6t + 9$$

(2) If the polytope P is simplicial, then f_{02} satisfies

$$f_{02} = 6t$$

Proof. (1) The first statement follows immediately from the fact that among all the 3-polytope with five vertices only the bipyramid over a triangle contains a unique non-edge. Since in this case every 2-dimensional face of P is a triangle, it is easy to see that

$$f_2 = \frac{4t+6}{2} = 2t+3,\ \ f_{02} = 3f_2 = 6t+9$$

(2) On the other hand, if the polytope P is simplicial, clearly we have $f_2 = 2t$, and thus f_{02} satisfies $f_{02} = 6t$. Hence we are done. □

2.2 Flag vector pairs (f_1, f_{03})

In proving our main results given in Chapter 3, the following inequalities in [2] play a crucial role. Here a 4-polytope P is called **2-simple** if each edge of the polytope P is contained in 3 facets of P, while a polytope is called **neighborly** if any pair of vertices of P is connected by an edge, forming a complete graph. More generally, a d-polytope P is called h-**simple** if each $(d-1-h)$-face of P is contained in $h+1$ facets of P. Hence any d-polytope P is $(d-1)$-simple if each vertex of P is contained in d facets of P, and any $(d-1)$-simple d-polytope is called just a **simple polytope**.

Theorem 2.7 The flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a

4-polytope P satisfies the following inequalities

$$f_1+1+\sqrt{1+8f_1} \le f_{03} \le 5f_1-3(1+\sqrt{1+8f_1})$$

where the lower (resp. upper) bound of f_{02} can be achieved by 2-simple (resp. neighborly) 4-polytopes.

Proof. This theorem has been proved in [2, Theorem 6.1]. □

Theorem 2.8 The flag vector pair $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the following inequality

$$f_1+1+\sqrt{1+8f_1} \le f_{03} \le 4f_1-2(1+\sqrt{1+8f_1})$$

Proof. This theorem has been proved in [2, Theorem 6.2]. □

Note that for any $f_1 \ge 10$ which is always true for any 4-polytopes, we have

$$4f_1-2(1+\sqrt{1+8f_1}) \le 5f_1-3(1+\sqrt{1+8f_1}).$$

Therefore, for any non-neighborly 4-polytopes P $(f_1 \ge 11)$ Theorem 2.8 gives better upper bound for f_{03} in terms of f_1 than those given in Theorem 2.7

Finally, we give a definition of a new fatness function $\tilde{\phi}(P)$ which is our main concern of this thesis. For this, first note that by the well-known Euler-Poincare equation, we have

$$f_0(P)-f_1(P)+f_2(P)-f_3(P)=0.$$

Thus it is straightforward to find the ratio

$$\frac{f_1(P)+f_3(P)}{f_0(P)+f_2(P)},$$

which is exactly equal to 1. On the other hand, it is not clear how to find the upper and lower bounds for other ratios such as

$$\frac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)} \text{ and } \frac{f_2(P)+f_3(P)}{f_0(P)+f_1(P)}.$$

In [2] and [7], the authors considered the ratio $\phi_4(P)=\dfrac{f_1(P)+f_2(P)}{f_0(P)+f_3(P)}$ as a fatness function for a given 4-polytope P, and gave some non-trivial upper and lower bounds for the fatness function ϕ_4 in certain cases. However, the upper and lower bounds for the ratio $\dfrac{f_2(P)+f_3(P)}{f_0(P)+f_1(P)}$ seem to be unknown so far.

In view of these contexts, in this thesis we define the ratio $\dfrac{f_2(P)+f_3(P)}{f_0(P)+f_1(P)}$ as a new fatness function $\tilde{\phi}_4$ for 4-polytopes, and give some non-trivial upper and lower bounds for $\tilde{\phi}_4$.

For later use, we state the definition of our new fatness function $\tilde{\phi}_4$ for 4-polytopes, as follows.

Definition 2.9 The new fatness $\tilde{\phi}_4(P)$ of 4-dimensional polytopes P is defined to be

$$\widetilde{\phi_4}(P)=\frac{f_2(P)+f_3(P)}{f_0(P)+f_1(P)}.$$

Ⅲ. Main results

In this chapter, we prove our main results for our new fatness of 4-dimensional polytopes for the cases of any 4-polytopes, non-neighborly 4-polytopes, 4-polytopes with a unique non-edge or exactly two non-edges.

Theorem 3.1 Let P be a convex 4-polytope. Then the following inequality holds:

$$\widetilde{\phi}_4(P) > \frac{1}{3}.$$

Proof. For the proof, first note that by Euler-Poincare equation we have

$$f_2 = -f_0 + f_1 + f_3.$$

Thus the fatness function $\widetilde{\phi}_4$ satisfies

$$\widetilde{\phi}_4 = \frac{f_2+f_3}{f_0+f_1} = \frac{f_1+f_3-f_0+f_3}{f_0+f_1} = \frac{f_1+2f_3-f_0}{f_0+f_1}. \tag{3.1}$$

Since $f_1 \geq 2f_0$, we have $f_0 \leq \frac{1}{2}f_1$, Therefore, the preceding expression is calculated to be $f_0+f_1 \leq \frac{3}{2}f_1$. Hence,

$$\widetilde{\phi}_4 = \frac{f_1+2f_3-f_0}{f_0+f_1} \geq \frac{f_1+2f_3-\frac{1}{2}f_1}{\frac{3}{2}f_1} = \frac{\frac{1}{2}f_1+2f_3}{\frac{3}{2}f_1} = \frac{f_1+4f_3}{3f_1}. \tag{3.2}$$

On the other hand, it follows from [3, Theorem 10.4.1] or Theorems 2.1 and 2.2 that we have

$$f_3^2-3f_3-2f_0 \geq 0 \ \text{ and } \ f_0^2-f_0-2f_1 \geq 0,$$

which implies

$$f_3 \geq \frac{3+\sqrt{9+8f_0}}{2} \ \text{ and } \ f_0 \geq \frac{1+\sqrt{1+8f_1}}{2}. \tag{3.3}$$

Therefore, f_3 satisfies

(3.4) $$f_3 \geq \frac{3+\sqrt{13+4\sqrt{1+8f_1}}}{2}.$$

By combining (3.2) with (3.4), we can obtain

$$\widetilde{\phi_4} \geq \frac{f_1+6+2\sqrt{13+4\sqrt{1+8f_1}}}{3f_1} = \frac{1}{3}+\frac{6+2\sqrt{13+4\sqrt{1+8f_1}}}{3f_1} > \frac{1}{3}.$$

□

Theorem 3.2 Let P be a convex 4-polytope. Then the following inequality holds:

$$\widetilde{\phi_4}(P) < \frac{7}{2}.$$

Proof. For the proof, we can use (3.1).

$$\widetilde{\phi_4} = \frac{f_2+f_3}{f_0+f_1} = \frac{f_1+2f_3-f_0}{f_0+f_1}.$$

By using (3.3), we can obtain

(3.5) $$\widetilde{\phi_4} = \frac{f_1+2f_3-f_0}{f_0+f_1} \leq \frac{f_1+2f_3-\frac{1+\sqrt{1+8f_1}}{2}}{\frac{1+\sqrt{1+8f_1}}{2}+f_1}.$$

By using the inequality $f_{03} \leq 5f_1-3(1+\sqrt{1+8f_1})$ from Theorem 2.7 with $f_3 \leq \frac{1}{4}f_{03}$ from Theorem 2.3 (1). it is easy to obtain

(3.6) $$f_3 \leq \frac{1}{4}(5f_1-3(1+\sqrt{1+8f_1})).$$

By combining (3.5) with (3.6)

$$\widetilde{\phi_4} \leq \frac{f_1+\frac{1}{2}(5f_1-3(1+\sqrt{1+8f_1}))-\frac{1+\sqrt{1+8f_1}}{2}}{\frac{1+\sqrt{1+8f_1}}{2}+f_1}$$

$$= \frac{(7f_1-4)-4\sqrt{1+8f_1}}{(2f_1+1)+\sqrt{1+8f_1}}, \; f_1 \geq 10.$$

Let $g(x)$ be a function given by

$$g(x) = \frac{(7x-4)-4\sqrt{1+8x}}{(2x+1)+\sqrt{1+8x}}, \; x \geq 10.$$

Then g is an increasing function and satisfies $g(10)=1$ and $1 \leq g(x) < \frac{7}{2} = 3.5$ (refer to Figure 3.1). To be more precise, we can show that g is an increasing function, as follows. To do so, it is easy to see that $g'(x)$ is given by

$$g'(x) = \frac{A}{(2x+1+\sqrt{1+8x})^2}.$$

Here A is equal to

$$A = (7 - \frac{16}{\sqrt{1+8x}})(2x+1+\sqrt{1+8x}) - (2+\frac{4}{\sqrt{1+8x}})(7x-4-4\sqrt{1+8x})$$

$$= 15x + 15\sqrt{1+8x} - \frac{60x}{\sqrt{1+8x}} = 15x + \frac{15+60x}{\sqrt{1+8x}} > 0.$$

This for any $x > 0$, we have $g'(x) > 0$. That is, the function g is indeed an increasing function.

Figure 3.1

Consequently, we have $\widetilde{\phi}_4(P) \le g(f_1) < \frac{7}{2}$. This completes the proof Theorem 3.2. □

Theorem 3.3 Let P be a convex 4-polytope. Then the following inequalities hold:

$$\frac{1}{3} < \widetilde{\phi}_4(P) < \frac{7}{2}.$$

Proof. Combining Theorem 3.1 and Theorem 3.2 proves the result. □

Theorem 3.4 Let P be a convex non-neighborly 4-polytope. Then the following inequality holds:

$$\frac{1}{3} < \widetilde{\phi}_4(P) < 3.$$

Proof. i) $\widetilde{\phi}_4(P) > \frac{1}{3}$: The proof of Theorem 3.4 is identical to that of Theorem 3.1

ii) $\widetilde{\phi}_4(P) < 3$: The proof of Theorem 3.4 is very similar to that of Theorem 3.2 with the inequality

$$f_{03} \le 4f_1 - 2(1 + \sqrt{1+8f_1}),$$

stated in Theorem 2.8. Calculate similarly to the Theorem 3.2. Then we have

$$\widetilde{\phi}_4 \le \frac{f_1 + \frac{1}{2}(4f_1 - 2(1+\sqrt{1+8f_1})) - \frac{1+\sqrt{1+8f_1}}{2}}{\frac{1+\sqrt{1+8f_1}}{2} + f_1}$$

$$= \frac{(6f_1 - 3) - 3\sqrt{1+8f_1}}{(2f_1+1) + \sqrt{1+8f_1}}, \quad f_1 \ge 10.$$

Let $h(x)$ be a function given by

$$h(x)=\frac{(6x-3)-3\sqrt{1+8x}}{(2x+1)+\sqrt{1+8x}},\ \ x\geq 10.$$

Then h is an increasing function and satisfies $h(10)=1$ and $1\leq h(x)<3$ (refer to Figure 3.2). More precisely, let us show that h is an increasing function. Note that $h'(x)$ is given by

$$h'(x)=\frac{B}{(2x+1+\sqrt{1+8x})^2}.$$

Here B is equal to

$$B=(6-\frac{12}{\sqrt{1+8x}})(2x+1+\sqrt{1+8x})-(2+\frac{4}{\sqrt{1+8x}})(6x-3-3\sqrt{1+8x})$$

$$=12x+12\sqrt{1+8x}-\frac{48x}{\sqrt{1+8x}}=12x+\frac{12+48x}{\sqrt{1+8x}}>0.$$

Thus for any $x>0$, we have $h'(x)>0$. Namely, the function $h(x)$ is an increasing function for any $x>0$.

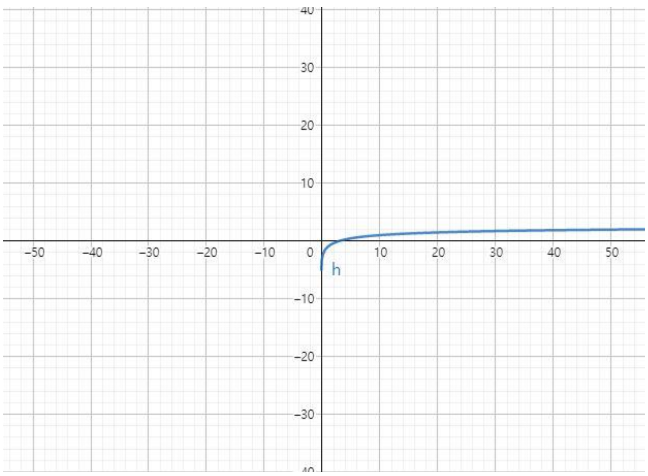

Figure 3.2

Therefore, we have $\widetilde{\phi_4}(P)\leq h(f_1)<3$. This completes the proof of Theorem 3.4. □

Theorem 3.5 Let P be a 4-polytope with a unique non-edge. Then the following inequalities hold:

$$0 < \widetilde{\phi}_4(P) < 3.$$

Proof. For the proof, we make use of the fact from Lemma 2.6 that if P is not simplicial, then P should be a 4-polytope with only one bipyramid facet and remaining tetrahedron facets. As before, let t denote the number of all tetrahedral facets of P.

Assume first that P is not simplicial. Then we have

$$f_3(P) = 1 + t, \;\; f_2(P) = 2t + 3, \;\; f_{03}(P) = 5 + 4t.$$

Thus it follows from the relation

$$\begin{aligned} 6t + 9 = f_{02}(P) &= -2f_0(P) + 2f_1(P) + f_{03}(P) \\ &= -2f_0(P) + 2f_1(P) + 5 + 4t \end{aligned}$$

that we have

$$f_1(P) = f_0(P) + t + 2.$$

Hence, since $f_0 \geq 5$ and $t \geq 5$ by Theorem 2.1, it is straightforward to show that

$$\begin{aligned} 0 < \widetilde{\phi}_4(P) = \frac{f_2 + f_3}{f_0 + f_1} &= \frac{2t + 3 + 1 + t}{f_0(P) + f_0(P) + 2 + t} = \frac{3t + 4}{2f_0(P) + t + 2} \\ &= 3 + \frac{-6f_0(P) - 2}{2f_0 + t + 2} < 3. \end{aligned}$$

Assume next that P is simplicial. Then, it is easy to obtain

$$f_3(P) = t, \;\; f_2(P) = 2t, \;\; f_{03}(P) = 4t, \;\; f_1(P) = f_0(P) + t.$$

Thus, since $f_0 \geq 5$ and $t \geq 5$, once again we have

$$0 < \widetilde{\phi}_4(P) = \frac{f_2 + f_3}{f_0 + f_1} = \frac{3t}{2f_0 + t} = 3 + \frac{-6f_0}{2f_0 + t} < 3.$$

This complete the proof of Theorem 3.5. □

Theorem 3.6 Let P be a 4-polytope with two non-edge. Then the following inequalities hold:

$$0 < \widetilde{\phi}_4(P) < 3.$$

Proof. 1) Assume first that P is simplicial. Let t denote the number of all tetrahedral facets of P. Then it is easy to obtain

$$f_3(P)=t,\ f_2(P)=2t,\ f_{03}(P)=4t,\ f_1(P)=f_0(P)+t.$$

Indeed, if follows from the identity

$$f_{02}(P)=-2f_0(P)+2f_1(P)+f_{03}(P)$$

and $f_{02}(P)=3f_2(P)=6t$ that we have $6t=-2f_0(P)+2f_1(P)+4t$. Thus, we have $f_1(P)=f_0(P)+t$. Note that $f_0\geq 5$ and $t\geq 5$. Hence we have

$$0<\widetilde{\phi}_4(P)=\frac{f_2+f_3}{f_0+f_1}=\frac{3t}{2f_0+t}=3+\frac{-6f_0}{2f_0+t}<3.$$

2) Assume next that P is not simplicial. Then, since P is assumed to have only two non-edge, we have two cases to consider:

① There are exactly two bipyramids as facets such that each bipyramid contains exactly one non-edge.

② There are exactly two square pyramids as facets such that two apices are connected. by one edge.

In case of ①, P is a 4-polytope with exactly two bipyramid facets and other remaining tetrahedral facets. Hence, we have

$$f_3(P)=t+2,\ f_2(P)=2t+6,\ f_{03}(P)=4t+10.$$

By the formula $f_{02}(P)=-2f_0(P)+2f_1(P)+f_{03}(P)$, we have

$$f_{02}(P)=3f_2(p)=6t+18=-2f_0(P)+2f_1(P)+4t+10.$$

Thus $f_1(P)=f_0(P)+t+4$. Hence, it is easy to obtain

$$0<\widetilde{\phi}_4(P)=\frac{f_2+f_3}{f_0+f_1}=\frac{2t+6+t+2}{f_0(P)+f_0(P)+t+4}$$

$$=\frac{3t+8}{2f_0(P)+t+4}=3+\frac{-6f_0-4}{2f_0+t+4}<3.$$

In case of ②, P is a 4-polytope with exactly two square pyramid facets and other tetrahedral facets. Note also that we have

$$f_3(P)=t+2,\ f_2(P)=2t+4,\ f_{03}(P)=4t+10,$$

$f_{02}(P)=6t+12=-2f_0(P)+2f_1(P)+4t+10$, and also $f_1(P)=f_0(P)+t+1$.

Therefore, it is easy to show

$$0<\widetilde{\phi}_4(P)=\frac{f_2+f_3}{f_0+f_1}=\frac{2t+4+t+2}{f_0(P)+f_0(P)+t+1}$$

$$=\frac{3t+6}{2f_0(P)+t+1}=3+\frac{-6f_0+3}{2f_0+t+1}<3.$$

This completes the proof of Theorem 3.6. □

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