[UCI]I804:24011-200000360180

2021학년도 2월
교육학석사(수학교육)학위논문

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이 논문을 교육학석사(수학전공)학위 청구논문으로 제출함

2020년 10월

조선대학교 교육대학원

수학교육전공

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설유미의 교육학 석사학위 논문을 인준함

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2020년 12월

조선대학교 교육대학원

목 차

국문초록

플래그벡터 순서쌍 (f_0, f_{02})를 만족하는 4차원 다면체의 구체적인 예의 나열에 관한 연구

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2018년에 Sjöberg와 Ziegler는 4차원 다면체의 플래그벡터 순서쌍 (f_0, f_{03})을 완벽하게 결정하는 연구결과를 발표하였다. Sjöberg와 Ziegler는 이 연구결과를 얻기 위해 Altshuler와 Steinberg의 최대 8개의 꼭짓점을 갖는 4차원 다면체에 관한 연구결과를 사용하였다. 또한 이를 바탕으로 2019년에 Kim and Park는 Sjöberg와 Ziegler의 연구결과를 심도 있게 이해하고, 기존의 연구 방법을 확장하여 4차원 다면체의 플래그벡터 순서쌍 (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), (f_1, f_{03})의 범위에 관한 새로운 결과를 제시하였다. 이에 본 논문은 Kim과 Park의 결과를 필요충분조건으로 확장하기 위한 기초 연구를 위해 Kim과 Park의 연구결과가 제시하는 범위를 만족하는 구체적인 예를 찾고 나열하였다. 그 결과, $m = 3f_0(f_0 - 3) - f_{02}$의 값이

$$9, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28$$

로 나타남을 확인하였고, 특히 $m = 9$를 만족하는 4차원 다면체의 구체적인 예가 존재함을 보였다.

I. Introduction

For a d-dimensional polytope P (or, in short, d-polytope), let $f_i = f_i(P)$ denote the number of i-dimensional faces of P for $0 \leq i \leq d-1$. Then the f-vector of P is defined to

$$(f_0(P), f_1(P), \ldots, f_{d-1}(P)).$$

It is one of the well-known and fundamental combinatorial invariants of polytopes, which is our major concern of thesis.

In order to understand geometric properties of a given polytope more deeply, one can generalize the concept of f-vectors in various ways. One of them is to use the notion of the so-called flag vector, which is another useful combinatorial invariant for convex polytopes. To be more precise, let S be a subset of $\{0,1,2,\ldots,d-1\}$, and let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \ \ldots \dim F_r\} = S.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0,1,2,\ldots d-1\}$. For example, $f_{02}(P)$ will mean $f_{\{0,2\}}(P)$. The flag vector of P is defined to be

$$(f_S)_{S \subseteq \{0, \ \ldots, \ d-1\}}.$$

Then clearly the f-vector $f(P)$ is just a vector which is formed by some part of components of the whole flag vector $(f_S)_{S \subseteq \{0, \ \ldots, \ d-1\}}$. That is, for example, if we take $S = \{i\}$ for each $0 \leq i \leq d-1$, then we have

$$f_S(P) = f_i(P).$$

Recently, in [13] Sjöberg and Ziegler has proved surprising results that completely determine the flag vector pair (f_0, f_{03}) of any 4-dimensional polytopes. In order to obtain such results, they crucially used Altshuler and Steinberg's results of a 4-dimensional polytopes with up to 8 vertices([2], [3]). Furthermore, Sjöberg and Ziegler used methods such as stacking, general stacking on cyclic polytopes, facet splitting, and truncating to find out the structure of specific 4-dimensional polytopes.

Based on the results of Sjöberg and Ziegler, in [10] Kim and Park proved some necessary conditions for the ranges of flag vector pairs such as $(f_{0,}f_{02})$, $(f_{02,}f_{03})$, $(f_{1,}f_{02})$, $(f_{1,}f_{03})$ of 4-dimensional polytopes. However, currently it is not clear that their results give rise to necessary and sufficient conditions for flag vector pairs $(f_{0,}f_{02})$, $(f_{02,}f_{03})$, $(f_{1,}f_{02})$, $(f_{1,}f_{03})$ to be satisfied by 4-dimensional polytopes.

In view of this, the goal of this thesis is to enumerate various and concrete examples of 4-dimensional polytopes which satisfy necessary conditions for the ranges of flag vector pair $(f_{0,}f_{02})$ proved by Kim and Park in [10]. This will provide some initial step towards a necessary and sufficient condition for the range of flag vector pair $(f_{0,}f_{02})$ as well as the validity for the results given in [10]. In order to achieve our goal, we make use of the examples of 4-dimensional polytopes with the number of vertices equal to 7 or 8 listed in the paper [13] of Sjöberg and Ziegler.

The thesis is organized as follows.

In Chapter 2, we first summarize some basic definitions, notation, and useful facts which are necessary for explaining our main results given in Chapters 3 and 4. We refer the reader to [1], [5], [7], [8], [9], [10], [11], [12], and [14] for more details.

In Chapter 3, we summarize some important results previously obtained by Kim and Park in [10] which are our main concern of this thesis.

Finally, in Chapter 4 we enumerate specific examples which satisfy the range of the flag vector pair (f_0, f_{02}), in detail.

Ⅱ. Preliminaries

This chapter sets up the notation and definitions used in this thesis. In addition, this chapter briefly describes some important facts used to understand this thesis.

A polytope refers to a shape that extends a shape such as a polygon or polyhedron to an arbitrary dimension. The polytope defined in n-dimension is called n-polytope. For example, polygons are 2-polytope, while polyhedra are 3-polytopes.

To be more precise, a convex polytope is the convex hull of a finite set of points in some Euclidean space $\mathbb{R}^n$. More generally, for $1 \le i \le m$ let l_i denote a linear functional in $\mathbb{R}^n$ and let a_i be a real number. Then a convex polyhedron P is an intersection of finitely many half-spaces in $\mathbb{R}^n$ given by

$$P = \left\{ p \in \mathbb{R}^n \mid < l_i, p > \ge -a_i,\ i = 1, 2, \ldots, m \right\}.$$

In this thesis, we will study only the 4-polytopes.

Recall that f_0 is the number of vertices, f_1 is the number of edges, f_{02} is the number of faces that make up the vertex, and f_{03} is the number of facets that make up the vertex.

Now we list some examples of polytopes with small polytopal pairs (f_0, f_{03}) for $f_{03} \le 80$ with simplex facet and/or simple vertex as in the paper [13] of Sjöberg and Ziegler, which will play an important role in finding some explicit examples which satisfy the results given in Theorem 3.1.

In Table 2.1 below, the second column explains the way to find the polytope. The polytopes P_i are ones with 7 or 8 vertices which are known from the well-known classification result of all polytopes with up to 8 vertices, and P_i^* denotes the dual of the polytope P_i.

We now explain some terminologies in Table 2.1. First, we begin with the definition of a cyclic polytope. To do so, let us define the moment curve in $\mathbb{R}^d$ by

$$\alpha : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto (t, t^2, \ldots, t^d) \in \mathbb{R}^d.$$

For any $n > d$, the standard d-th cyclic polytope with n vertices, denoted by $C_d(t_1, t_2, \ldots, t_n)$, is defined as the convex hull in R^d of n different points $\alpha(t_1), \ldots, \alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$. The set of all the faces (including the improper faces) of a (convex) polytope P is a partially ordered set (or poset), when partially ordered by inclusion. Two polytopes are said to be combinatorial equivalent, or of the same combinatorial type, if they have isomorphic face posets. Cyclic polytopes $C_d(n)$ are precisely those which are combinatorial equivalent to the standard cyclic polytope $C_d(t_1, t_2, \ldots t_n)$.

Let P be a 4-polytope with a facet F and let v be a point beyond F and beneath all other facets of P. Let Q be the convex hull of P and v, i.e., $Q = \mathrm{conv}(\{v\} \cup P)$. In this case, we say that Q is a 4-polytope obtained by stacking. Hence, for example, by skacking onto square pyramid P we can obtain a new 4-polytope Q which is the convex hull of P and a new vertex v.

On the other hand, a pyramid over triangular bipyramid just means the polytope obtained by taking the pyramid over a 3-dimensional triangular bipyramid.

For the facet splitting, consider a facet F of a 4-polytope P and a hyperplane H which intersects the relative interior of F in a polygon X.

If the only vertices of P happen to be simple vertices on one side of H, then we can obtain a new polytope Q by separating the facet F into two new facets by the polygon X. In this case, we say that Q is obtained from P by splitting a facet. So, for example, splitting bipyramid means that we obtain a new polytope by splitting one facet of bipyramid.

[Table 2.1] Some polytopal pairs

(f_0, f_{03})	Description	(f_0, f_{03})	Description
Polytopes with $\triangle_3$-facet and simple vertex		$(11,45)$	P_5^*
$(5,20)$	4-simplex	$(11,49)$	P_{13}^*
$(6,26)$	2-fold pyramid over quadrangle	$(11,52)$	dual of $(9,52)$
$(6,29)$	pyramid over triangular bipyramid	$(11,55)$	dual of $(10,55)$
$(7,29)$	pyramid over triangular prism	$(12,52)$	P_{14}^*
$(7,32)$	2-fold pyramid over pentagon	$(13,55)$	P_{15}^*
$(7,35)$	P_1	polytopes with $\triangle_3$-facet	
$(7,36)$	P_2	$(6,36)$	cyclic polytope $C_4(6)$
$(7,39)$	P_3	$(7,42)$	P_4
$(7,45)$	P_5	$(7,46)$	P_6
$(8,35)$	P_1^*	$(7,49)$	P_7
$(8,36)$	P_2^*	$(7,52)$	$R_2(6)$
$(8,38)$	2-fold pyramid over hexagon	$(7,56)$	cyclic polytope $C_4(7)$
$(8,39)$	P_8	$(8,43)$	P_{10}
$(8,42)$	P_9	$(8,60)$	P_{17}
$(8,45)$	P_{11}	$(8,63)$	P_{19}
$(8,46)$	P_{12}	$(8,65)$	P_{20}
$(8,49)$	P_{13}	$(8,66)$	P_{21}
$(8,52)$	P_{14}	$(8,68)$	P_{22}
$(8,55)$	P_{15}	$(8,69)$	P_{23}
$(8,59)$	P_{16}	$(8,70)$	P_{24}
$(8,62)$	P_{18}	$(8,72)$	P_{25}
$(9,39)$	P_3^*	$(8,73)$	P_{26}
$(9,42)$	P_9^*	$(8,76)$	P_{27}
$(9,45)$	split bipyramid in $(9,42)$	$(8,80)$	cyclic polytope $C_4(8)$
$(9,46)$	split bipyramid in $(9,43)$	$(9,79)$	stack onto square pyramid in $(8,63)$
$(9,49)$	split bipyramid in $(9,46)$	Polytopes with simple vertex	
$(9,52)$	stack onto square pyramid in $(8,36)$	$(9,36)$	dual of cyclic polytope $C_4(6)$
$(10,45)$	P_{11}^*	$(9,43)$	P_{10}^*
$(10,46)$	P_{12}^*	$(10,42)$	P_4^*
$(10,49)$	dual of $(9,49)$	$(11,46)$	P_6^*
$(10,52)$	split bipyramid in $(10,49)$	$(12,49)$	P_7^*
$(10,55)$	stack onto square pyramid in $(9,39)$	$(13,52)$	$R_2(6)^*$

[Table 2.2] list all polytopes P_i with 7 or 8 vertices from [Table 2.1] which were used in the construction of all possible flag vector pairs (f_0, f_{03}) in the paper [13] of Sjöberg and Ziegler. They will be used in finding specific examples in Chapter 4 that satisfy the results proved by Kim and Park in [10].

The polytopes in [Table 2.2] are listed by their facet list. In fact, in [6] Fukuda, Miyata, and Moriyama provide a complete list of all 31 polytopes with 7 vertices and 1294 polytops with 8 vertices. The third column in [Table 2.2] such as $7.x$ means that the polytope can be found as the x-th polytope listed in the classification of 4-polytopes with 7 vertices.

[Table 2.2] 4-Polytopes P_i with 7 and 8 vertices

polytope	facet list	row
P_1	[654321] [65430] [6520] [6420] [5310] [5210] [4310] [4210]	7.3
P_2	[65432] [65431] [65210] [64210] [5320][5310] [4320] [4310]	7.21
P_3	[65432] [65431] [65210] [6421] [5320] [5310] [4320] [4310] [4210]	7.22
P_4	[65432] [65410] [6531] [6431] [5420] [5321] [5210] [4320] [4310] [3210]	7.11
P_5	[65432] [6541] [6531] [6431] [5421] [5320] [5310] [5210] [4320] [4310] [4210]	7.16
P_6	[65432] [65431] [6521] [6420] [6410] [6210] [5320] [5310] [5210] [4320] [4310]	7.24
P_7	[65432] [6541] [6531] [6430] [6410] [6310] [5421] [5320] [5310] [5210] [4320] [4210]	7.13
P_8	[765432] [765410] [76321] [75310] [64210] [5430] [4320] [3210]	8.186
P_9	[765432] [76541] [76310] [75310] [64210] [6320] [5420] [5410] [5320]	8.285
P_{10}	[76543] [76542] [76321] [75310] [75210] [64310] [64210] [5430] [5420]	8.1145
P_{11}	[765432] [76541] [76310] [54310] [7531] [6421] [6320] [6210] [4320] [4210]	8.241
P_{12}	[765432] [76541] [76320] [75310] [54310] [7610] [6421] [6210] [4320] [4210]	8.353
P_{13}	[765432] [76541] [73210] [63210] [7631] [7520] [7510] [6420] [6410] [5420] [5410]	8.201
P_{14}	[765432] [76541] [76310] [7531] [6430] [6410] [5420] [5410] [5321] [5210] [4320] [3210]	8.306
P_{15}	[765432] [76510] [7641] [7541] [6530] [6421] [6321] [6310] [5420] [5410] [5320] [4210] [3210]	8.117
P_{16}	[76543] [76521] [76420] [7542] [6531] [6431] [6410] [6210] [5432] [5320] [5310] [5210] [4320] [4310]	8.676
P_{17}	[76543] [76542] [73210] [63210] [7632] [7531] [7520] [7510] [6431] [6420] [6410] [5431] [5420] [5410]	8.909
P_{18}	[76543] [76521] [7642] [7542] [6530] [6510] [6432] [6320] [6210] [5430] [5421] [5410] [4321] [4310] [3210]	8.778
P_{19}	[76543] [76542] [73210] [7631] [7621] [7530] [7520] [6431] [6420] [6410] [6210] [5431] [5420] [5410] [5310]	8.910
P_{20}	[76543] [7652] [7642] [7531] [7521] [7431] [7421] [6530] [6521] [6510] [6430] [6420] [6210] [5310] [4310] [4210]	8.805
P_{21}	[76543] [76542] [7632] [7531] [7521] [7320] [7310] [7210] [6431] [6420] [6410] [6320] [6310] [5431] [5421] [4210]	8.1227
P_{22}	[7654] [7653] [7643] [7542] [7532] [7431] [7421] [7321] [6540] [6530] [6431] [6410] [6310] [5420] [5320] [4210] [3210]	8.1262
P_{23}	[76543] [7652] [7642] [7531] [7521] [7431] [7421] [6530] [6521] [6510] [6430] [6420] [6210] [5310] [4321] [4320] [3210]	8.806
P_{24}	[76543] [76542] [7631] [7621] [7531] [7520] [7510] [7210] [6430] [6420] [6321] [6320] [5431] [5420] [5410] [4310] [3210]	8.1041
P_{25}	[7654] [7653] [7643] [7542] [7532] [7431] [7421] [7321] [6542] [6530] [6520] [6430] [6420] [5321] [5310] [5210] [4310] [4210]	8.1263
P_{26}	[76543] [7652] [7642] [7541] [7521] [7420] [7410] [7210] [6530] [6521] [6510] [6432] [6320] [6210] [5431] [5310] [4320] [4310]	8.815
P_{27}	[7654] [7653] [7643] [7542] [7532] [7431] [7421] [7321] [6542] [6530] [6520] [6431] [6420] [6410] [6310] [5321] [5310] [5210] [4210]	8.1266

Ⅲ. Results of Kim and Park revisited

In this section, we briefly review Theorem 3.1 of Kim and Park and its proof in [10] which is necessary for our concrete enumeration of flag vector pairs (f_0, f_{02}) for certain 4-polytopes.

Theorem 3.1 The flag vector pair $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ of a 4-polytope P satisfies the following two conditions:
(1) $30 \leq 6f_0 \leq f_{02} \leq 3f_0(f_0 - 3)$.
(2) $f_0 \geq 6$ and for $m \in \{1,2,3,4,5,7,8,10,11\}$, $f_{02} \neq 3f_0(f_0-3) - m$.

Lemma 3.2 The flag vector of every 4-polytope P satisfies the following identity

$$2f_0(P) - 2f_1(P) + f_{02}(P) - f_{03}(P) = 0.$$

Proof. For the proof, we apply the generalized Dehn-Sommerville equation [4] with $S = \{0\}$, $i = 0$, $k = 4$. Then we can obtain

$$\sum_{j=1}^{3} (-1)^{j-1} f_{0j} = f_0(1 - (-1)^{4-0-1}).$$

That is, we have

$$f_{01} - f_{02} + f_{03} = 2f_0.$$

By using the identity $f_{01} = 2f_1$, it is now straightforward to show

$$2f_0 - 2f_1 + f_{02} - f_{03} = 0,$$

as desired.

□

As an immediate consequence, we have the following result that is equivalent to Theorem 3.1 (1).

Proposition 3.3 The flag vector of every 4-polytope P satisfies the inequalities

$$30 \le 6f_0(P) \le f_{02}(P) \le 3f_0(P)(f_0(P)-3).$$

Proof. Recall that by a result of Sjöberg and Ziegler [13] we have

$$20 \le 4f_0 \le f_{03} \le 2f_0(f_0-3).$$

By combining the above inequalities with the identity given in Lemma 3.2, it is easy to obtain

$$2f_0(f_0-3) \ge f_{03} = 2f_0 - 2f_1 + f_{02}.$$

Hence, we can show

$$\begin{aligned} f_{02} &\le -2f_0 + 2f_1 + 2f_0(f_0-3) \\ &\le -2f_0 + f_0(f_0-1) + 2f_0(f_0-3) \\ &= 3f_0(f_0-3). \end{aligned} \tag{3.1}$$

Also, it follows from $f_{03} \ge 4f_0$ and $f_1 \ge 2f_0$ that the identity $f_{03} = 2f_0 - 2f_1 + f_{02}$ implies

$$f_{02} \ge 2f_0 + 2f_1 \ge 6f_0 \ge 30. \tag{3.2}$$

By (3.1) and (3.2), we now have

$$30 \le 6f_0 \le f_{02} \le 3f_0(f_0-3).$$

□

Note that $f_{02}(P) = 3f_0(P)(f_0(P)-3)$ if and only if P is neighborly. Thus, if $f_{02}(P) < 3f_0(P)(f_0(P)-3)$, then P is not neighborly, i.e.,

$$f_1(P) \le \frac{1}{2}f_0(P)(f_0(P)-1)-1 = \binom{f_0(P)}{2}-1. \tag{3.3}$$

Lemma 3.4 For each $m \in \{1,2,3,4,5\}$, we have

$$f_{02} \ne 3f_0(f_0-3)-m.$$

Proof. We prove this lemma by contradiction. So suppose

$$f_{02} = 3f_0(f_0-3)-m.$$

for some positive integer m with $1 \le m \le 5$. Then it follows from Sjöberg and Ziegler [13] that we have

$$3f_0(f_0-3)-m=f_{02}=-2f_0+2f_1+f_{03}$$
$$\leq -2f_0+2f_1+2f_0(f_0-3)-4.$$

This implies the following inequality

$$2f_1 \geq f_0(f_0-1)+4-m.$$

That is, we should have

$$f_1 \geq \frac{f_0(f_0-1)}{2}+\frac{4-m}{2}=\binom{f_0}{2}+\frac{4-m}{2}. \tag{3.4}$$

Note that, since $f_{02}(P)$ is assumed to be equal to $3f_0(P)(f_0(P)-3)-m$ for $1 \leq m \leq 5$, P is not neighborly. Thus, it follows from (3.3) that we have

$$f_1(P) < \binom{f_0(P)}{2}-1 \ . \tag{3.5}$$

By (3.4) and (3.5), it is now immediate to obtain

$$\binom{f_0}{2}-\frac{m-4}{2} \leq f_1 < \binom{f_0}{2}-1.$$

Therefore, if $1 \leq m \leq 5$, then we should have

$$\binom{f_0}{2}-1 < \binom{f_0}{2}-\frac{1}{2} \leq f_1 < \binom{f_0}{2}-1.$$

Clearly this is a contradiction.

□

The cases of $m=7,8,10,11$ can be dealt with separately in a similar fashion as above, so that we leave the proofs of those cases to a reader (refer to the paper [10] of Kim and Park for more details).

By combining all of the results obtained so far, we can finish the proof of Theorem 3.1.

□

Note that the bipyramid P over the tetrahedron contains a unique non-edge so that P satisfies

$$(f_0(P), f_1(P), f_{02}(P), f_{03}(P)) = (6, 14, 48, 32) \text{ and } f_{02} = 3f_0(f_0 - 3) - 6.$$

Hence, there exists a 4-polytope for which $m=6$ in Theorem 3.1 is actually attained. This corrects the statement of Theorem 3.1 given in [10], where the case of $m=6$ was also ruled out.

Ⅳ. Examples of 4-polytopes with (f_0, f_{02}) : $P_k\ (16 \le k \le 26)$

The aim of this chapter is to find some concrete examples satisfying the results obtained in the paper [10] of Kim and Park, and thus to provide some positive evidence for their results. In order to achieve our goal, we make use of the examples listed by Fukuda, Miyata, and Moriyama in [6].

In order to label all of the vertices of a 4-polytope with a given facet list, we first label a bipyramid over a triangle or a square pyramid in such a way that the labeling of the remaining tetrahedra fits well with that of a bipyramid over a triangle or a square pyramid. Note also that a 4-polytope with a given facet list cannot have only one square pyramid as a facet. This is because the square pyramid has a square as a facet, while a tetrahedron or a bipyramid over a triangle has only a triangle as a facet.

4.1 P_{16} case

In this section, we deal with P_{16} case. For 8 eight vertices labeled with 0,1,2,⋯,7, P_{16} is a 4-polytope with the following facet list:

[76543] [76521] [76420] [7542] [6531] [6431] [6410]
[6210] [5432] [5320] [5310] [5210] [4320] [4310]

(see Table 2.2 for more details). Thus it has 11 tetrahedra and 3 bipyramids over a triangle or 3 square pyramids. It turns out that there are eight possibilities for P_{16} with such a facet list.

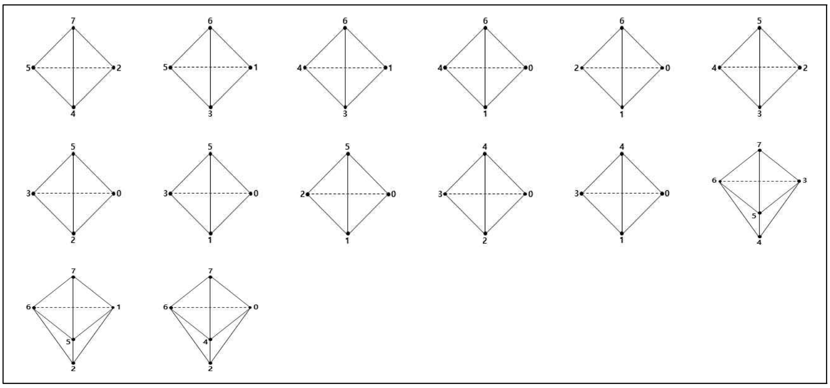
Below, we list all eight possibilities for P_{16} with the facet list, and by explicitly calculating the value

$$m = 3f_0(f_0 - 3) - f_{02},$$

we show that each case fits well with Theorem 3.1 of Kim and Park and thus supports Theorem 3.1 positively. We will explain how to obtain the value m only for the first case, in detail, and leave the details of other cases to a reader.

(1) $f_1(P_{16}) = 28$ with the followings edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,$$
$$14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 67, 37, 17, 07$$



[Figure 4.1] The first case of all facets of P_{16}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 28,$$
$$f_{02}(P_{16}) = -2f_0(P_{16}) + 2f_1(P_{16}) + f_{03}(P_{16}),$$

Thus it is easy to obtain

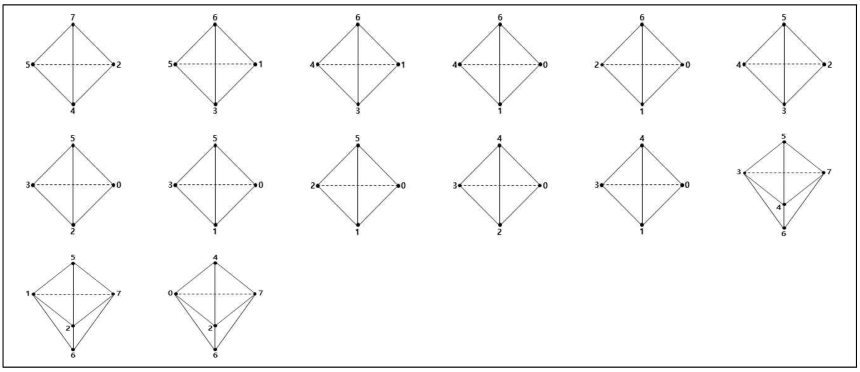
$$f_{02}(P_{16}) = -2\times 8 + 2\times 28 + 59 = 99.$$

It follows from the equation $f_{02}(P_{16}) = 3f_0(P_{16})(f_0(P_{16})-3) - m$ that we have $m = 21$. Consequently, this case provides an example P_{16} of a 4-polytope which satisfies

$$f_{02}(P_{16}) = 3f_0(P_{16})(f_0(P_{16})-3) - 21.$$

(2) $f_1(P_{16}) = 27$ with the following edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,$$
$$14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 67, 17, 07$$



[Figure 4.2] The second case of all facets of P_{16}

In this case, we have

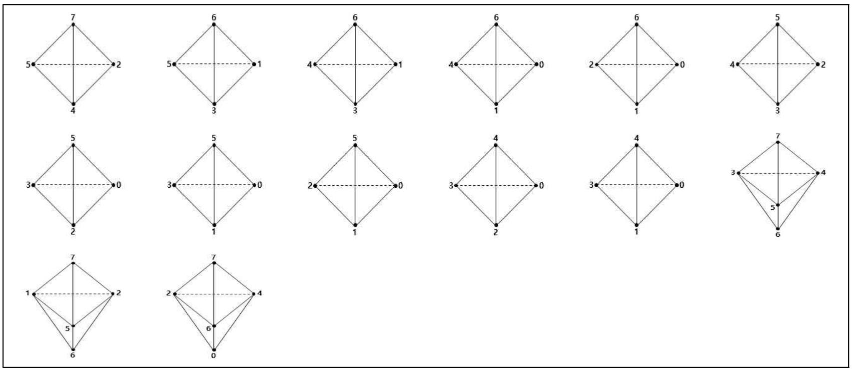
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 27.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16}) + 2f_1(P_{16}) + f_{03}(P_{16}) = 97$, and

$$m = 23 = 3f_0(P_{16})(f_0(P_{16}) - 3) - f_{02}(P_{16}).$$

(3) $f_1(P_{16}) = 27$ with the following edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,$$
$$14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 37, 17, 67$$



[Figure 4.3] The third case of all facets of P_{16}

In this case, we have

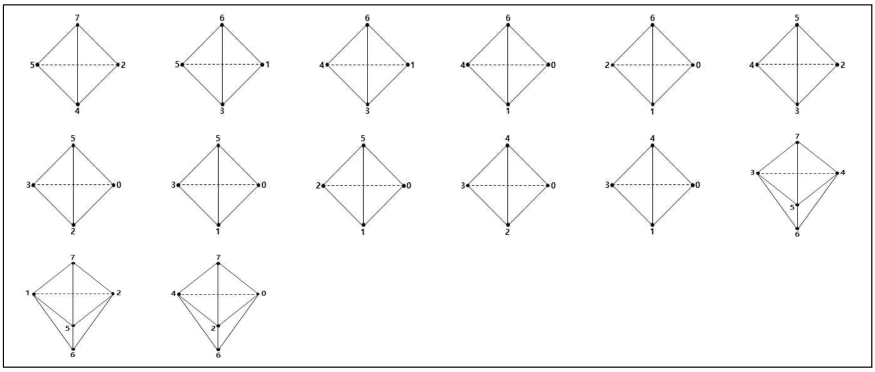
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 27.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16}) + 2f_1(P_{16}) + f_{03}(P_{16}) = 97$, and

$$m = 23 = 3f_0(P_{16})(f_0(P_{16}) - 3) - f_{02}(P_{16}).$$

(4) $f_1(P_{16}) = 27$ with the following edges:

$$\begin{aligned}&57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,\\&14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 37, 17, 07\end{aligned}$$



[Figure 4.4] The fourth case of all facets of P_{16}

In this case, we have

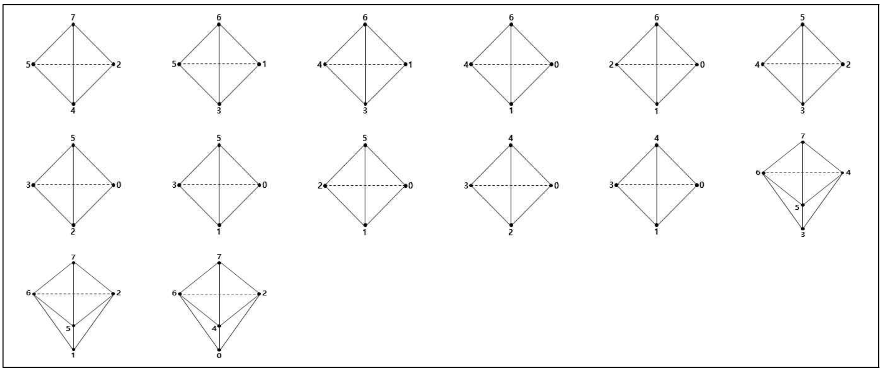
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 27.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16}) + 2f_1(P_{16}) + f_{03}(P_{16}) = 97$, and

$$m = 23 = 3f_0(P_{16})(f_0(P_{16}) - 3) - f_{02}(P_{16}).$$

(5) $f_1(P_{16}) = 25$ with the following edges:

$$\begin{aligned}&57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46,\\&34, 14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 67\end{aligned}$$



[Figure 4.5] The fifth case of all facets of P_{16}

In this case, we have

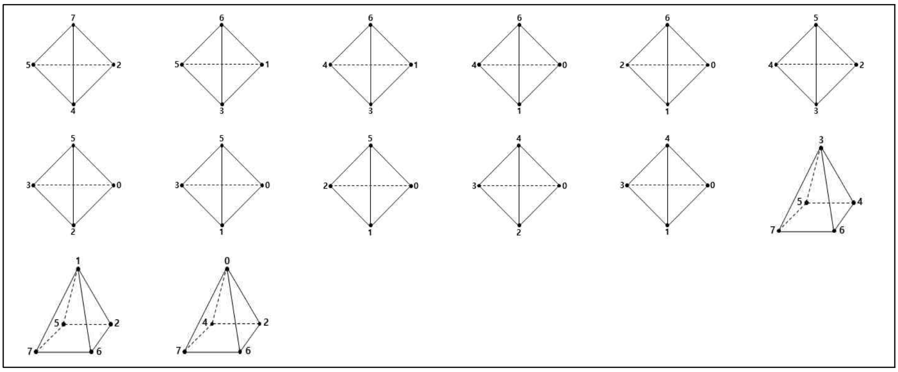
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 25.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16})+2f_1(P_{16})+f_{03}(P_{16})=93$, and

$$m=27=3f_0(P_{16})(f_0(P_{16})-3)-f_{02}(P_{16}).$$

(6) $f_1(P_{16}) = 28$ with the following edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,$$
$$14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 67, 37, 17, 07$$



[Figure 4.6] The sixth case of all facets of P_{16}

In this case, we have

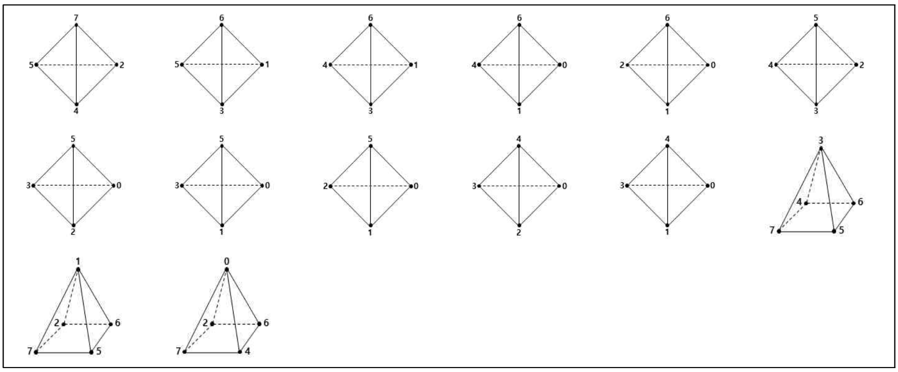
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 28.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16})+2f_1(P_{16})+f_{03}(P_{16})=99$, and

$$m=21=3f_0(P_{16})(f_0(P_{16})-3)-f_{02}(P_{16}).$$

(7) $f_1(P_{16}) = 27$ with the following edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46, 34,$$
$$14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 37, 17, 07$$



[Figure 4.7] The seventh case of all facets of P_{16}

In this case, we have

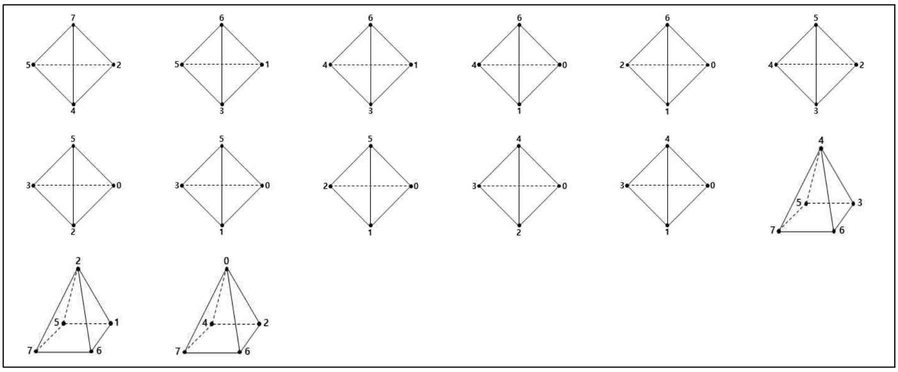
$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 27.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16})+2f_1(P_{16})+f_{03}(P_{16})=97$, and

$$m=23=3f_0(P_{16})(f_0(P_{16})-3)-f_{02}(P_{16}).$$

(8) $f_1(P_{16}) = 26$ with the following edges:

$$57, 47, 27, 45, 25, 24, 56, 36, 16, 35, 15, 13, 46,$$
$$34, 14, 06, 04, 01, 26, 12, 02, 23, 05, 03, 67, 07$$



[Figure 4.8] The eighth case of all facets of P_{16}

In this case, we have

$$f_0(P_{16}) = 8,\ f_{03}(P_{16}) = 59,\ f_1(P_{16}) = 26.$$

Thus, $f_{02}(P_{16}) = -2f_0(P_{16})+2f_1(P_{16})+f_{03}(P_{16})=95$, and

$$m=25=3f_0(P_{16})(f_0(P_{16})-3)-f_{02}(P_{16}).$$

All of the above cases are consistent with the results of the thesis [10] of Kim and Park. According to the results of the paper [13] of Sjöberg-Ziegler, if $f_0=8$, then f_1 should be equal to 28, which is the maximum possible value. But in this case f_{02} as well as f_{03} must also have a maximum value. Note that the maximum value of f_{02} is 120, while the maximum value of f_{03} is 80. The first and sixth cases are ones which have $f_1=28$. But our computations show that in their cases both f_{02} and f_{03} do not have the maximum value. This implies that the first and sixth cases should be excluded from our results. Furthermore, notice that the second, third, fourth, and seventh cases have the same value m that is equal to 23. Therefore our results can be summarized, as follows.

[Table 4.1] P_{16}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{16}	8	25	93	59	$m=27$
	8	26	95	59	$m=25$
	8	27	97	59	$m=23$

4.2 P_{17} case

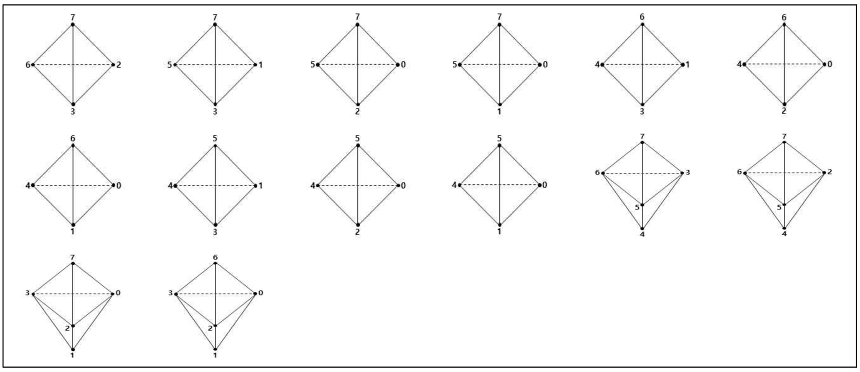
In this section, we deal with P_{17} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{17} is a 4-polytope with the following facet list:

$$[76543]\,[76542]\,[73210]\,[63210]\,[7632]\,[7531]\,[7520]$$
$$[7510]\,[6431]\,[6420]\,[6410]\,[5431]\,[5420]\,[5410]$$

(see Table 2.2 for more details). Thus it has 10 tetrahedra and 4 bipyramids over a triangle or 4 square pyramids. It turns out that there are seven possibilities for P_{17} with such a facet list, as follows.

(1) $f_1(P_{17})=27$ with the followings edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25, 05,$$
$$02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 56, 03, 12$$



[Figure 4.9] The first case of all facets of P_{17}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 27,$$
$$f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}),$$

Thus it is easy to obtain

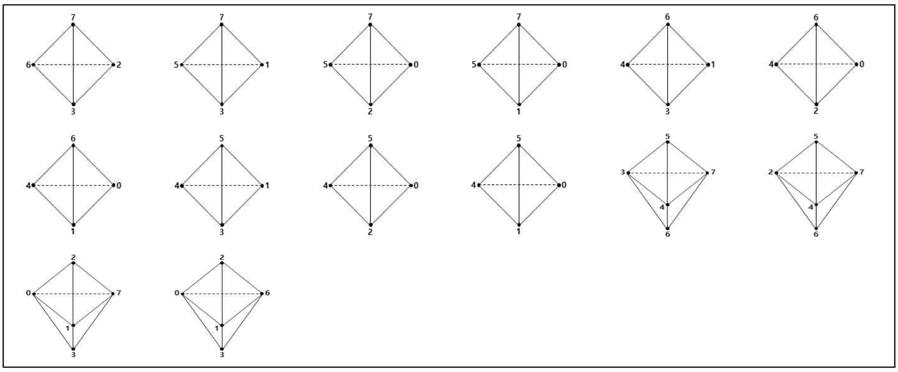
$$f_{02}(P_{17}) = -2\times 8 + 2\times 27 + 60 = 98.$$

It follows from the equation $f_{02}(P_{17}) = 3f_0(P_{17})(f_0(P_{17})-3)-m$ that we have $m=22$. Consequently, this case provides an example P_{17} of a 4-polytope which satisfies

$$f_{02}(P_{17}) = 3f_0(P_{17})(f_0(P_{17})-3)-22.$$

(2) $f_1(P_{17}) = 27$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25, 05,$$
$$02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 47, 03, 12$$



[Figure 4.10] The second case of all facets of P_{17}

In this case, we have

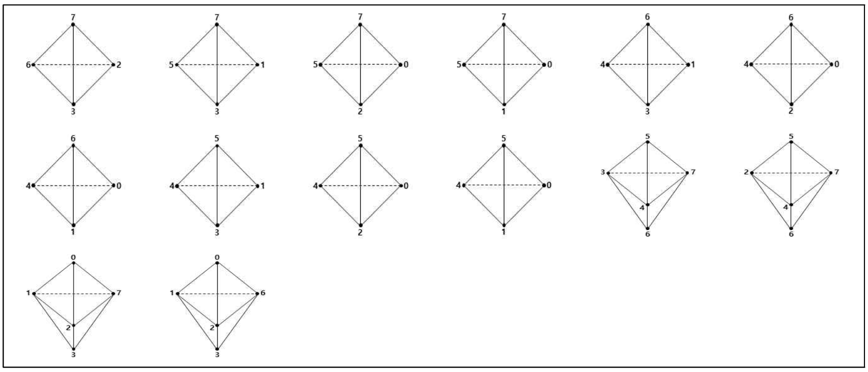
$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 27.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}) = 98$, and

$$m = 22 = 3f_0(P_{17})(f_0(P_{17}) - 3) - f_{02}(P_{17}).$$

(3) $f_1(P_{17}) = 26$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25,$$
$$05, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 47, 12$$



[Figure 4.11] The third case of all facets of P_{17}

In this case, we have

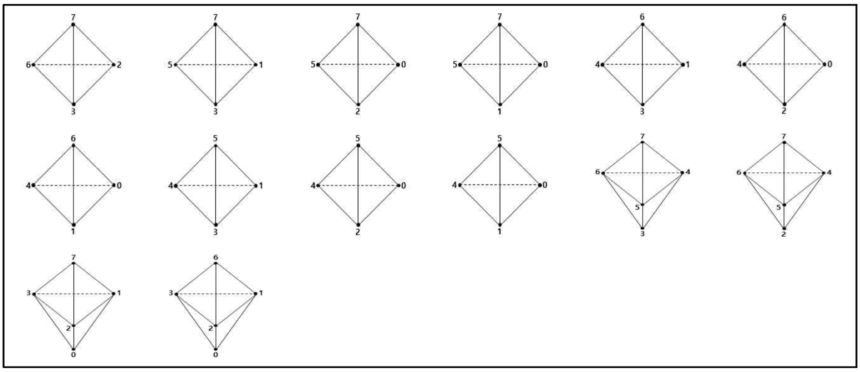
$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 26.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}) = 96$, and

$$m = 24 = 3f_0(P_{17})(f_0(P_{17}) - 3) - f_{02}(P_{17}).$$

(4) $f_1(P_{17}) = 28$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25, 05,$$
$$02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 47, 56, 03, 12$$



[Figure 4.12] The fourth case of all facets of P_{17}

In this case, we have

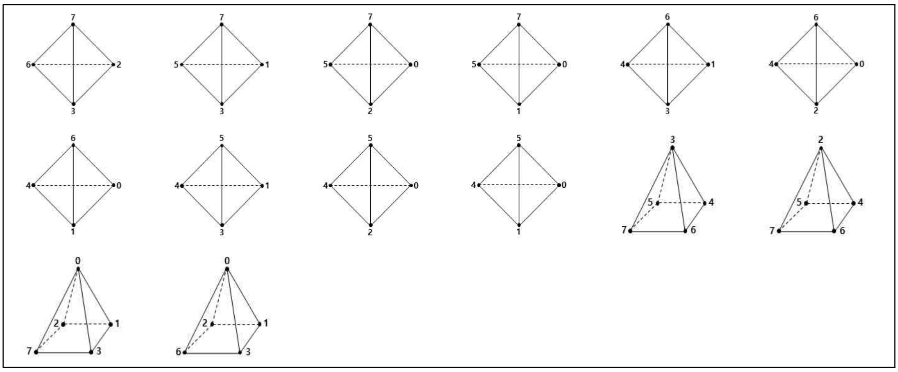
$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 28.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}) = 100$, and

$$m = 20 = 3f_0(P_{17})(f_0(P_{17}) - 3) - f_{02}(P_{17}).$$

(5) $f_1(P_{17}) = 26$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25,$$
$$05, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 03, 12$$



[Figure 4.13] The fifth case of all facets of P_{17}

In this case, we have

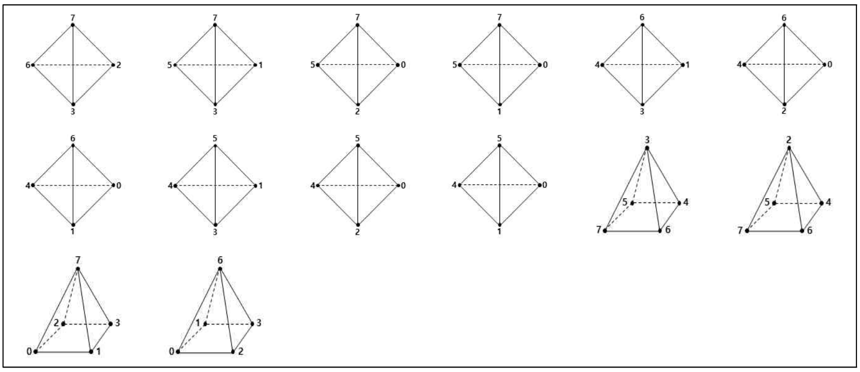
$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 26.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}) = 96$, and

$$m = 24 = 3f_0(P_{17})(f_0(P_{17}) - 3) - f_{02}(P_{17}).$$

(6) $f_1(P_{17}) = 24$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07,$$
$$25, 05, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45$$



[Figure 4.14] The sixth case of all facets of P_{17}

In this case, we have

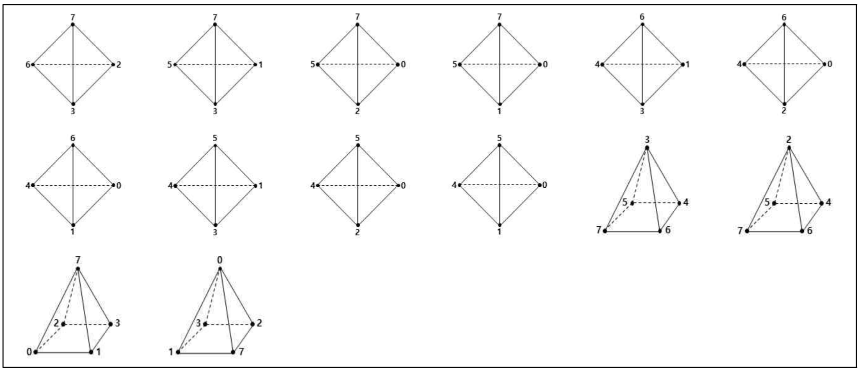
$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 24.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17}) + 2f_1(P_{17}) + f_{03}(P_{17}) = 92$, and

$$m = 28 = 3f_0(P_{17})(f_0(P_{17}) - 3) - f_{02}(P_{17}).$$

(7) $f_1(P_{17}) = 25$ with the following edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 07, 25,$$
$$05, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 03$$



[Figure 4.15] The seventh case of all facets of P_{17}

In this case, we have

$$f_0(P_{17}) = 8,\ f_{03}(P_{17}) = 60,\ f_1(P_{17}) = 25.$$

Thus, $f_{02}(P_{17}) = -2f_0(P_{17})+2f_1(P_{17})+f_{03}(P_{17})=94$, and

$$m=26=3f_0(P_{17})(f_0(P_{17})-3)-f_{02}(P_{17}).$$

These results can be summarized, as follows.

[Table 4.2] P_{17}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{17}	8	24	92	60	$m=28$
	8	25	94	60	$m=26$
	8	26	96	60	$m=24$
	8	27	98	60	$m=22$

4.3 P_{18} case

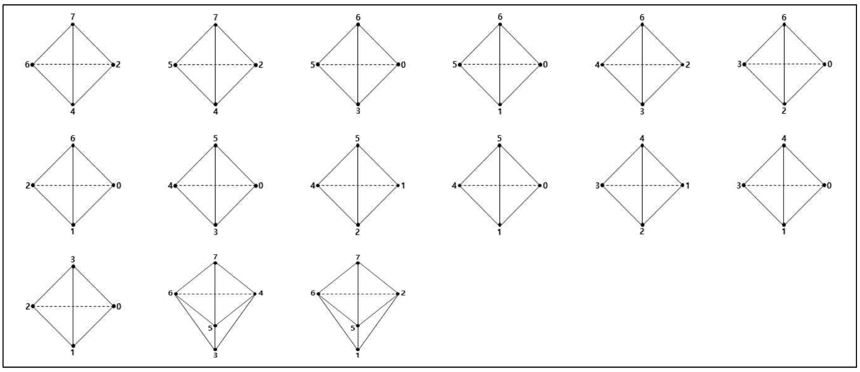
In this section, we deal with P_{18} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{18} is a 4-polytope with the following facet list:

[76543] [76521] [7642] [7542] [6530] [6510] [6432] [6320]
[6210] [5430] [5421] [5410] [4321][4310] [3210]

(see Table 2.2 for more details). Thus it has 13 tetrahedra and 2 bipyramids over a triangle or 2 square pyramids. It turns out that there are six possibilities for P_{18} with such a facet list, as follows.

(1) $f_1(P_{18})=25$ with the followings edges:

67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35,
05, 03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13



[Figure 4.16] The first case of all facets of P_{18}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 25,$$
$$f_{02}(P_{18}) = -2f_0(P_{18}) + 2f_1(P_{18}) + f_{03}(P_{18}),$$

Thus it is easy to obtain

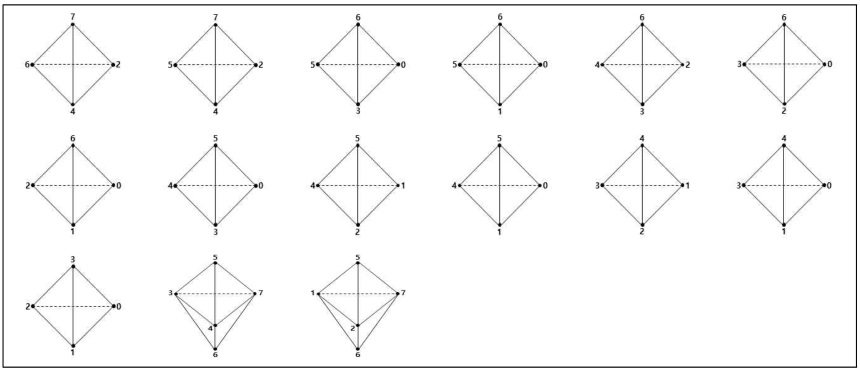
$$f_{02}(P_{18}) = -2\times 8 + 2\times 25 + 62 = 96.$$

It follows from the equation $f_{02}(P_{18}) = 3f_0(P_{18})(f_0(P_{18}) - 3) - m$ that we have $m = 24$. Consequently, this case provides an example P_{18} of a 4-polytope which satisfies

$$f_{02}(P_{18}) = 3f_0(P_{18})(f_0(P_{18}) - 3) - 24.$$

(2) $f_1(P_{18}) = 27$ with the followings edges:

$$67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35, 05,$$
$$03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13, 37, 17$$



[Figure 4.17] The second case of all facets of P_{18}

In this case, we have

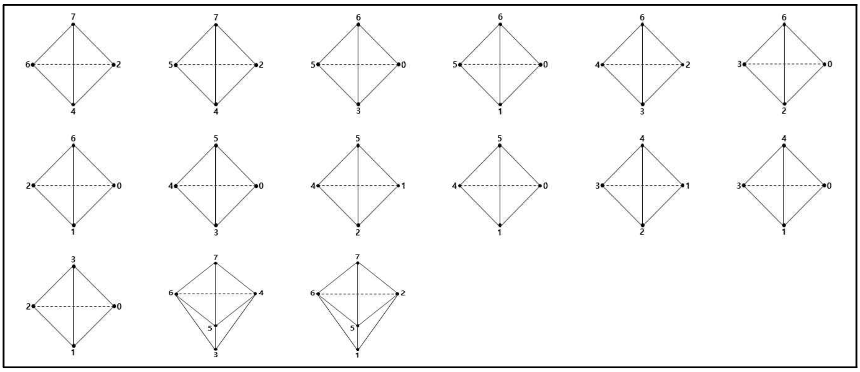
$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 27.$$

Thus, $f_{02}(P_{18}) = -2f_0(P_{18})+2f_1(P_{18})+f_{03}(P_{18})=100$, and

$$m=20=3f_0(P_{18})(f_0(P_{18})-3)-f_{02}(P_{18}).$$

(3) $f_1(P_{18}) = 25$ with the followings edges:

$$67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35,$$
$$05, 03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13$$



[Figure 4.18] The third case of all facets of P_{18}

In this case, we have

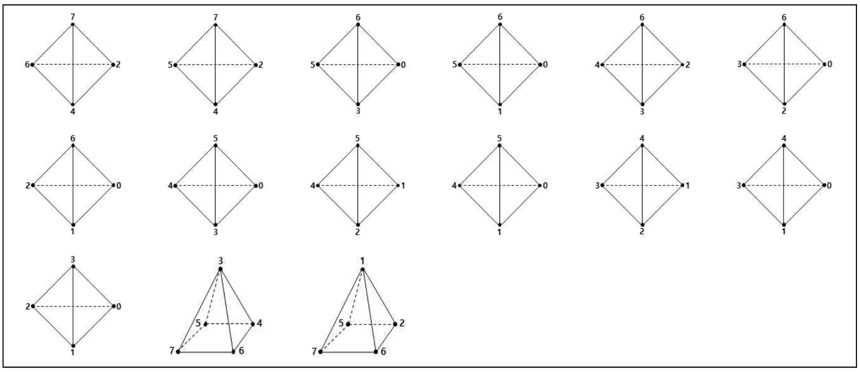
$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 25.$$

Thus, $f_{02}(P_{18}) = -2f_0(P_{18})+2f_1(P_{18})+f_{03}(P_{18})=96$, and

$$m=24=3f_0(P_{18})(f_0(P_{18})-3)-f_{02}(P_{18}).$$

(4) $f_1(P_{18}) = 27$ with the followings edges:

$$67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35, 05,$$
$$03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13, 37, 17$$



[Figure 4.19] The fourth case of all facets of P_{18}

In this case, we have

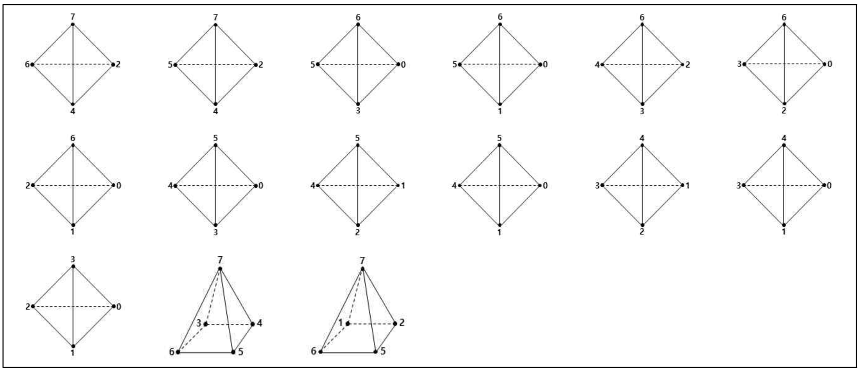
$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 27.$$

Thus, $f_{02}(P_{18}) = -2f_0(P_{18}) + 2f_1(P_{18}) + f_{03}(P_{18}) = 100$, and

$$m = 20 = 3f_0(P_{18})(f_0(P_{18}) - 3) - f_{02}(P_{18}).$$

(5) $f_1(P_{18}) = 25$ with the followings edges:

$$67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35,$$
$$05, 03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13$$



[Figure 4.20] The fifth case of all facets of P_{18}

In this case, we have

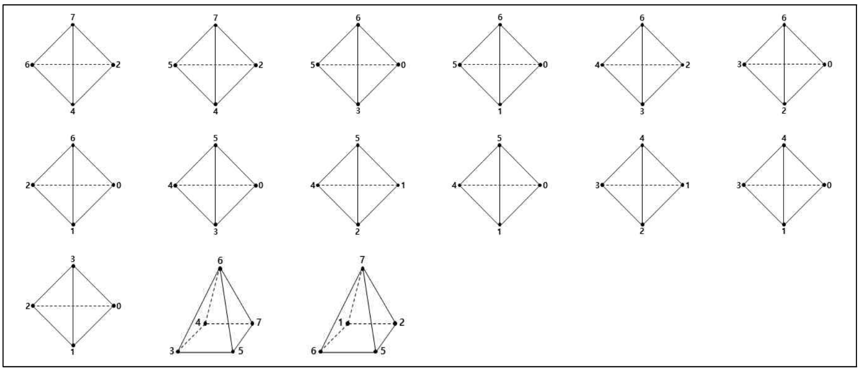
$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 25.$$

Thus, $f_{02}(P_{18}) = -2f_0(P_{18}) + 2f_1(P_{18}) + f_{03}(P_{18}) = 96$, and

$$m = 24 = 3f_0(P_{18})(f_0(P_{18}) - 3) - f_{02}(P_{18}).$$

(6) $f_1(P_{18}) = 26$ with the followings edges:

$$67, 47, 27, 46, 26, 24, 57, 45, 25, 56, 36, 06, 35,$$
$$05, 03, 16, 15, 01, 34, 23, 02, 12, 04, 14, 13, 17$$



[Figure 4.21] The sixth case of all facets of P_{18}

In this case, we have

$$f_0(P_{18}) = 8,\ f_{03}(P_{18}) = 62,\ f_1(P_{18}) = 26.$$

Thus, $f_{02}(P_{18}) = -2f_0(P_{18}) + 2f_1(P_{18}) + f_{03}(P_{18}) = 98$, and

$$m = 22 = 3f_0(P_{18})(f_0(P_{18}) - 3) - f_{02}(P_{18}).$$

These results can be summarized, as follows.

[Table 4.3] P_{18}

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_{18}	8	25	96	62	$m = 24$
	8	26	98	62	$m = 22$
	8	27	100	62	$m = 20$

4.4 P_{19} case

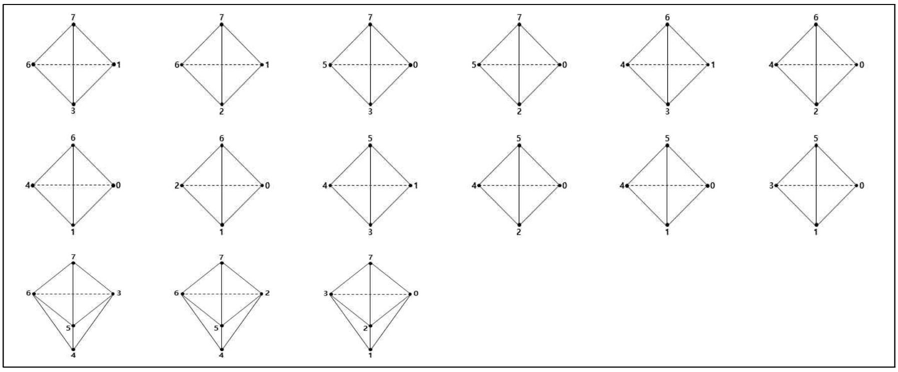
In this section, we deal with P_{19} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{19} is a 4-polytope with the following facet list:

[76543] [76542] [73210] [7631] [7621] [7530] [7520] [6431]
[6420] [6410] [6210] [5431] [5420] [5410] [5310]

(see Table 2.2 for more details). Thus it has 12 tetrahedra and 3 bipyramids over a triangle or 3 square pyramids. It turns out that there are four possibilities for P_{19} with such a facet list.

(1) $f_1(P_{19}) = 27$ with the followings edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 07, 35, 05, 03,$$
$$25, 02, 46, 34, 14, 06, 24, 04, 01, 45, 15, 56, 23$$



[Figure 4.22] The first case of all facets of P_{19}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{19}) = 8,\ f_{03}(P_{19}) = 63,\ f_1(P_{19}) = 27,$$
$$f_{02}(P_{19}) = -2f_0(P_{19}) + 2f_1(P_{19}) + f_{03}(P_{19}),$$

Thus it is easy to obtain

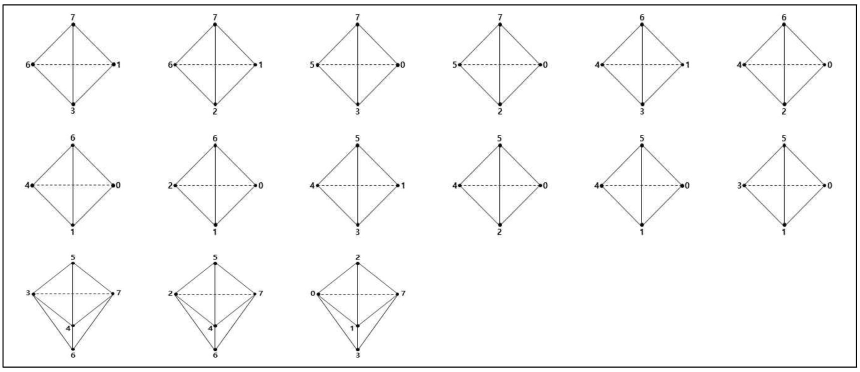
$$f_{02}(P_{19}) = -2\times 8 + 2\times 27 + 63 = 101\,.$$

It follows from the equation $f_{02}(P_{19}) = 3f_0(P_{19})(f_0(P_{19})-3)-m$ that we have $m=19$. Consequently, this case provides an example P_{19} of a 4 -polytope which satisfies

$$f_{02}(P_{19}) = 3f_0(P_{19})(f_0(P_{19})-3)-19.$$

(2) $f_1(P_{19}) = 26$ with the following edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 07, 35, 05,$$
$$03, 25, 02, 46, 34, 14, 06, 24, 04, 01, 45, 15, 47$$



[Figure 4.23] The second case of all facets of P_{19}

In this case, we have

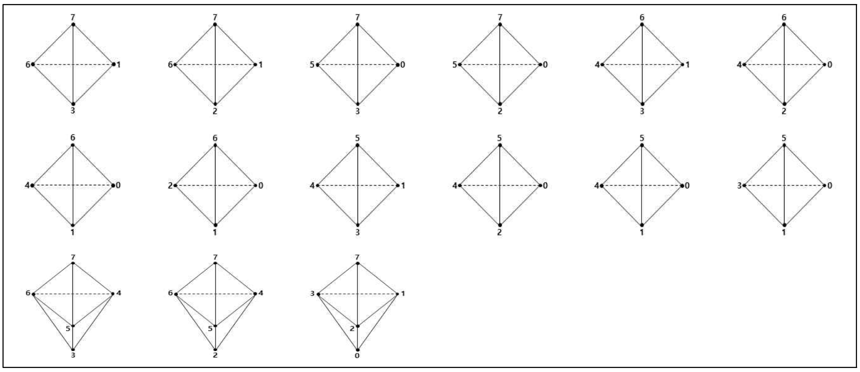
$$f_0(P_{19}) = 8,\ f_{03}(P_{19}) = 63,\ f_1(P_{19}) = 26.$$

Thus, $f_{02}(P_{19}) = -2f_0(P_{19}) + 2f_1(P_{19}) + f_{03}(P_{19}) = 99$, and

$$m = 21 = 3f_0(P_{19})(f_0(P_{19}) - 3) - f_{02}(P_{19}).$$

(3) $f_1(P_{19}) = 28$ with the following edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 07, 35, 05, 03,$$
$$25, 02, 46, 34, 14, 06, 24, 04, 01, 45, 15, 47, 56, 23$$



[Figure 4.24] The third case of all facets of P_{19}

In this case, we have

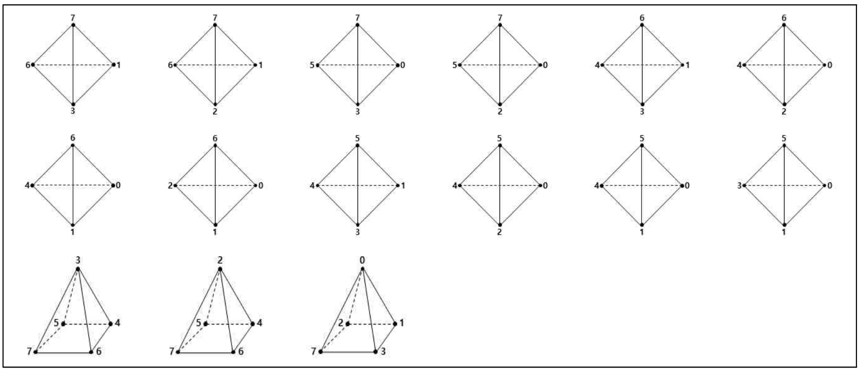
$$f_0(P_{19}) = 8,\ f_{03}(P_{19}) = 63,\ f_1(P_{19}) = 28.$$

Thus, $f_{02}(P_{19}) = -2f_0(P_{19}) + 2f_1(P_{19}) + f_{03}(P_{19}) = 103$, and

$$m = 17 = 3f_0(P_{19})(f_0(P_{19}) - 3) - f_{02}(P_{19}).$$

(4) $f_1(P_{19}) = 25$ with the following edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 07, 35, 05,$$
$$03, 25, 02, 46, 34, 14, 06, 24, 04, 01, 45, 15$$



[Figure 4.25] The fourth case of all facets of P_{19}

In this case, we have

$$f_0(P_{19}) = 8,\ f_{03}(P_{19}) = 63,\ f_1(P_{19}) = 25.$$

Thus, $f_{02}(P_{19}) = -2f_0(P_{19}) + 2f_1(P_{19}) + f_{03}(P_{19}) = 97$, and

$$m = 23 = 3f_0(P_{19})(f_0(P_{19}) - 3) - f_{02}(P_{19}).$$

These results can be summarized, as follows.

[Table 4.4] P_{19}

	f_0	f_1	f_{02}	f_{03}	$m = 3f_0(f_0 - 3) - f_{02}$
P_{19}	8	25	97	63	$m = 23$
	8	26	99	63	$m = 21$
	8	27	101	63	$m = 19$

4.5 P_{20} case

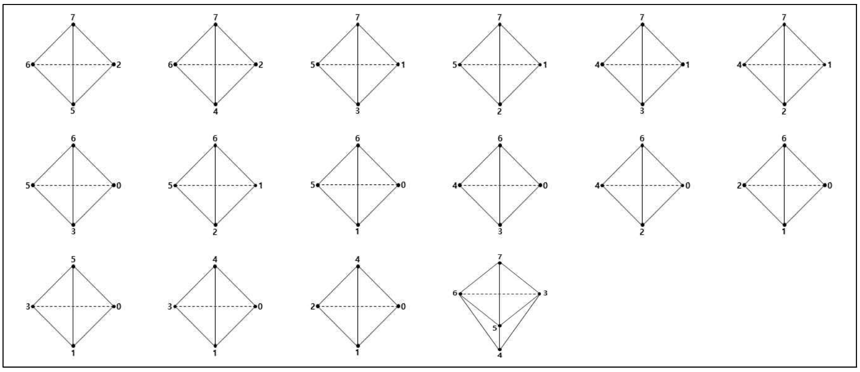
In this section, we deal with P_{20} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{20} is a 4-polytope with the following facet list:

$$[76543]\ [7652]\ [7642]\ [7531]\ [7521]\ [7431]\ [7421]\ [6530]$$
$$[6521]\ [6510]\ [6430]\ [6420]\ [6210]\ [5310]\ [4310]\ [4210]$$

(see Table 2.2 for more details). Thus it has 15 tetrahedra and 1 bipyramids over a triangle or 1 square pyramids. It turns out that there are twelve possibilities for P_{20} with such a facet list.

(1) $f_1(P_{20}) = 26$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.26] The first case of all facets of P_{20}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26,$$
$$f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}),$$

Thus it is easy to obtain

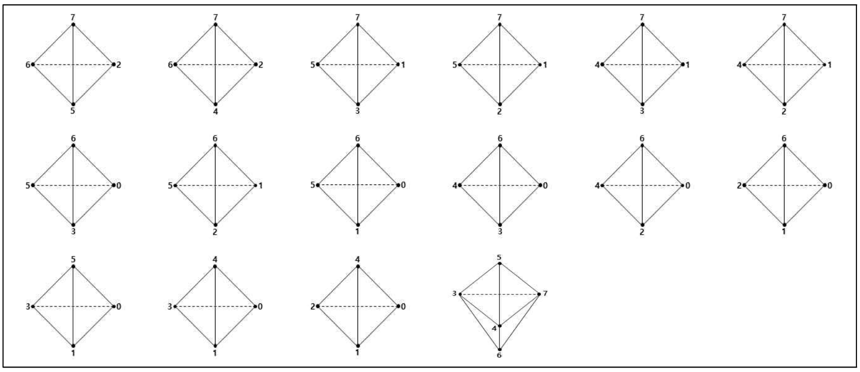
$$f_{02}(P_{20}) = -2\times 8 + 2\times 26 + 65 = 101\,.$$

It follows from the equation $f_{02}(P_{20}) = 3f_0(P_{20})(f_0(P_{20})-3) - m$ that we have $m=19$. Consequently, this case provides an example P_{20} of a 4 -polytope which satisfies

$$f_{02}(P_{20}) = 3f_0(P_{20})(f_0(P_{20})-3) - 19.$$

(2) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.27] The second case of all facets of P_{20}

In this case, we have

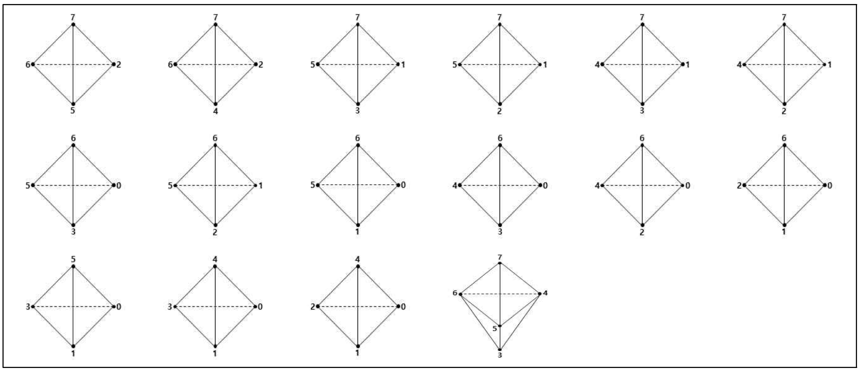
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(3) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.28] The third case of all facets of P_{20}

In this case, we have

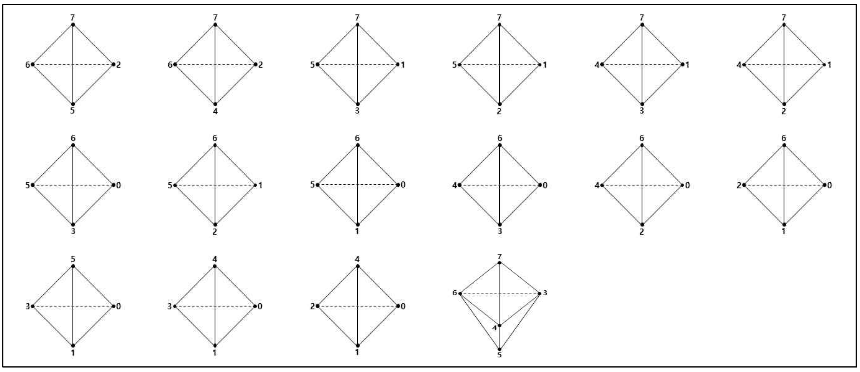
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(4) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.29] The fourth case of all facets of P_{20}

In this case, we have

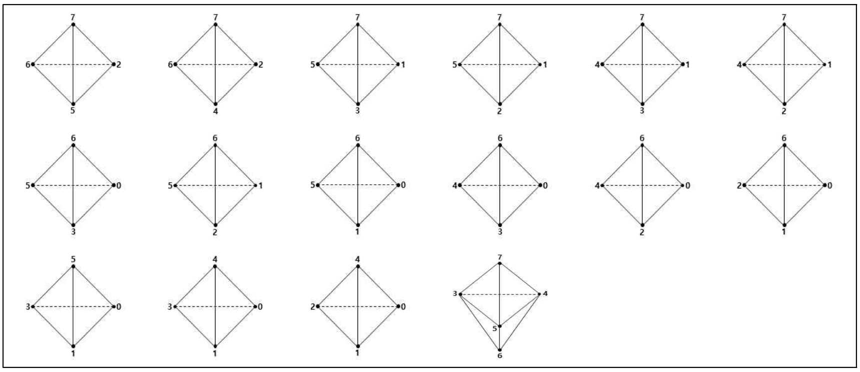
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(5) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.30] The fifth case of all facets of P_{20}

In this case, we have

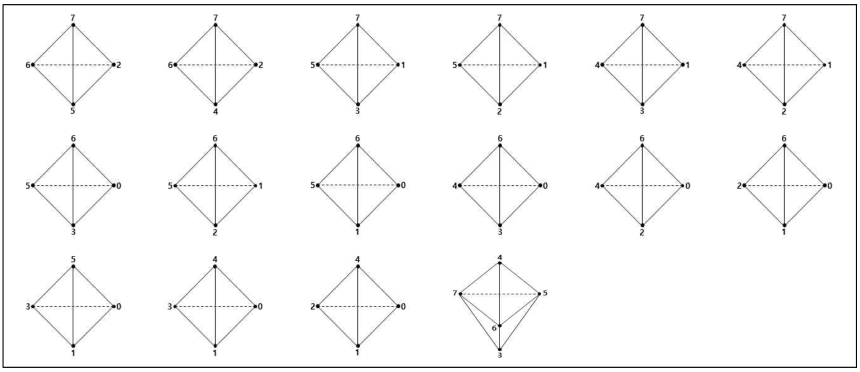
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20})+2f_1(P_{20})+f_{03}(P_{20})=101$, and

$$m=19=3f_0(P_{20})(f_0(P_{20})-3)-f_{02}(P_{20}).$$

(6) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.31] The sixth case of all facets of P_{20}

In this case, we have

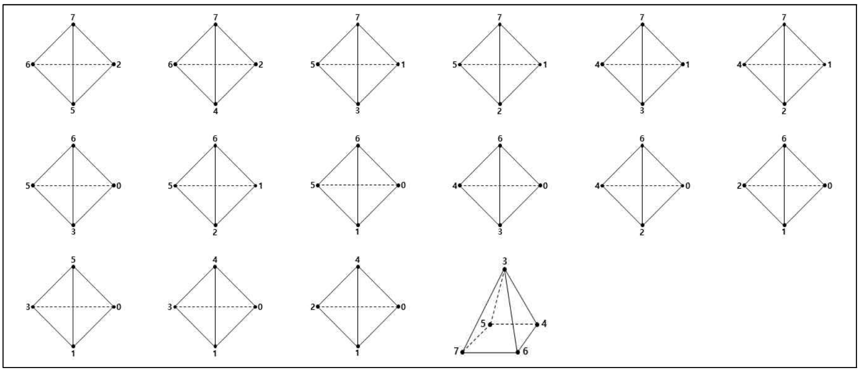
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20})+2f_1(P_{20})+f_{03}(P_{20})=101$, and

$$m=19=3f_0(P_{20})(f_0(P_{20})-3)-f_{02}(P_{20}).$$

(7) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.32] The seventh case of all facets of P_{20}

In this case, we have

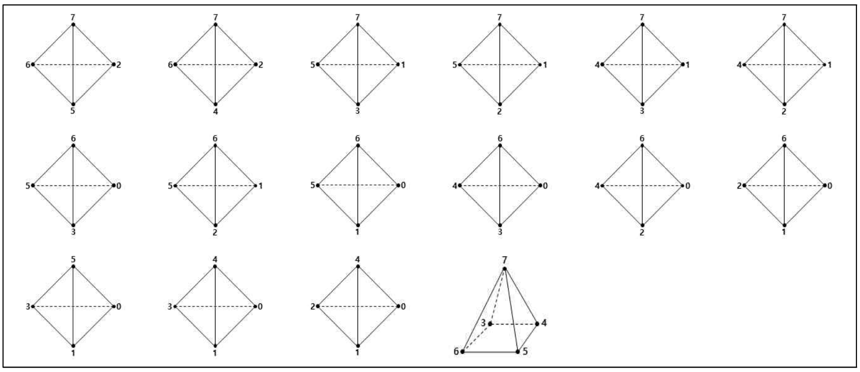
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20})+2f_1(P_{20})+f_{03}(P_{20})=101$, and

$$m=19=3f_0(P_{20})(f_0(P_{20})-3)-f_{02}(P_{20}).$$

(8) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.33] The eighth case of all facets of P_{20}

In this case, we have

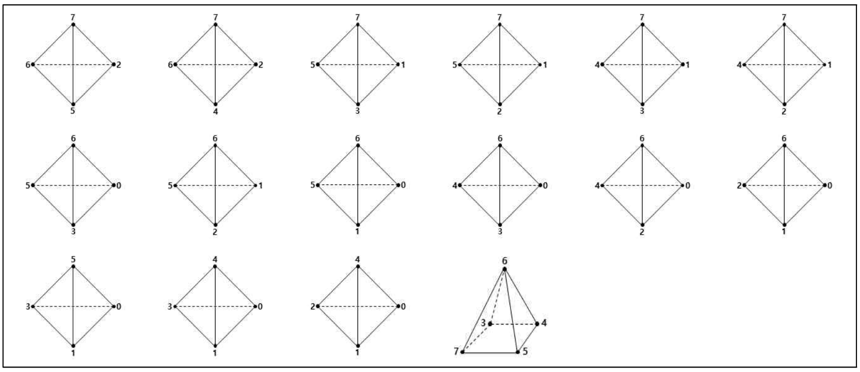
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20})+2f_1(P_{20})+f_{03}(P_{20})=101$, and

$$m=19=3f_0(P_{20})(f_0(P_{20})-3)-f_{02}(P_{20}).$$

(9) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.34] The ninth case of all facets of P_{20}

In this case, we have

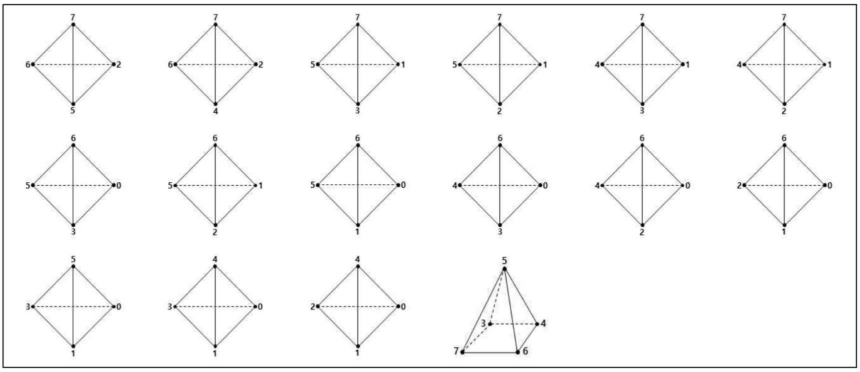
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(10) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.35] The tenth case of all facets of P_{20}

In this case, we have

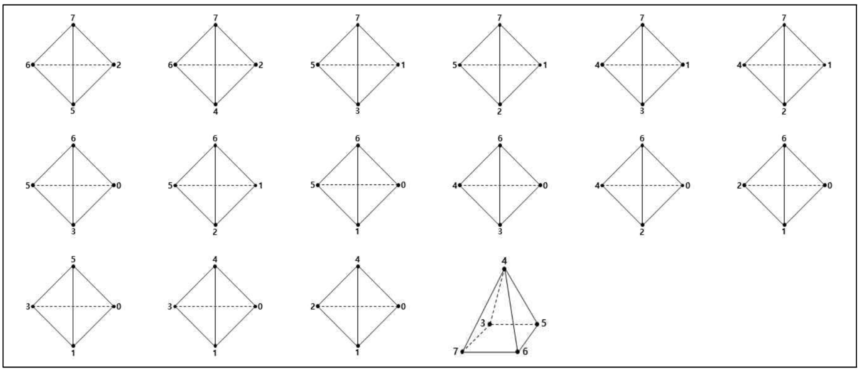
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(11) $f_1(P_{20}) = 26$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 45$$



[Figure 4.36] The eleventh case of all facets of P_{20}

In this case, we have

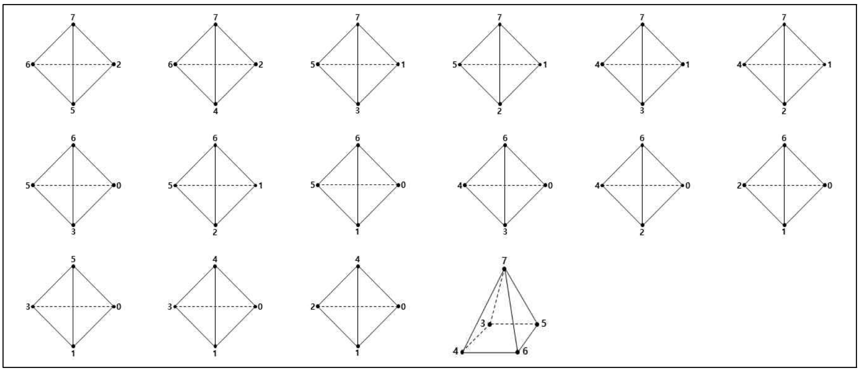
$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 26.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20}) + 2f_1(P_{20}) + f_{03}(P_{20}) = 101$, and

$$m = 19 = 3f_0(P_{20})(f_0(P_{20}) - 3) - f_{02}(P_{20}).$$

(12) $f_1(P_{20}) = 25$ with the following edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02$$



[Figure 4.37] The twelfth case of all facets of P_{20}

In this case, we have

$$f_0(P_{20}) = 8,\ f_{03}(P_{20}) = 65,\ f_1(P_{20}) = 25.$$

Thus, $f_{02}(P_{20}) = -2f_0(P_{20})+2f_1(P_{20})+f_{03}(P_{20})=99$, and

$$m=21=3f_0(P_{20})(f_0(P_{20})-3)-f_{02}(P_{20}).$$

These results can be summarized, as follows.

[Table 4.5] P_{20}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{20}	8	25	99	65	$m=21$
	8	26	101	65	$m=19$

4.6 P_{21} case

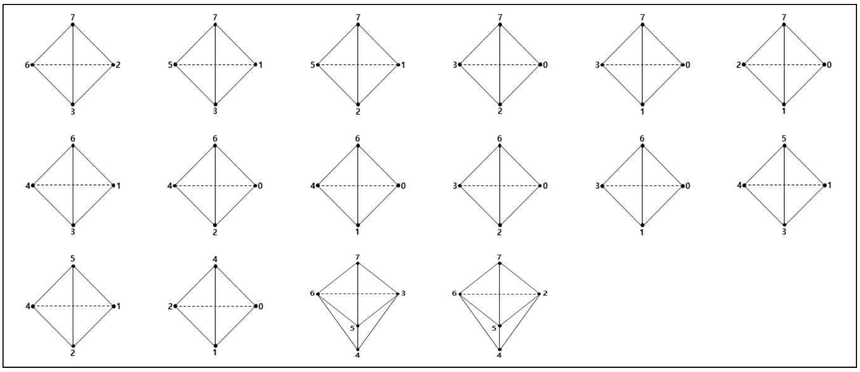
In this section, we deal with P_{21} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{21} is a 4-polytope with the following facet list:

[76543] [76542] [7632] [7531] [7521] [7320] [7310] [7210]
[6431] [6420] [6410] [6320] [6310] [5431] [5421] [4210]

(see Table 2.2 for more details). Thus it has 14 tetrahedra and 2 bipyramids over a triangle or 2 square pyramids. It turns out that there are five possibilities for P_{21} with such a facet list.

(1) $f_1(P_{21})=26$ with the followings edges:

67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 25, 12,
07, 03, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 56



[Figure 4.38] The first case of all facets of P_{21}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{21}) = 8,\ f_{03}(P_{21}) = 66,\ f_1(P_{21}) = 26,$$
$$f_{02}(P_{21}) = -2f_0(P_{21}) + 2f_1(P_{21}) + f_{03}(P_{21}),$$

Thus it is easy to obtain

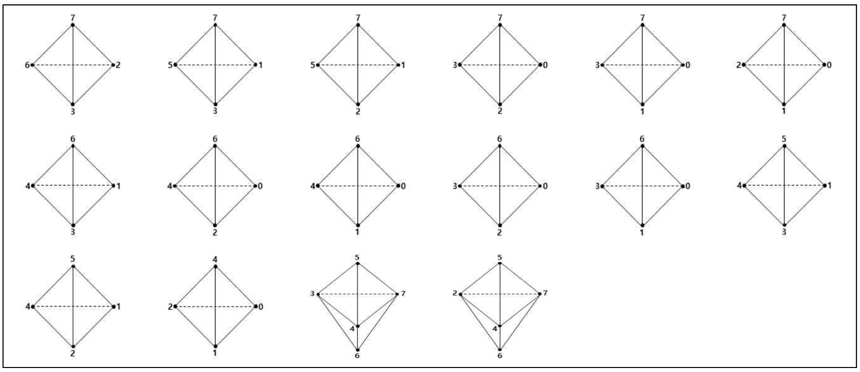
$$f_{02}(P_{21}) = -2\times 8 + 2\times 26 + 66 = 102.$$

It follows from the equation $f_{02}(P_{21}) = 3f_0(P_{21})(f_0(P_{21})-3)-m$ that we have $m=18$. Consequently, this case provides an example P_{21} of a 4-polytope which satisfies

$$f_{02}(P_{21}) = 3f_0(P_{21})(f_0(P_{21})-3)-18.$$

(2) $f_1(P_{21}) = 26$ with the followings edges:

67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 25, 12,
07, 03, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 47



[Figure 4.39] The second case of all facets of P_{21}

In this case, we have

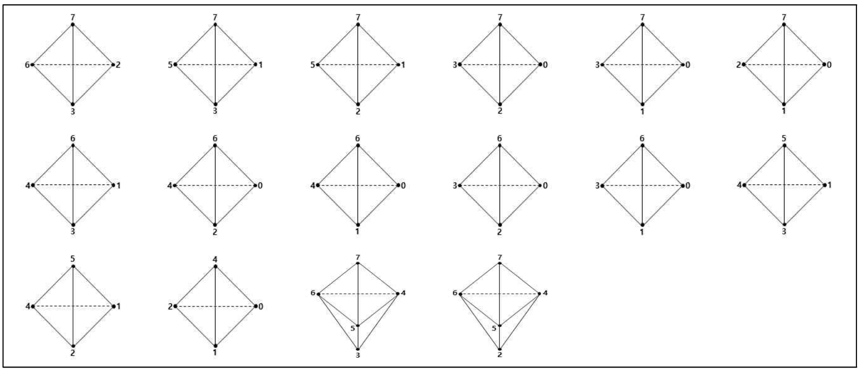
$$f_0(P_{21}) = 8,\ f_{03}(P_{21}) = 66,\ f_1(P_{21}) = 26.$$

Thus, $f_{02}(P_{21}) = -2f_0(P_{21}) + 2f_1(P_{21}) + f_{03}(P_{21}) = 102$, and

$$m = 18 = 3f_0(P_{21})(f_0(P_{21})-3) - f_{02}(P_{21}).$$

(3) $f_1(P_{21}) = 27$ with the followings edges:

67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 25, 12, 07,
03, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 56, 47



[Figure 4.40] The third case of all facets of P_{21}

In this case, we have

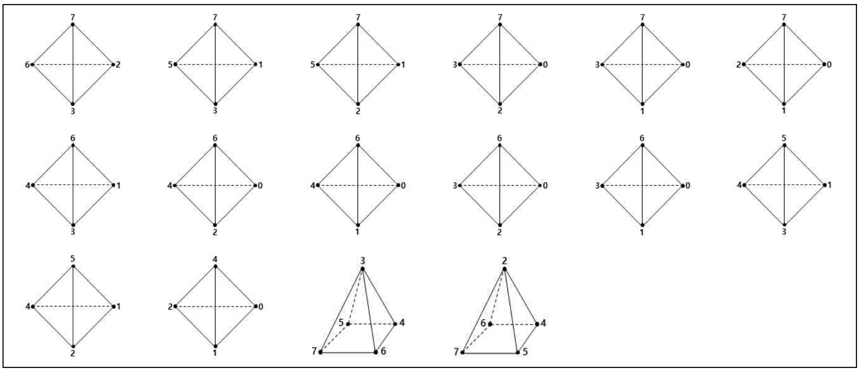
$$f_0(P_{21}) = 8,\ f_{03}(P_{21}) = 66,\ f_1(P_{21}) = 27.$$

Thus, $f_{02}(P_{21}) = -2f_0(P_{21}) + 2f_1(P_{21}) + f_{03}(P_{21}) = 104$, and

$$m = 16 = 3f_0(P_{21})(f_0(P_{21}) - 3) - f_{02}(P_{21}).$$

(4) $f_1(P_{21}) = 25$ with the followings edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 25, 12,$$
$$07, 03, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45$$



[Figure 4.41] The fourth case of all facets of P_{21}

In this case, we have

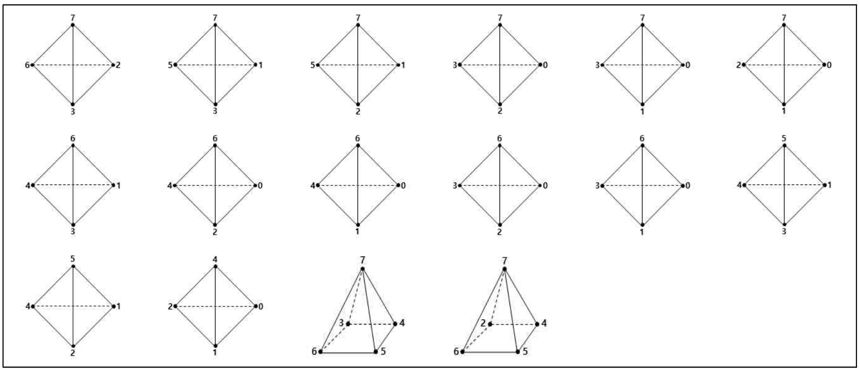
$$f_0(P_{21}) = 8,\ f_{03}(P_{21}) = 66,\ f_1(P_{21}) = 25.$$

Thus, $f_{02}(P_{21}) = -2f_0(P_{21}) + 2f_1(P_{21}) + f_{03}(P_{21}) = 100$, and

$$m = 20 = 3f_0(P_{21})(f_0(P_{21}) - 3) - f_{02}(P_{21}).$$

(5) $f_1(P_{21}) = 27$ with the followings edges:

$$67, 37, 27, 36, 26, 23, 57, 17, 35, 15, 13, 25, 12, 07,$$
$$03, 02, 01, 46, 16, 34, 14, 06, 24, 04, 45, 56, 47$$



[Figure 4.42] The fifth case of all facets of P_{21}

In this case, we have

$$f_0(P_{21}) = 8,\ f_{03}(P_{21}) = 66,\ f_1(P_{21}) = 27.$$

Thus, $f_{02}(P_{21}) = -2f_0(P_{21}) + 2f_1(P_{21}) + f_{03}(P_{21}) = 104$, and

$$m = 16 = 3f_0(P_{21})(f_0(P_{21}) - 3) - f_{02}(P_{21}).$$

These results can be summarized, as follows.

[Table 4.6] P_{21}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{21}	8	25	100	66	$m = 20$
	8	26	102	66	$m = 18$
	8	27	104	66	$m = 16$

4.7 P_{22} case

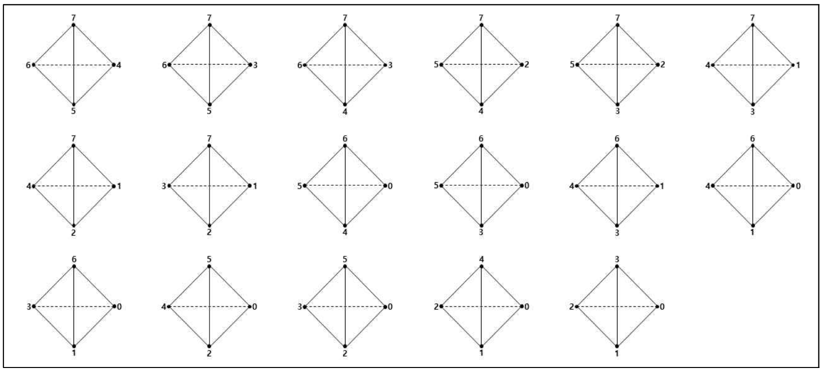
In this section, we deal with P_{22} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{22} is a 4-polytope with the following facet list:

[7654] [7653] [7643] [7542] [7532] [7431] [7421] [7321] [6540]
[6530] [6431] [6410] [6310] [5420] [5320] [4210] [3210]

(see Table 2.2 for more details). Thus it has only 17 tetrahedra. P_{22} is simplicial, i.e., all facets are 3-simplexes. It turns out that there is only one possibility for P_{22} with such a facet list.

(1) $f_1(P_{22}) = 25$ with the followings edges:

$$67, 57, 47, 56, 46, 45, 37, 36, 35, 34, 27, 25, 24,$$
$$23, 17, 14, 13, 12, 06, 05, 04, 03, 16, 01, 02$$



[Figure 4.43] The first case of all facets of P_{22}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{22}) = 8,\ f_{03}(P_{22}) = 68,\ f_1(P_{22}) = 25,$$
$$f_{02}(P_{22}) = -2f_0(P_{22}) + 2f_1(P_{22}) + f_{03}(P_{22}),$$

Thus it is easy to obtain

$$f_{02}(P_{22}) = -2\times 8 + 2\times 25 + 68 = 102.$$

It follows from the equation $f_{02}(P_{22}) = 3f_0(P_{22})(f_0(P_{22})-3)-m$ that we have $m=18$. Consequently, this case provides an example P_{22} of a 4-polytope which satisfies

$$f_{02}(P_{22}) = 3f_0(P_{22})(f_0(P_{22})-3)-18.$$

Since all facets are only tetrahedron, there is only one of the above for P_{22}. This result can be summarized, as follows.

[Table. 7] P_{22}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{22}	8	25	102	68	$m=18$

4.8 P_{23} case

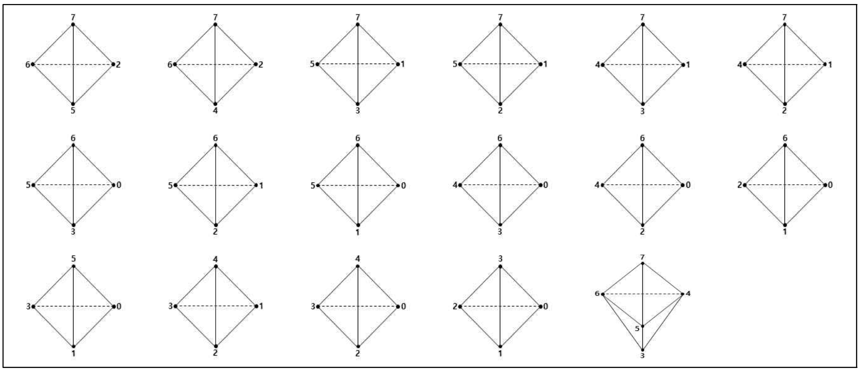
In this section, we deal with P_{23} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{23} is a 4-polytope with the following facet list:

$$[76543]\ [7652]\ [7642]\ [7531]\ [7521]\ [7431]\ [7421]\ [6530]\ [6521]$$
$$[6510]\ [6430]\ [6420]\ [6210]\ [5310]\ [4321]\ [4320]\ [3210]$$

(see Table 2.2 for more details). Thus it has 16 tetrahedra and 1 bipyramids over a triangle or 1 square pyramids. It turns out that there are six possibilities for P_{23} with such a facet list.

(1) $f_1(P_{23})=27$ with the followings edges:

$$67,57,27,56,26,25,47,46,24,37,17,35,15,13,$$
$$12,34,14,36,06,05,03,16,01,04,02,23,45$$



[Figure 4.44] The first case of all facets of P_{23}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{23})=8,\ f_{03}(P_{23})=69,\ f_1(P_{23})=27,$$
$$f_{02}(P_{23})=-2f_0(P_{23})+2f_1(P_{23})+f_{03}(P_{23}),$$

Thus it is easy to obtain

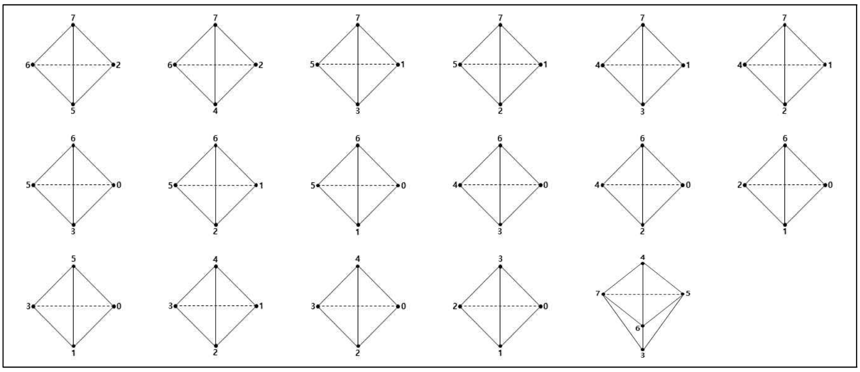
$$f_{02}(P_{23}) = -2\times 8+2\times 27+69=107.$$

It follows from the equation $f_{02}(P_{23}) = 3f_0(P_{23})(f_0(P_{23})-3)-m$ that we have $m=13$. Consequently, this case provides an example P_{23} of a 4-polytope which satisfies

$$f_{02}(P_{23}) = 3f_0(P_{23})(f_0(P_{23})-3)-13.$$

(2) $f_1(P_{23}) = 27$ with the followings edges:

67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15, 13,
12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 23, 45



[Figure 4.45] The second case of all facets of P_{23}

In this case, we have

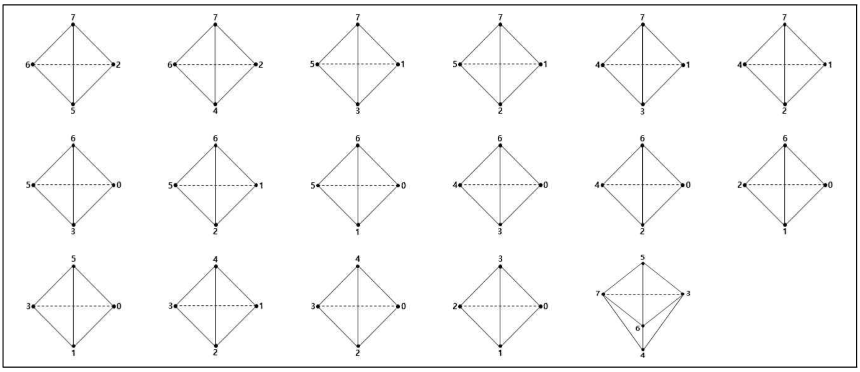
$$f_0(P_{23}) = 8,\ f_{03}(P_{23}) = 69,\ f_1(P_{23}) = 27.$$

Thus, $f_{02}(P_{23}) = -2f_0(P_{23})+2f_1(P_{23})+f_{03}(P_{23})=107$, and

$$m=13=3f_0(P_{23})(f_0(P_{23})-3)-f_{02}(P_{23}).$$

(3) $f_1(P_{23}) = 26$ with the followings edges:

67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,
13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 23



[Figure 4.46] The third case of all facets of P_{23}

In this case, we have

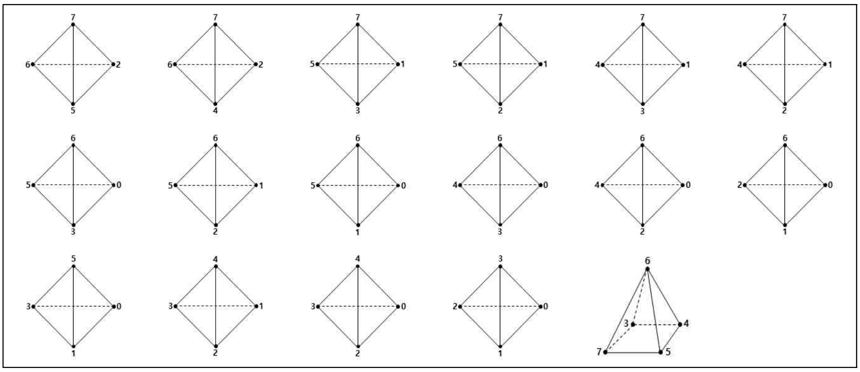
$$f_0(P_{23}) = 8,\ f_{03}(P_{23}) = 69,\ f_1(P_{23}) = 26.$$

Thus, $f_{02}(P_{23}) = -2f_0(P_{23}) + 2f_1(P_{23}) + f_{03}(P_{23}) = 105$, and

$$m = 15 = 3f_0(P_{23})(f_0(P_{23}) - 3) - f_{02}(P_{23}).$$

(4) $f_1(P_{23}) = 27$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15, 13,$$
$$12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 23, 45$$



[Figure 4.47] The fourth case of all facets of P_{23}

In this case, we have

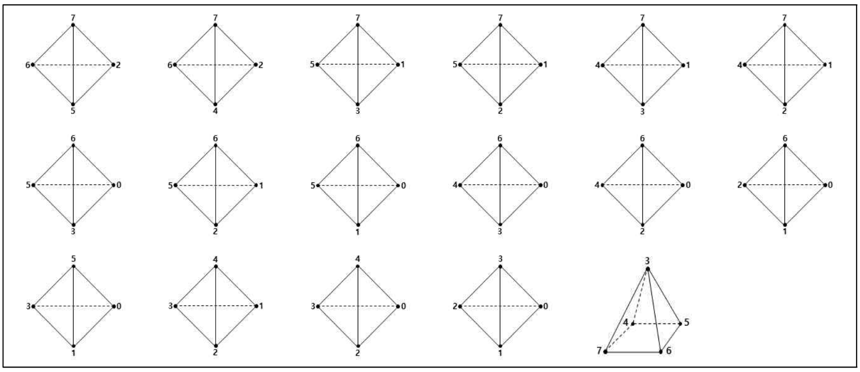
$$f_0(P_{23}) = 8,\ f_{03}(P_{23}) = 69,\ f_1(P_{23}) = 27.$$

Thus, $f_{02}(P_{23}) = -2f_0(P_{23}) + 2f_1(P_{23}) + f_{03}(P_{23}) = 107$, and

$$m = 13 = 3f_0(P_{23})(f_0(P_{23}) - 3) - f_{02}(P_{23}).$$

(5) $f_1(P_{23}) = 27$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15, 13,$$
$$12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 23, 45$$



[Figure 4.48] The fifth case of all facets of P_{23}

In this case, we have

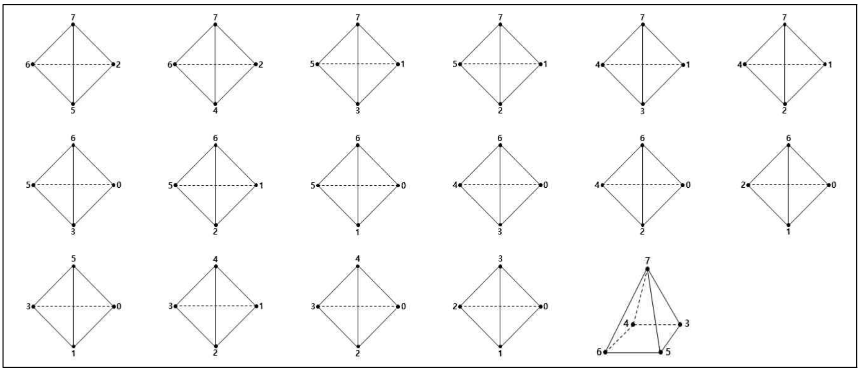
$$f_0(P_{23}) = 8,\ f_{03}(P_{23}) = 69,\ f_1(P_{23}) = 27.$$

Thus, $f_{02}(P_{23}) = -2f_0(P_{23}) + 2f_1(P_{23}) + f_{03}(P_{23}) = 107$, and

$$m = 13 = 3f_0(P_{23})(f_0(P_{23}) - 3) - f_{02}(P_{23}).$$

(6) $f_1(P_{23}) = 26$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 37, 17, 35, 15,$$
$$13, 12, 34, 14, 36, 06, 05, 03, 16, 01, 04, 02, 23$$



[Figure 4.49] The sixth case of all facets of P_{23}

In this case, we have

$$f_0(P_{23}) = 8,\ f_{03}(P_{23}) = 69,\ f_1(P_{23}) = 26.$$

Thus, $f_{02}(P_{23}) = -2f_0(P_{23})+2f_1(P_{23})+f_{03}(P_{23})=105$, and
$$m=15=3f_0(P_{23})(f_0(P_{23})-3)-f_{02}(P_{23}).$$

These results can be summarized, as follows.

[Table 4.8] P_{23}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{23}	8	26	105	69	$m=15$
	8	27	107	69	$m=13$

4.9 P_{24} case

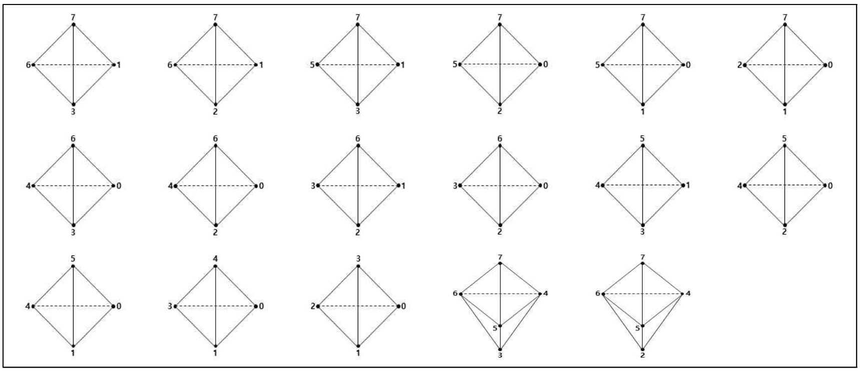
In this section, we deal with P_{24} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{24} is a 4-polytope with the following facet list:

[76543] [76542] [7631] [7621] [7531] [7520] [7510] [7210] [6430]
[6420] [6321] [6320] [5431] [5420] [5410] [4310] [3210]

(see Table 2.2 for more details). Thus it has 15 tetrahedra and 2 bipyramids over a triangle or 2 square pyramids. It turns out that there are four possibilities for P_{24} with such a facet list.

(1) $f_1(P_{24}) = 28$ with the followings edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 35, 15, 07, 25,$$
$$05, 02, 01, 46, 06, 34, 04, 03, 24, 23, 45, 14, 47, 56$$



[Figure 4.50] The first case of all facets of P_{24}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{24}) = 8,\ f_{03}(P_{24}) = 70,\ f_1(P_{24}) = 28,$$
$$f_{02}(P_{24}) = -2f_0(P_{24}) + 2f_1(P_{24}) + f_{03}(P_{24}),$$

Thus it is easy to obtain

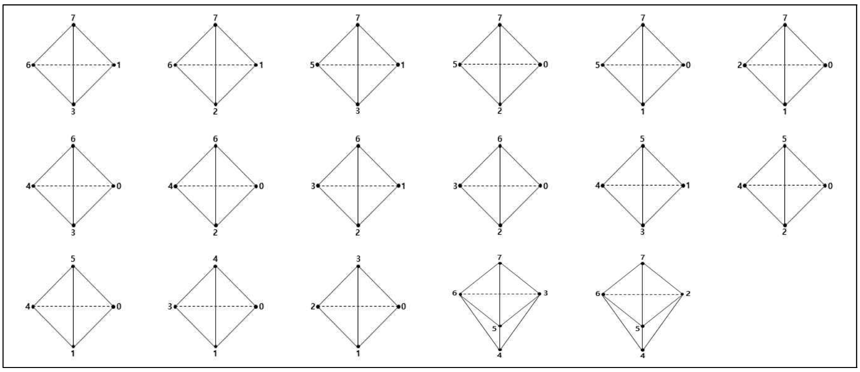
$$f_{02}(P_{24}) = -2\times 8 + 2\times 28 + 70 = 110.$$

It follows from the equation $f_{02}(P_{24}) = 3f_0(P_{24})(f_0(P_{24})-3) - m$ that we have $m=10$. Consequently, this case provides an example P_{24} of a 4-polytope which satisfies

$$f_{02}(P_{24}) = 3f_0(P_{24})(f_0(P_{24})-3) - 10.$$

(2) $f_1(P_{24}) = 27$ with the followings edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 35, 15, 07, 25,$$
$$05, 02, 01, 46, 06, 34, 04, 03, 24, 23, 45, 14, 56$$



[Figure 4.51] The second case of all facets of P_{24}

In this case, we have

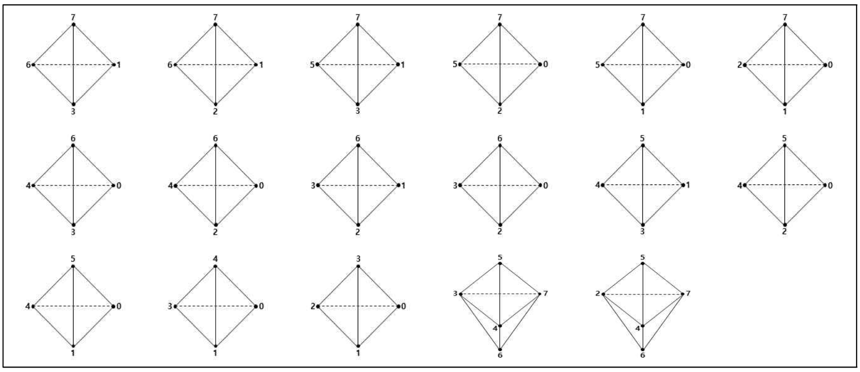
$$f_0(P_{24}) = 8,\ f_{03}(P_{24}) = 70,\ f_1(P_{24}) = 27.$$

Thus, $f_{02}(P_{24}) = -2f_0(P_{24}) + 2f_1(P_{24}) + f_{03}(P_{24}) = 108$, and

$$m = 12 = 3f_0(P_{24})(f_0(P_{24})-3) - f_{02}(P_{24}).$$

(3) $f_1(P_{24}) = 27$ with the followings edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 35, 15, 07, 25,$$
$$05, 02, 01, 46, 06, 34, 04, 03, 24, 23, 45, 14, 47$$



[Figure 4.52] The third case of all facets of P_{24}

In this case, we have

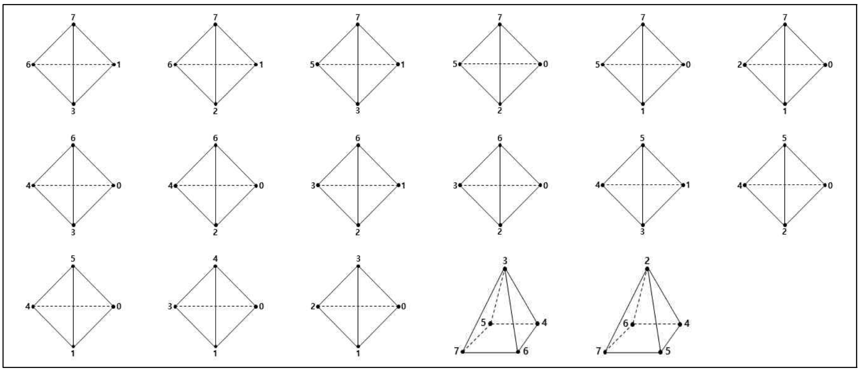
$$f_0(P_{24}) = 8,\ f_{03}(P_{24}) = 70,\ f_1(P_{24}) = 27.$$

Thus, $f_{02}(P_{24}) = -2f_0(P_{24}) + 2f_1(P_{24}) + f_{03}(P_{24}) = 108$, and

$$m = 12 = 3f_0(P_{24})(f_0(P_{24}) - 3) - f_{02}(P_{24}).$$

(4) $f_1(P_{24}) = 26$ with the followings edges:

$$67, 37, 17, 36, 16, 13, 27, 26, 12, 57, 35, 15, 07,$$
$$25, 05, 02, 01, 46, 06, 34, 04, 03, 24, 23, 45, 14$$



[Figure 4.53] The fourth case of all facets of P_{24}

In this case, we have

$$f_0(P_{24}) = 8,\ f_{03}(P_{24}) = 70,\ f_1(P_{24}) = 26.$$

Thus, $f_{02}(P_{24}) = -2f_0(P_{24}) + 2f_1(P_{24}) + f_{03}(P_{24}) = 106$, and

$$m = 14 = 3f_0(P_{24})(f_0(P_{24}) - 3) - f_{02}(P_{24}).$$

These results can be summarized, as follows.

[Table 4.9] P_{24}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{24}	8	26	106	70	$m=14$
	8	27	108	70	$m=12$

4.10 P_{25} case

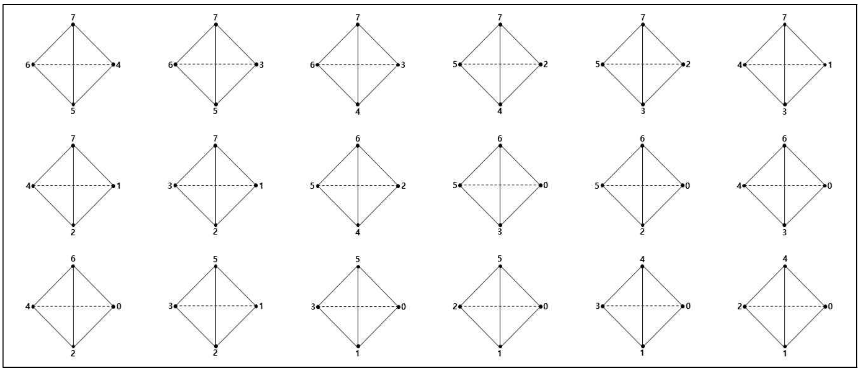
In this section, we deal with P_{25} case. For 8 eight vertices labeled with $0,1,2,\cdots,7$, P_{25} is a 4-polytope with the following facet list:

$$[7654]\ [7653]\ [7643]\ [7542]\ [7532]\ [7431]\ [7421]\ [7321]\ [6542]$$
$$[6530]\ [6520]\ [6430]\ [6420]\ [5321]\ [5310]\ [5210]\ [4310]\ [4210]$$

(see Table 2.2 for more details). Thus it has only 18 tetrahedra. P_{25} is simplicial, i.e., all facets are 3-simplexes. It turns out that there is only one possibility for P_{25} with such a facet list.

(1) $f_1(P_{25})=26$ with the followings edges:

$$67, 57, 47, 56, 46, 45, 37, 36, 35, 34, 27, 25, 24,$$
$$23, 17, 14, 13, 12, 26, 06, 05, 03, 02, 04, 15, 01$$



[Figure 4.54] The first case of all facets of P_{25}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{25})=8,\ f_{03}(P_{25})=72,\ f_1(P_{25})=26,$$

$$f_{02}(P_{25}) = -2f_0(P_{25})+2f_1(P_{25})+f_{03}(P_{25}),$$

Thus it is easy to obtain

$$f_{02}(P_{25}) = -2\times 8+2\times 26+72=108.$$

It follows from the equation $f_{02}(P_{25}) = 3f_0(P_{25})(f_0(P_{25})-3)-m$ that we have $m=12$. Consequently, this case provides an example P_{25} of a 4-polytope which satisfies

$$f_{02}(P_{25}) = 3f_0(P_{25})(f_0(P_{25})-3)-12.$$

Since all facets are only tetrahedron, there is only one of the above for P_{25}. This result can be summarized, as follows.

[Table 4.9] P_{25}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{25}	8	26	108	72	$m=12$

4.11 P_{26} case

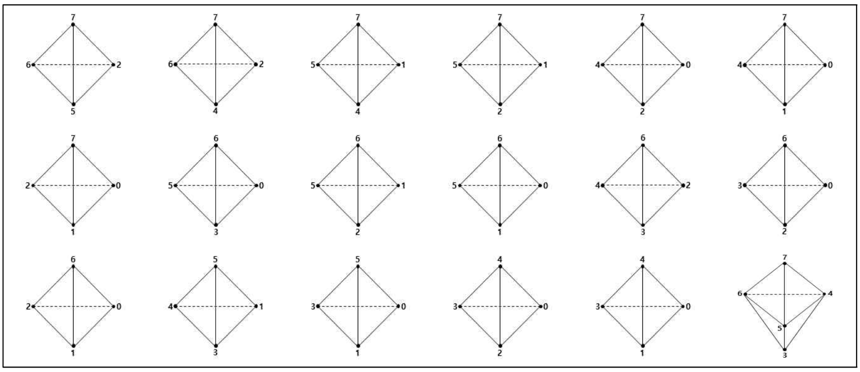
In this section, we deal with P_{26} case. For 8 eight vertices labeled with 0,1,2,⋯,7, P_{26} is a 4-polytope with the following facet list:

[76543] [7652] [7642] [7541] [7521] [7420] [7410] [7210] [6530]
[6521] [6510] [6432] [6320] [6210] [5431] [5310] [4320] [4310]

(see Table 2.2 for more details). Thus it has 17 tetrahedra and 1 bipyramids over a triangle or 1 square pyramids. It turns out that there are four possibilities for P_{26} with such a facet list.

(1) $f_1(P_{26}) = 27$ with the followings edges:

67, 57, 27, 56, 26, 25, 47, 46, 24, 17, 45, 15, 14, 12,
07, 04, 02, 01, 36, 06, 35, 05, 03, 16, 34, 23, 13



[Figure 4.55] The first case of all facets of P_{26}

The value of f_0 and f_{03} can be obtained from Table 2.1, as follows.

$$f_0(P_{26}) = 8,\ f_{03}(P_{26}) = 73,\ f_1(P_{26}) = 27,$$
$$f_{02}(P_{26}) = -2f_0(P_{26}) + 2f_1(P_{26}) + f_{03}(P_{26}),$$

Thus it is easy to obtain

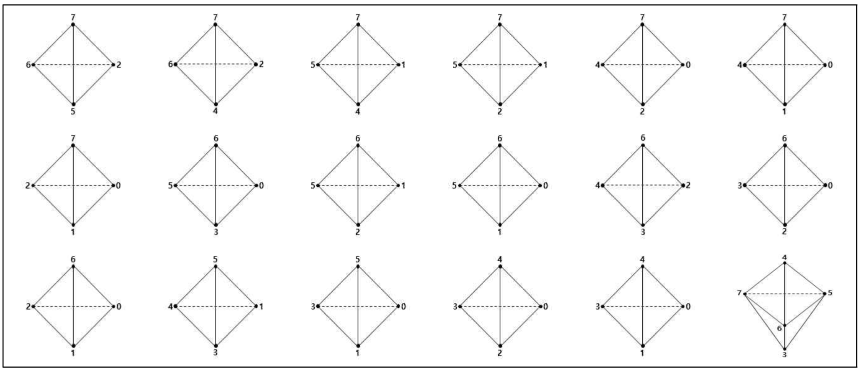
$$f_{02}(P_{26}) = -2\times 8 + 2\times 27 + 73 = 111.$$

It follows from the equation $f_{02}(P_{26}) = 3f_0(P_{26})(f_0(P_{26})-3)-m$ that we have $m=9$. Consequently, this case provides an example P_{26} of a 4-polytope which satisfies

$$f_{02}(P_{26}) = 3f_0(P_{26})(f_0(P_{26})-3)-9.$$

(2) $f_1(P_{26}) = 28$ with the followings edges:

67, 57, 27, 56, 26, 25, 47, 46, 24, 17, 45, 15, 14, 12,
07, 04, 02, 01, 36, 06, 35, 05, 03, 16, 34, 23, 13, 37



[Figure 4.56] The second case of all facets of P_{26}

In this case, we have

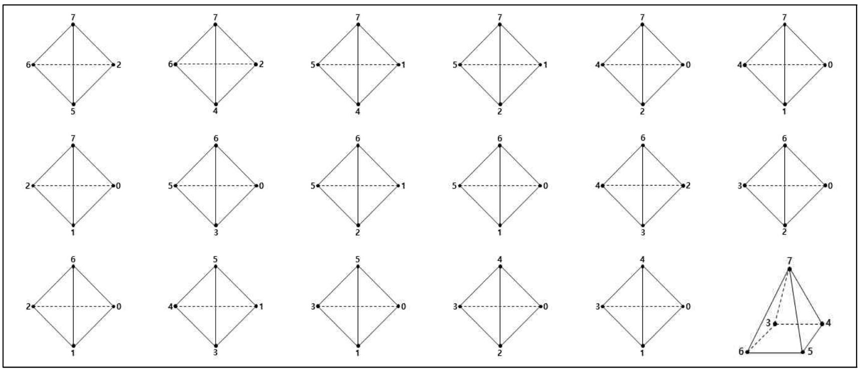
$$f_0(P_{26}) = 8,\ f_{03}(P_{26}) = 73,\ f_1(P_{26}) = 28.$$

Thus, $f_{02}(P_{26}) = -2f_0(P_{26}) + 2f_1(P_{26}) + f_{03}(P_{26}) = 113$, and

$$m = 7 = 3f_0(P_{26})(f_0(P_{26}) - 3) - f_{02}(P_{26}).$$

(3) $f_1(P_{26}) = 28$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 17, 45, 15, 14, 12,$$
$$07, 04, 02, 01, 36, 06, 35, 05, 03, 16, 34, 23, 13, 37$$



[Figure 4.57] The third case of all facets of P_{26}

In this case, we have

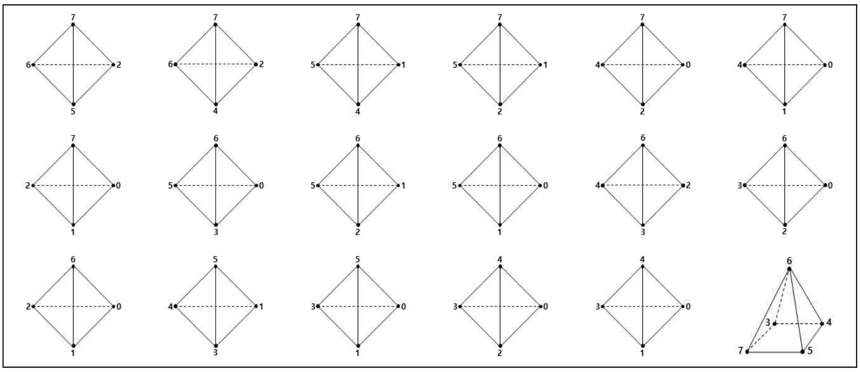
$$f_0(P_{26}) = 8,\ f_{03}(P_{26}) = 73,\ f_1(P_{26}) = 28.$$

Thus, $f_{02}(P_{26}) = -2f_0(P_{26}) + 2f_1(P_{26}) + f_{03}(P_{26}) = 113$, and

$$m = 7 = 3f_0(P_{26})(f_0(P_{26}) - 3) - f_{02}(P_{26}).$$

(4) $f_1(P_{26}) = 28$ with the followings edges:

$$67, 57, 27, 56, 26, 25, 47, 46, 24, 17, 45, 15, 14, 12,$$
$$07, 04, 02, 01, 36, 06, 35, 05, 03, 16, 34, 23, 13, 37$$



[Figure 4.58] The fourth case of all facets of P_{26}

In this case, we have

$$f_0(P_{26}) = 8,\ f_{03}(P_{26}) = 73,\ f_1(P_{26}) = 28.$$

Thus, $f_{02}(P_{26}) = -2f_0(P_{26})+2f_1(P_{26})+f_{03}(P_{26})=113$, and

$$m=7=3f_0(P_{26})(f_0(P_{26})-3)-f_{02}(P_{26}).$$

These results can be summarized, as follows.

[Table 4.10] P_{26}

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{26}	8	27	111	73	$m=9$

4.12 Summary of Our Results

As mentioned above, our goal of this paper is to set up a first step to find a necessary and sufficient condition for the flag vector pair (f_0, f_{02}) of a 4-polytope to satisfy. To do so, we enumerated many specific examples that satisfy the range of the flag vector pairs (f_0, f_{02}) suggested by the paper [10] of Kim and Park.

As a consequence, we found that there are some concrete examples of 4-polytopes satisfying $m = 3f_0(f_0 - 3) - f_{02}$ with

$$9, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28.$$

It is interesting to note that at least in our list there is no example of a 4-polytope whose value of $3f_0(f_0 - 3) - f_{02}$ is exactly equal to 17. Further, it is worth mentioning that there is a concrete example of 4-polytope whose $3f_0(f_0 - 3) - f_{02}$ is equal to 9. Notice also that in our list there are no examples of a 4-polytope whose value of $3f_0(f_0 - 3) - f_{02}$ lies in the set $\{1,2,3,4,5,7,8,10,11\}$. This, in particular, implies that our results perfectly fit well with the result of Kim and Park in [10] (see Theorem 3.1).

Table 4.11 below shows our overall results of concrete examples that we found in this thesis, based on the list provided by Fukuda, Miyata, and Moriyama in [6] (see [Table 2.2]).

[Table 4.11] Summary of our results

	f_0	f_1	f_{02}	f_{03}	$m=3f_0(f_0-3)-f_{02}$
P_{16}	8	25	93	59	$m=27$
	8	26	95	59	$m=25$
	8	27	97	59	$m=23$
P_{17}	8	24	92	60	$m=28$
	8	25	94	60	$m=26$
	8	26	96	60	$m=24$
	8	27	98	60	$m=22$
P_{18}	8	25	96	62	$m=24$
	8	26	98	62	$m=22$
	8	27	100	62	$m=20$
P_{19}	8	25	97	63	$m=23$
	8	26	99	63	$m=21$
	8	27	101	63	$m=19$
P_{20}	8	25	99	65	$m=21$
	8	26	101	65	$m=19$
P_{21}	8	25	100	66	$m=20$
	8	26	102	66	$m=18$
	8	27	104	66	$m=16$
P_{22}	8	25	102	68	$m=18$
P_{23}	8	26	105	69	$m=15$
	8	27	107	69	$m=13$
P_{24}	8	26	106	70	$m=14$
	8	27	108	70	$m=12$
P_{25}	8	26	108	72	$m=12$
P_{26}	8	27	111	73	$m=9$

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