

Summability Results for Mapping Matrices

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Abstract

For topological vector spaces X and Y , let $F_0(X, Y) = \{f \in Y^X : f(0) = 0\}$. Then it is an extremely large family and the family of linear operators is a very small subfamily of $F_0(X, Y)$. In this paper, we establish the characterizations of $F_0(X, Y)$ - matrix families $(l^\infty(X), l^\infty(Y))$, $(c_0(X), c_0(Y))$ and $(c_0(X), l^\infty(Y))$.

Keywords: Summability, $F_0(X, Y)$ -Matrix Family, Braked Space.

Mathematics Subject Classification: 40A05, 40C05, 54A20

1. Introduction

For topological vector spaces X and Y , $F_0(X, Y) = \{f \in Y^X : f(0) = 0\}$ is an extremely large family and the family of linear operators is a very small subfamily of $F_0(X, Y)$.

For sequence families $\lambda(X) \subset X^{\mathbb{N}}$, $\mu(Y) \subset Y^{\mathbb{N}}$ and mappings $f_{ij} \in F_0(X, Y) (i, j \in \mathbb{N})$, the matrix $(f_{ij})_{i,j \in \mathbb{N}} \in (\lambda(X), \mu(Y))$ means that $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges whenever $i \in \mathbb{N}$, $(x_j) \in \lambda(X)$ and $\left\{ \sum_{j=1}^{\infty} f_{ij}(x_j) \right\}_{i=1}^{\infty} \in \mu(Y)$, $\forall (x_j) \in \lambda(X)$.

Let $l^\infty(X) = \{(x_j) \in X^{\mathbb{N}} : \{x_j\} \text{ is bounded}\}$ and $c_0(X) = \{(x_j) \in X^{\mathbb{N}} : x_j \rightarrow 0\}$.

In 2001 and 2002, a mapping family $QH_f(X, Y)$ including all linear operators and many non-linear mappings were defined in the literature^[1,2] which gave the characterization of $QH_f(X, Y)$ - matrix families $(c_0(X), c_0(Y))$, $(c_0(X), l^\infty(Y))$ and $(l^\infty(X), l^\infty(Y))$. However, $QH_f(X, Y)$ is also a small subfamily of $F_0(X, Y)$ and so the quality lower of the results in the literature^[1,2] is lower than Theorem B^[3].

If $Y = (Y, \|\cdot\|)$ is a seminormed space and $\{f_j\} \subset F_0(X, Y)$, then for $n \in \mathbb{N}$ and $B \subset X$ let $R_n = (f_n, f_{n+1}, f_{n+2}, \dots)$ and

$$\|R_n\|_B = \sup_{m \in \mathbb{N}, \{x_j\} \subset B} \left\| \sum_{k=0}^{m-1} f_{n+k}(x_j) \right\| = \sup_{m \geq n, \{x_j\} \subset B} \left\| \sum_{j=n}^m f_j(x_j) \right\|.$$

Obviously, $\|R_n\|_B = \sup \left\{ \left\| \sum_{k \in \Delta} f_{n+k}(x_j) \right\| : \Delta \subset \mathbb{N} \text{ finite}, \{x_j\} \subset B \right\}$ whenever $0 \in B$. If both X and Y are seminormed, then $\|R_n\| = \|R_n\|_{\{x \in X : \|x\| \leq 1\}}$ is just the group norm of $(f_j)_{j \geq n}$ ^[4].

For $f_{ij} \in F_0(X, Y)$, $i, j \in \mathbb{N}$, R_{i_n} denotes the sequence $(f_{i_n}, f_{i_{n+1}}, f_{i_{n+2}}, \dots)$.

In the paper^[3], we have established the characterizations of $F_0(X, Y)$ - matrix families $(l^\infty(X), c_0(Y))$ and $(l^\infty(X), l^\infty(Y))$ as follows.

Proposition 1. $f_{ij} \in F_0(X, Y)$ for $i, j \in \mathbb{N}$. Then $(f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0(Y))$ if and only if

(1) $\lim_i f_{ij}(x) = 0, \forall j \in \mathbb{N}, x \in X$ and

(2) for every bounded $B \subset X$, $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges

uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proposition 2. Let Y be a Banach space and $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$. Then $(f_{i,j})_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0(Y))$ if and only if

(1) $\lim_i f_{ij}(x) = 0, \forall j \in \mathbb{N}, x \in X$ and

(2) for every bounded $B \subset X$, $\lim_m \sup_{i \in \mathbb{N}} \|R_{i_m}\|_B = 0$.

If, in addition, X is seminormed and each $f_{ij} : X \rightarrow Y$ is linear, then $(f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0(Y))$ if and only if (1) and

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(Received : January 26, 2018, Accepted : February 19, 2018)

(3) $\lim_m \sup_{i \in \mathbb{N}} \|R_{im}\| = 0$.

Proposition 3. $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$. Then the following (a), (b) and (c) are equivalent.

- (a) $(f_{i,j})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y))$.
- (b) $(s_i f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0(Y)), \forall (s_i) \in c_0 = \{(s_i) \in \mathbb{C}^{\mathbb{N}} : s_i \rightarrow 0\}$.
- (c) (4) $\{f_{ij}(x)\}_{i=1}^\infty$ is bounded, $\forall j \in \mathbb{N}, x \in X$ and
 (5) for every bounded $B \subset X$ and $(s_i) \in c_0$, $\sum_{j=1}^\infty s_j f_{ij}(x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proposition 4. If Y is seminormed and $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$, then $(f_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ if and only if

- (4) $\{f_{ij}(x)\}_{i=1}^\infty$ is bounded, $\forall j \in \mathbb{N}, x \in X$,
- (6) $\sum_{j=1}^\infty f_{ij}(x_j)$ converges, $\forall i \in \mathbb{N}, (x_j) \in l^\infty(X)$ and
- (7) for every $(x_j) \in l^\infty(X)$ and integer sequences $i_1 < i_2 < \dots, m_1 < n_1 < m_2 < n_2 < \dots$,

$$\sup_{k \in \mathbb{N}} \left\| \sum_{j=m_k}^{n_k} f_{i_j k}(x_j) \right\| < +\infty.$$

In this paper we would like to give the additional characterizations of $(l^\infty(X), l^\infty(Y)), (c_0(X), c_0(Y))$ and $(c_0(X), l^\infty(Y))$ for matrices in $F_0(X, Y)$.

The characterizations we ascertain in this paper is different from the summability results of matrices of linear operators during 1950-1992 that C. Swartz^[5] gave an epoch-making result of Theorem A^[3].

2. Main Results

Throughout this paper, X and Y are topological vector spaces, and let $L(X, Y) = \{T \in Y^X : T \text{ is linear and continuous}\}$.

We begin with the corollary of Proposition 3 which is a more clear-cut characterization of the family $(l^\infty(X), l^\infty(Y))$ for matrices of linear operators on Banach spaces as follows.

Corollary 1. Let X, Y be Banach spaces and $T_{ij} \in L(X, Y), \forall i, j \in \mathbb{N}$. Then $(T_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ if and only if

- (8) $\sup_{i \in \mathbb{N}} \|T_{ij}\| < +\infty, \forall j \in \mathbb{N}$ and
- (9) $\lim_m \|R_{im}\| = \lim_m \|(T_{im}, T_{i,m+1}, \dots)\| = 0, \forall i \in \mathbb{N}$ and

$$\sup_{i \in \mathbb{N}} \|R_{i1}\| = \sup_{i, n \in \mathbb{N}, \|x_j\| \leq 1} \left\| \sum_{j=1}^n T_{ij}(x_j) \right\| < +\infty.$$

Proof. Suppose $(T_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y))$. Since $\{T_{ij}(x)\}_{i=1}^\infty$ is bounded for every $x \in X$ and $j \in \mathbb{N}$, (8) follows from the resonance theorem. Since $\sum_{j=1}^\infty T_{ij}(x_j)$ converges whenever $(x_j) \in l^\infty(X)$ and $i \in \mathbb{N}$, $\lim_m \|R_{im}\| = 0, \forall i \in \mathbb{N}$ (see^[4], p. 21, Proposition 3.3).

Suppose $\sup_i \|R_{i1}\| = +\infty$. Then $\|R_{i1}\| = \sup_{n \in \mathbb{N}, \|x_j\| \leq 1} \left\| \sum_{j=1}^n T_{ij}(x_j) \right\| > 1 + \sup_i \|T_{i1}\|$ for some $i_1 \in \mathbb{N}$ and, hence, $\left\| \sum_{j=1}^{n_1} T_{i_1 j}(x_{1j}) \right\| > 1 + \sup_i \|T_{i1}\|$ for some $n_1 > 1$ and $\{x_{1j} : 1 \leq j \leq n_1\} \subset B = \{x \in X : \|x\| \leq 1\}$. Since $\sum_{j=1}^\infty T_{ij}(x_j)$ converges for each $\{x_j\} \subset B$ and $i \in \mathbb{N}$, it follows from Lemma 1[3] that there exists an $m_0 > n_1 + 1$ such that $\left\| \sum_{j=m_0+1}^\infty T_{ij}(x_j) \right\| < \frac{1}{2}, \forall 1 \leq i \leq i_1, \{x_j\} \subset B$. By $\sup_i \|R_{i1}\| = +\infty$ again, there is an $i_2 \in \mathbb{N}$ for which $\|R_{i_2 1}\| = \sup_{n \in \mathbb{N}, \|x_j\| \leq 1} \left\| \sum_{j=1}^n T_{i_2 j}(x_j) \right\| > 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \|T_{ij}\|$ and, hence, $\left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| > 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \|T_{ij}\|$ for some $n_2 \in \mathbb{N}$ and $\{x_{2j} : 1 \leq j \leq n_2\} \subset B$. Obviously, $n_2 > m_0$. If $i_2 \leq i_1$, then

$$\begin{aligned} \left\| \sum_{j=1}^{m_0} T_{i_2 j}(x_{2j}) \right\| &\geq \left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| - \left\| \sum_{j=m_0+1}^{n_2} T_{i_2 j}(x_{2j}) \right\| \\ &> 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \|T_{ij}\| - \frac{1}{2} \\ &\geq 2 + \left\| \sum_{j=1}^{m_0} T_{i_2 j}(x_{2j}) \right\|. \end{aligned}$$

This is a contradiction and we have that $i_2 > i_1$. Since $n_2 > m_0 > n_1 + 1$,

$$\begin{aligned} \left\| \sum_{j=n_1+1}^{n_2} T_{i_2 j}(x_{2j}) \right\| &\geq \left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| - \left\| \sum_{j=1}^{n_1} T_{i_2 j}(x_{2j}) \right\| \\ &> 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \|T_{ij}\| - \left\| \sum_{j=1}^{n_1} T_{i_2 j}(x_{2j}) \right\| \\ &\geq 2 + \frac{1}{2}, \end{aligned}$$

i.e., letting $m_2 = n_1 + 1$, $\left\| \sum_{j=m_2}^{n_2} T_{ij}(x_{2j}) \right\| > 2$.

Continuing by induction, we have integer sequences $i_1 < i_2 < i_3 < \dots$, $1 = m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \dots$ and $\{x_{kj} : m_k \leq j \leq n_k, k \in \mathbb{N}\} \subset B$ such that

$$\left\| \sum_{j=m_k}^{n_k} T_{ij}(x_{kj}) \right\| > k, \quad k = 1, 2, 3, \dots$$

Letting

$$x_j = \begin{cases} x_{kj}, & m_k \leq j \leq n_k, \quad k = 1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$(x_j) \in l^\infty(X)$ and $\sup_k \left\| \sum_{j=m_k}^{n_k} T_{ij}(x_j) \right\| = +\infty$. This contradicts Proposition 4. Hence, $\sup_{i \in \mathbb{N}} \|R_{i1}\| < +\infty$.

Conversely, suppose that (8) and (9) hold. Then $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges whenever $(x_j) \in l^\infty(X)$ and $i \in \mathbb{N}$ (see [4], p. 21). Since

$$\begin{aligned} \sup_{\|x_j\| \leq 1} \left\| \sum_{j=m}^n T_{ij}(x_j) \right\| &\leq \sup_{\|x_j\| \leq 1} \left\| \sum_{j=1}^n T_{ij}(x_j) \right\| \\ &\leq \sup_{k \in \mathbb{N}, \|x_j\| \leq 1} \left\| \sum_{j=1}^k T_{ij}(x_j) \right\| \leq \sup_{i \in \mathbb{N}} \|R_{i1}\|, \end{aligned}$$

$\forall m \leq n, i \in \mathbb{N}$, (8) + (9) \Rightarrow (4) + (6) + (7) and $(T_{ij})_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ by Proposition 4. \square

X is said to be braked if for every $(x_j) \in c_0(X)$ there exist $(t_j) \in c_0$ and $(z_j) \in c_0(X)$ such that $x_j = t_j z_j$ for all j (see [6], p. 43). Metrizable spaces are braked and the non-metrizable (l^1, weak) is also braked. Especially, each (LF) space is not metrizable but braked.

Each $t \in \mathbb{C}$ gives a continuous linear operator $t : X \rightarrow X$ by letting $t(x) = tx, x \in X$. In fact, if $t = 0$, then $t : X \rightarrow X$ is the zero operator, and for $t \neq 0, t : X \rightarrow X$ is a linear homeomorphism from X onto X . Then for $f \in Y^X$ and $t \in \mathbb{C}, f \circ t \in Y^X$ and $(f \circ t)(x) = f(tx), x \in X$.

Theorem 1. If X is braked and $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$, then the following (d), (e) and (f) are equivalent.

- (d) $(f_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), c_0(Y))$.
- (e) $(f_{ij} \circ t_j)_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0(Y)), \forall (t_j) \in c_0$.

- (f) (1) $\lim_i f_{ij}(x) = 0, \forall j \in \mathbb{N}, x \in X$ and (10) for every $(t_j) \in c_0$ and bounded $B \subset X, \sum_{j=1}^{\infty} f_{ij}(t_j x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proof. Obviously, (d) \Rightarrow (e). (e) \Rightarrow (f) by Proposition 1.

(e) \Rightarrow (d): Let $(x_j) \in c_0(X)$. Since X is braked, $(x_j) = (t_j z_j)$ where $(t_j) \in c_0, (z_j) \in c_0(X) \subset l^\infty(X)$. By (e), $\left\{ \sum_{j=1}^{\infty} f_{ij}(x_j) \right\}_{i=1}^{\infty} = \left\{ \sum_{j=1}^{\infty} (f_{ij} \circ t_j)(z_j) \right\}_{i=1}^{\infty} \in c_0(Y)$. \square

The braked condition of X can not be omitted in Theorem 1.

Example 1. $X = (l^\infty, \sigma(l^\infty, l^1))$ is not braked : $e_j = (0, \dots, 0, 1^{(j)}, 0, 0, \dots) \rightarrow 0$ in X but $\lambda_j e_j \not\rightarrow 0$ in X for every $\lambda_j \rightarrow \infty$. For $i, j \in \mathbb{N}$ define $f_{ij} : X \rightarrow \mathbb{C}$ by

$$f_{ij}((\alpha_k)_{k=1}^{\infty}) = \begin{cases} \alpha_i, & i=j; \\ 0, & i \neq j; \end{cases} (\alpha_k) \in l^\infty.$$

If $(t_j) \in c_0$, then $(f_{ij} \circ t_j)_{i,j \in \mathbb{N}} \in (l^\infty(X), c_0)$. In fact, if $\left\{ (\alpha_{jk})_{k=1}^{\infty} \right\}_{j=1}^{\infty} \in l^\infty(X)$, then $\sup_{j,k \in \mathbb{N}} |\alpha_{jk}| = M < +\infty$ by the resonance theorem and, hence, $\sum_{j=1}^{\infty} f_{ij}(t_j (\alpha_{jk})_{k=1}^{\infty}) = f_{ii}(t_i (\alpha_{ik})_{k=1}^{\infty}) = t_i \alpha_{ii} \rightarrow 0$.

However, $(f_{ij})_{i,j \in \mathbb{N}} \notin (c_0(X), c_0) : (e_j) \in c_0(X)$ but $\sum_{j=1}^{\infty} f_{ij}(e_j) = f_{ii}(e_i) = 1, \forall i \in \mathbb{N}$.

Combining Theorem 1 and Proposition 2, we have

Corollary 2. If X is braked and Y is a Banach space and $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$, then $(f_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), c_0(Y))$ if and only if

- (1) $\lim_i f_{ij}(x) = 0, \forall j \in \mathbb{N}, x \in X$ and
- (2') for every $(t_j) \in c_0$ and bounded $B \subset X$, $\lim_m \sup_{i \in \mathbb{N}} \| (f_{im} \circ t_m, f_{im+1} \circ t_{m+1}, \dots) \|_B = \lim_m \sup_i \sup_{n \geq m, \{x_j\} \subset B} \left\| \sum_{j=m}^n f_{ij}(t_j x_j) \right\| = 0$.

For the case of linear operators, we have a more clear-cut result as follows.

Corollary 3. If X is braked and Y is a Banach space and $T_{ij} : X \rightarrow Y$ is linear, $\forall i, j \in \mathbb{N}$, then $(T_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), c_0(Y))$ if and only if

- (1) $\lim_i T_{ij}(x) = 0, \forall j \in \mathbb{N}, x \in X$ and
- (2*) for every bounded $B \subset X$ there is an $m_0 \in \mathbb{N}$

such that

$$\sup_{i \in \mathbb{N}} \|(T_{i m_0}, T_{i m_0+1}, \dots)\|_B < +\infty.$$

If, in addition, X is seminormed, then (2*) can be replaced by the following (2**) $\exists m_0 \in \mathbb{N}$ such that $\sup_{i \in \mathbb{N}} \|(T_{i m_0}, T_{i m_0+1}, \dots)\| < +\infty$.

Proof. Let $c_0 = \{(t_j) \in c_0 : t_j = 0 \text{ eventually}\}$ and $(t_j) \in c_0 \setminus c_0$. Then $\delta_m = \sup_{j \geq m} |t_j| > 0, \delta_m \rightarrow 0$. Observe that $B \subset X$ is bounded if and only if $B_0 = \{tx : |t| \leq 1, x \in B\}$ is bounded.

Since $\|(T_{i m}, T_{i m+1}, \dots)\|_{B_0} \geq \|(T_{i n}, T_{i n+1}, \dots)\|_{B_0}$ for $m < n$, if (2*) holds and $B \subset X$ is bounded, then

$$\begin{aligned} & \lim_m \sup_{i \in \mathbb{N}} \sup \left\{ \left\| \sum_{j=m}^n T_{ij}(t_j x_j) \right\| : n \geq m, x_j \in B \text{ for } m \leq j \leq n \right\} \\ &= \lim_m \delta_m \sup_{i \in \mathbb{N}} \sup \left\{ \left\| \sum_{j=m}^n T_{ij} \left(\frac{t_j}{\delta_m} x_j \right) \right\| : n \geq m, x_j \in B \text{ for } m \leq j \leq n \right\} \\ &\leq \lim_m \delta_m \sup_{i \in \mathbb{N}} \sup \left\{ \left\| \sum_{j=m}^n T_{ij}(z_j) \right\| : n \geq m, z_j \in B_0 \text{ for } m \leq j \leq n \right\} \\ &\leq \lim_m \delta_m \sup_{i \in \mathbb{N}} \sup \left\{ \left\| \sum_{j=m_0}^n T_{ij}(z_j) \right\| : n \geq m_0, z_j \in B_0 \text{ for } m_0 \leq j \leq n \right\} = 0, \end{aligned}$$

i.e., (2*) \Rightarrow (2').

Conversely, if (2*) fails, then there is a bounded $B \subset X$ such that

$$\begin{aligned} & \sup_{i \in \mathbb{N}} \|R_{im}\|_B = \\ & \sup_{i \in \mathbb{N}} \|(T_{i m}, T_{i m+1}, \dots)\|_B = +\infty, \forall m \in \mathbb{N}. \end{aligned}$$

Since $\sup_{i \in \mathbb{N}} \|R_{i1}\|_B = +\infty$, there exist $i_1, n_1 \in \mathbb{N}$ and $x_{1j} \in B$ for $1 \leq j \leq n_1$ such that $\left\| \sum_{j=1}^{n_1} T_{i_1 j}(x_{1j}) \right\| > 1$.

But $\sup_{i \in \mathbb{N}} \|R_{i n_1}\| = +\infty$ so there exist integers $i_2, n_2 \geq n_1 + 1$ and $x_{2j} \in B$ for $n_1 + 1 \leq j \leq n_2$ such that $\left\| \sum_{j=n_1+1}^{n_2} T_{i_2 j}(x_{2j}) \right\| > 2$. Proceeding inductively, we have integer sequences $\{i_k\}, 0 = n_0 < n_1 < n_2 < n_3 < \dots$ and $\{x_{k+1j} : n_k + 1 \leq j \leq n_{k+1}, k = 0, 1, 2, \dots\} \subset B$ such that

$$\left\| \sum_{j=n_k+1}^{n_{k+1}} T_{i_{k+1} j}(x_{k+1j}) \right\| > k+1, k = 0, 1, 2, 3, \dots$$

Now let $t_j = \frac{1}{k+1}$ if $n_k < j \leq n_{k+1}, k = 0, 1, 2, \dots$, then $(t_j) \in c_0$ and

$$\begin{aligned} & \sup_{i \in \mathbb{N}} \sup_{n \geq n_{k+1}, \{z_j\} \subset B} \left\| \sum_{j=n_k+1}^n T_{ij}(t_j z_j) \right\| \\ & \geq \left\| \sum_{j=n_k+1}^{n_{k+1}} T_{i_{k+1} j} \left(\frac{1}{k+1} x_{k+1j} \right) \right\| > 1, k = 0, 1, 2, \dots \\ & \geq \left\| \sum_{j=n_k+1}^{n_{k+1}} T_{i_{k+1} j} \left(\frac{1}{k+1} x_{k+1j} \right) \right\| > 1, k = 0, 1, 2, \dots \end{aligned}$$

This contradicts (2') so (2') \Rightarrow (2*) holds. \square

Theorem 2. $f_{ij} \in F_0(X, Y), \forall i, j \in \mathbb{N}$. If X is braked, then the following (g), (h) and (i) are equivalent.

- (g) $(f_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), l^\infty(Y))$.
- (h) $(f_{ij} \circ t_j)_{i,j \in \mathbb{N}} \in (l^\infty(X), l^\infty(Y)), \forall (t_j) \in c_0$.
- (i) (4) $\{f_{ij}(x)\}_{i=1}^\infty$ is bounded, $\forall j \in \mathbb{N}, x \in X$ and

(11) for every bounded $B \subset X$ and $(s_i), (t_j) \in c_0, \sum_{j=1}^\infty s_i f_{ij}(t_j x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proof. Since X is braked, (g) \Leftrightarrow (h) is obvious. By Proposition 3, (h) \Leftrightarrow (i). \square

Corollary 4. Let X, Y be Banach spaces and $T_{ij} \in L(X, Y), \forall i, j \in \mathbb{N}$. Then $(T_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), l^\infty(Y))$ if and only if

- (8) $\sup_{i \in \mathbb{N}} \|T_{ij}\| < +\infty, \forall j \in \mathbb{N}$ and
- (12) $\sup_{i \in \mathbb{N}} \|R_{i1}\| =$

$$\sup_{i,n \in \mathbb{N}, \|x_j\| \leq 1} \left\| \sum_{j=1}^n T_{ij}(x_j) \right\| < +\infty.$$

Proof. Suppose that (8) and (12) hold. Then $\sum_{j=1}^\infty T_{ij}(x_j)$ converges for every $(x_j) \in c_0(X)$ and $i \in \mathbb{N}$ (see [4], p. 19). If $0 \neq (x_j) \in c_0(X)$, then

$$\begin{aligned} & \left\| \sum_{j=1}^\infty T_{kj}(x_j) \right\| = \lim_n \left\| \sum_{j=1}^n T_{kj}(x_j) \right\| \\ & = \|(x_j)\|_\infty \lim_n \left\| \sum_{j=1}^n T_{kj} \left(\frac{x_j}{\|(x_j)\|_\infty} \right) \right\| \\ & \leq \|(x_j)\|_\infty \sup_{i \in \mathbb{N}} \|R_{i1}\|, \forall k \in \mathbb{N}, \end{aligned}$$

i.e., $\left\{ \sum_{j=1}^{\infty} T_{ij}(x_j) \right\}_{i=1}^{\infty} \in l^{\infty}(Y)$.

Conversely, suppose that $(T_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), l^{\infty}(Y))$ but $\sup_{i \in \mathbb{N}} \|R_{i1}\| = +\infty$. Observe that (8) holds bt the resonance theorem. As in the proof of Corollary 1, there exist integer sequences $i_1 < i_2 < \dots, m_1 < n_1 < m_2 < n_2 < \dots$ and $\{x_{kj} : m_k \leq j \leq n_k, k \in \mathbb{N}\} \subset B = \{x \in X : \|x\| \leq 1\}$ such that

$$\left\| \sum_{j=m_k}^{n_k} T_{ij}(x_{kj}) \right\| > k, k=1, 2, 3, \dots$$

Let

$$s_i = \begin{cases} 1/\sqrt{k}, & i=i_k, k=1, 2, 3, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

$$t_j = \begin{cases} 1/\sqrt{k}, & m_k \leq j \leq n_k, k=1, 2, 3, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $s_i \rightarrow 0, t_j \rightarrow 0$ but

$$\left\| \sum_{j=m_k}^{n_k} s_i T_{ij}(t_j x_{kj}) \right\| = \left\| \frac{1}{k} \sum_{j=m_k}^{n_k} T_{ij}(x_{kj}) \right\| > 1, k=1, 2, 3, \dots$$

This contradicts Theorem 2 so $\sup_{i \in \mathbb{N}} \|R_{i1}\| < +\infty$. \square

$(l^{\infty}(X), l^{\infty}(Y)) \subset (c_0(X), l^{\infty}(Y))$ and, in general, the containment is strict. We now characterize $(l^{\infty}(X), l^{\infty}(Y)) = (c_0(X), l^{\infty}(Y))$ as follows.

For a matrix family $(\lambda(X), \mu(Y))$, let

$$(\lambda(X), \mu(Y))|_{L(X, Y)} = \{(T_{ij}) \in (\lambda(X), \mu(Y)) : \text{each } T_{ij} : X \rightarrow Y \text{ is linear and continuous}\}.$$

Theorem 3. A Banach space Y contains no copy of c_0 if and only if $(l^{\infty}(X), l^{\infty}(Y))|_{L(X, Y)} = (c_0(X), l^{\infty}(Y))|_{L(X, Y)}$ for every Banach space X .

Proof. \Rightarrow : Let X be a Banach space and $(T_{ij})_{i,j \in \mathbb{N}} \in (c_0(X), l^{\infty}(Y))|_{L(X, Y)}$. If $(x_j) \in l^{\infty}(X)$ and $(t_j) \in c_0$, then $(t_j x_j) \in c_0(X)$ so $\sum_{j=1}^{\infty} t_j T_{ij}(x_j) = \sum_{j=1}^{\infty} T_{ij}(t_j x_j)$ converges, $\forall i \in \mathbb{N}$. Since Y contains no

copy of c_0 , it follows from Theorem 4^[7] that $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges for every $i \in \mathbb{N}$ and $(x_j) \in l^{\infty}(X)$.

By Corollary 4, $\sup_{i \in \mathbb{N}} \|(T_{i1}, T_{i2}, \dots)\| < +\infty$. For $0 \neq (x_j) \in l^{\infty}(X)$,

$$\left\| \sum_{j=1}^{\infty} T_{kj}(x_j) \right\| = \|(x_j)\|_{\infty} \lim_n \left\| \sum_{j=1}^n T_{kj} \left(\frac{x_j}{\|(x_j)\|_{\infty}} \right) \right\| \leq \|(x_j)\|_{\infty} \sup_{i \in \mathbb{N}} \|(T_{i1}, T_{i2}, \dots)\|, \forall k \in \mathbb{N},$$

i.e., $(T_{ij})_{i,j \in \mathbb{N}} \in (l^{\infty}(X), l^{\infty}(Y))$.

\Leftarrow : Suppose that Y contains a copy of c_0 . We may assume that $(c_0, \|\cdot\|_{\infty})$ is a subspace of Y .

Define $T_{ij} : \mathbb{C} \rightarrow Y$ by $T_{ij}(t) = t e_j, \forall i, j \in \mathbb{N}, t \in \mathbb{C}$. Then $\sum_{j=1}^{\infty} T_{ij}(t_j) = \sum_{j=1}^{\infty} t_j e_j$ converges in $(c_0, \|\cdot\|_{\infty}) \subset Y$ for each $(t_j) \in c_0$ and

$$\left\| \sum_{j=1}^{\infty} T_{ij}(t_j) \right\|_{\infty} = \left\| \sum_{j=1}^{\infty} t_j e_j \right\|_{\infty} = \|(t_j)\|_{\infty}, \forall i \in \mathbb{N}, \text{ i.e.,}$$

$(T_{ij})_{i,j \in \mathbb{N}} \in (c_0, l^{\infty}(Y))$. However, $\sum_{j=1}^{\infty} T_{ij}(1) = \sum_{j=1}^{\infty} e_j$ diverges for $(1, 1, 1, \dots) \in l^{\infty}$ so $(l^{\infty}, l^{\infty}(Y)) \neq (c_0, l^{\infty}(Y))$. \square

Acknowledgments

This research was supported by Kumoh National Institute of Technology.

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