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A Study on the Computation of the Number of Equivalence Classes and Its applications

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Abstract

In this paper, we study on the reinterpretation of Burnside's theorem that is a tool to compute the number of equivalence classes of a group action on a set, and by applying it, we find out the numbers of different patterns in coloring problems of various types of figures with symmetry and use them for computing the numbers of different kinds of chemical compounds.

Keywords : Equivalence Classes, Burnside's Theorem, Pattern, Action, Cycle Index
Mathematics Subject Classifications : 05A15, 20B30, 20A05

1. Introduction

One of the important goals in the study of group theory is to develop tools to compute the number of equivalence classes of a group action on a set. Among such tools, the first is a result of Frobenius, but it was implicit in the work of Cauchy many years earlier than Frobenius and later given again by Burnside in his pioneering book on group theory ([1]). We call the result Burnside's theorem.

Since Burnside's theorem has been introduced, proofs of this result are mentioned in many books or papers. However, the explanations are relatively simple or they used sophisticated tools.

The purpose of this paper is to reinterpret the procedure of the theory development for easier understanding, and to get various useful results by applying this theorem.

In section 2, by using the stabilizer, we introduce a result to compute the number of equivalence classes. And then, as a deformation of it, we obtain Burnside's theorem.

In section 3, as applications of Burnside's theorem, we find out the numbers of different patterns in coloring problems of various types of figures with symmetry and use them for computing the numbers of different kinds of chemical compounds.

In section 4, we study about the computation of the number of patterns using the cycle index.

We refer to the paper[1,2] for Burnside's theorem which has been studied in this context. For the study of group theory to compute the number of equivalence classes, we refer to[3,4,5], and[6,7,8] for other applications of counting problems.

2. Reinterpretation of Burnside's theorem

Let G be a group and $X\,(\neq \varnothing)$ a set. The function

$$\cdot\ :\ G\times X\ \to\ X,\ (g,\,x)\ \mapsto g\cdot x$$

is called an action on G if it satisfies the following two conditions :

(i) $\forall x\in X,\ e\cdot x = x$, where e is the identity of G

(ii) $\forall g,\,h\in G,\ x\in X,\ \ h\cdot(g\cdot x) = (hg)\cdot x$,

and if such an action on G is defined, we say that G acts on X.

Suppose G acts on a set $X(\neq\varnothing)$. Then the relation $\sim$ on X defined by

$$x\sim y\ \ \Leftrightarrow\ y = g\cdot x\ (g\in G)$$

is an equivalence relation, and for $x\in X$ the equivalence class of x is $\{y\in X\mid x\sim y\} = \{y\in X\mid y = g\cdot x,\ g\in G\} = \{g\cdot x\mid g\in G\}$ and we denote it by Gx.

Also, for $x\in X$ if we put $S_x = \{g\in G\mid g\cdot x = x\}$, the

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stabilizer of x, then the following two propositions are hold:

Proposition 1 If G acts on a set $X(\neq\varnothing)$, then S_x is a subgroup of G.

Proof Since $e\cdot x=x$, $e\in S_x\neq\varnothing$. Suppose $g, h\in S_x$. Then

$$g\cdot x=x,\quad h\cdot x=x$$

and so

$$(g^{-1}h)\cdot x=g^{-1}\cdot(h\cdot x)=g^{-1}\cdot x=x.$$

Thus, $g^{-1}h\in S_x$, and hence S_x is a subgroup of G. ▪

Proposition 2 If a finite group G acts on a finite set $X(\neq\varnothing)$, then $|Gx|=\dfrac{|G|}{|S_x|}$.

Proof We compare the numbers of elements of the two sets $Gx=\{g\cdot x\mid g\in G\}$ and $\&=\{gS_x\mid g\in G\}$.

Since $g\cdot x=h\cdot x\Leftrightarrow(h^{-1}g)\cdot x=x\Leftrightarrow h^{-1}g\in S_x$ $\Leftrightarrow gS_x=hS_x$, the function $\phi: Gx\to\&$ defined by $\phi(g\cdot x)=gS_x$ is well defined and one-one correspondence. Thus $|Gx|=|\&|$, and so $|Gx|=\dfrac{|G|}{|S_x|}$ because $|\&|=\dfrac{|G|}{|S_x|}$. ▪

The preceding Proposition 2 yields the following result.

Theorem 3 Suppose a finite group G acts on a finite set $X(\neq\varnothing)$ and the equivalence relation $\sim$ on X is defined by

$$x\sim y\ \Leftrightarrow\ y=g\cdot x\ (g\in G)$$

Then the number of elements in the set $X/\sim$ of all equivalence classes is

$$|X/\sim|=\frac{1}{|G|}\sum_{x\in X}|S_x|,\text{ where } S_x=\{g\in G\mid g\cdot x=x\}.$$

Proof Assume that there are k equivalence classes $G_{x_1}, G_{x_2},\cdots,G_{x_k}$ of X. For each $x\in G_{x_i}$, $Gx=G_{x_i}$. Thus

$$\sum_{x\in Gx_i}|S_x|=\sum_{x\in Gx_i}\frac{|G|}{|Gx|}=\sum_{x\in Gx_i}\frac{|G|}{|Gx_i|}=|G|\cdots(*)$$

On the other hand, $X=\bigcup_{i=1}^{k}Gx_i$ is a disjoint union. Thus, from $(*)$, $\sum_{x\in X}|S_x|=k|G|=|X/\sim||G|$. ▪

Using the above Theorem 3, to compute the number of $|X/\sim|$, we have to get $|S_x|$ for each element $x\in X$ and add them all. However, in general, this computation is very cumbersome, and therefore, the above Theorem can be changed as follows:

Theorem 4 (Burnside's theorem, [1]) The number $|X/\sim|$ as in Theorem 3 can be computed by

$$|X/\sim|=\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|,$$

where $\mathrm{Fix}(g)=\{x\in X\mid g\cdot x=x\}$.

Proof Since

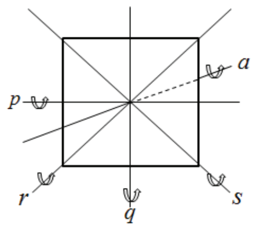
$$\sum_{x\in X}|S_x|=|\{(g,x)\in G\times X\mid g\cdot x=x\}|=\sum_{g\in G}|\mathrm{Fix}(g)|,$$

$$|X/\sim|=\frac{1}{|G|}\sum_{x\in X}|S_x|=\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|.\ ▪$$

The method used in the proofs of Theorem 3 and 4 is different from that of the proofs stated in[1].

Using the above Burnside's theorem, we can compute the number of different patterns in coloring problems of figures with symmetry.

For example, if we consider a square as a figure in the space and let D_4 be the total set of rotational replacements of this square in the space, $D_4=\{e,\ a,\ a^2,\ a^3,\ p,\ q,\ r,\ s\}$ makes a group and we call it the dihedral group of order 8.



a : 90° rotation,
p, q, r, s : 180° rotation

Fig. 1. Elements of D_4

Now, let $G=D_4$, $C=\{b, w\}$ be the set of colors, and $D=\{d_1, d_2, d_3, d_4\}$ be the set of 4-squares of 2×2 size chessboard to paint. We call the function $x: D\to C$ a coloring of D. Let X be the set of all colorings of D. Then the function

$$\cdot: G\times X\to X,\quad (g, x)\mapsto g\cdot x=x\circ g^{-1}$$

is an action on the set X and the equivalence relation $\sim$ on the X as in Theorem 3 is defined. Here, let's think that when the two elements of set X coincide by a proper rotational movement of G those two elements have the same pattern. Then, we can compute the number of different patterns using Burnside's theorem because it is the same as the number of the set $X/\sim$ of all equivalence classes. Therefore, by Table 1 below, the number of different patterns is

$$|X/\sim| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$$
$$= \frac{1}{8}\{2^4+2\cdot 2+3\cdot 2^2+2\cdot 2^3\}=6$$

Table 1. $|\mathrm{Fix}(g)|$ of 2×2 chessboard

$g\in G$	e	a	a^2	a^3	p	q	r	s
$\lvert\mathrm{Fix}(g)\rvert$	2^4	2	2^2	2	2^2	2^2	2^3	2^3

We see from the preceding example how coloring problems may be treated by using the Burnside's theorem.

3. Applications

In this section, using Burnside's theorem, we compute the numbers of different patterns in coloring of figures with various symmetries and use them to find out the numbers of different kinds of chemical compounds.

Theorem 5 If we paint each square of $n\times n$ size chessboard by duplicated choosing from c different colors, the number of different patterns is

$$\frac{1}{8}\left[c^{n^2}+\left\{3-2\left(n-2[\frac{n}{2}]\right)\right\}c^{\frac{n^2+n-2[\frac{n}{2}]}{2}}+2c^{\frac{n^2+3\left(n-2[\frac{n}{2}]\right)}{4}}+2\left(n-2[\frac{n}{2}]+1\right)c^{\frac{n^2+n}{2}}\right],$$

where [ ] is the Gauss symbol.

Proof We get $|\mathrm{Fix}(g)|$ of each element $g\in G=D_4$ of dihedral group $D_4=\{e,\ a,\ a^2,\ a^3,\ p,\ q,\ r,\ s\}$ as follows:

First, if n is an even number, $|\mathrm{Fix}(g)|$ corresponding to each $g\in D_4$ is as in Table 2.

Table 2. $|\mathrm{Fix}(g)|$ of $n\times n$ chessboard (n is even)

$g\in G$	e	a	a^2	a^3	p	q	r	s
$\lvert\mathrm{Fix}(g)\rvert$	c^{n^2}	$c^{\frac{n^2}{4}}$	$c^{\frac{n^2}{2}}$	$c^{\frac{n^2}{4}}$	$c^{\frac{n^2}{2}}$	$c^{\frac{n^2}{2}}$	$c^{\frac{n^2+n}{2}}$	$c^{\frac{n^2+n}{2}}$

Therefore, the number of different patterns is

$$\frac{1}{8}[c^{n^2}+3c^{\frac{n^2}{2}}+2c^{\frac{n^2}{4}}+2c^{\frac{n^2+n}{2}}].$$

Also, if n is an odd number, $|\mathrm{Fix}(g)|$ corresponding to each $g\in D_4$ is as in Table 3.

Table 3. $|\mathrm{Fix}(g)|$ of $n\times n$ chessboard (n is odd)

$g\in G$	e	a	a^2	a^3
$\lvert\mathrm{Fix}(g)\rvert$	c^{n^2}	$c^{\frac{n^2+3}{4}}$	$c^{\frac{n^2+1}{2}}$	$c^{\frac{n^2+3}{4}}$

p	q	r	s
$c^{\frac{n^2+n}{2}}$	$c^{\frac{n^2+n}{2}}$	$c^{\frac{n^2+n}{2}}$	$c^{\frac{n^2+n}{2}}$

Therefore, the number of different patterns is

$$\frac{1}{8}[c^{n^2}+2c^{\frac{n^2+3}{4}}+c^{\frac{n^2+1}{2}}+4c^{\frac{n^2+n}{2}}].$$

So, by combining the above two cases, when we describe in general the number of different patterns of $n\times n$ size, we get the result of the theorem. ▪

The following is to solve the coloring problems in rectangle type figure that is not a square.

Theorem 6 If we paint each square of $n\times m\ (n\neq m)$ size chessboard of by duplicated choosing from c different colors, the number of different patterns is

$$\frac{1}{4}\left[c^{m\times n}+c^{\frac{m\times n+(m-2\times\left[\frac{m}{2}\right])\times(n-2\times\left[\frac{n}{2}\right])}{2}}+c^{\frac{m+(m-2\times\left[\frac{m}{2}\right])\times n}{2}}+c^{\frac{m\times n+(n-2\times\left[\frac{n}{2}\right])}{2}}\right],$$

where [ ] is the Gauss symbol.

Proof We can compute |Fix(g)| for each $g\in G$ by classifying in 4 cases according to the positive integers $n,\ m$ as follows; where $G=\{e,\ a,\ p,\ q\}$ is the Klein's four group.

Case 1 n and m are even numbers

Table 4. $|\mathrm{Fix}(g)|$ of $n\times m$ chessboard (n and m are even)

$g\in G$	e	a	p	q
$\lvert\mathrm{Fix}(g\rvert$	$c^{m\times n}$	$c^{\frac{m\times n}{2}}$	$c^{\frac{m\times n}{2}}$	$c^{\frac{m\times n}{2}}$

Case 2 n is even, m is an odd number

Table 5. $|\mathrm{Fix}(g)|$ of $n\times m$ chessboard (n is even and m is odd)

$g\in G$	e	a	p	q
$\lvert\mathrm{Fix}(g)\rvert$	$c^{m\times n}$	$c^{\frac{m\times n}{2}}$	$c^{\frac{m\times n}{2}}$	$c^{\frac{m\times(n+1)}{2}}$

Case 3 n is odd, m is an even number

Table 6. $|\mathrm{Fix}(g)|$ of $n\times m$ chessboard (n is odd and m is even)

$g\in G$	e	a	p	q
$\lvert\mathrm{Fix}(g)\rvert$	$c^{m\times n}$	$c^{\frac{m\times n}{2}}$	$c^{\frac{(m+1)\times n}{2}}$	$c^{\frac{m\times n}{2}}$

Case 4 n and m are odd numbers

Table 7. $|\mathrm{Fix}(g)|$ of $n\times m$ chessboard (n and m are odd)

$g\in G$	e	a	p	q
$\lvert\mathrm{Fix}(g)\rvert$	$c^{m\times n}$	$c^{\frac{m\times n+1}{2}}$	$c^{\frac{(m+1)\times n}{2}}$	$c^{\frac{m\times(n+1)}{2}}$

Also, by combining the above four cases, $|\mathrm{Fix}(g)|$ corresponding to each element of the group is as in Table 8.

Table 8. $|\mathrm{Fix}(g)|$ of $n\times m$ chessboard

$g\in G$	e	a
$\lvert\mathrm{Fix}(g)\rvert$	$c^{m\times n}$	$c^{\frac{m\times n+(m-2\times\left[\frac{m}{2}\right])\times(n-2\times\left[\frac{n}{2}\right])}{2}}$

p	q
$c^{\frac{m+(m-2\times\left[\frac{m}{2}\right])\times n}{2}}$	$c^{\frac{m\times n+(n-2\times\left[\frac{n}{2}\right])}{2}}$

Therefore, the number of different patterns is

$$\frac{1}{4}\left[c^{m\times n}+c^{\frac{m\times n+(m-2\times\left[\frac{m}{2}\right])\times(n-2\times\left[\frac{n}{2}\right])}{2}}+c^{\frac{m+(m-2\times\left[\frac{m}{2}\right])\times n}{2}}+c^{\frac{m\times n+(n-2\times\left[\frac{n}{2}\right])}{2}}\right].\ ▪$$

Theorem 7 If we paint each vertex of a regular n-polygon by duplicated choosing from different c numbers of colors, the number of different patterns is as follows:

(1) if n is even, $\frac{1}{2n}\left\{c^n+\sum_{i=1}^{n-1}c^{(n,i)}+\frac{n}{2}c^{\frac{n+2}{2}}+\frac{n}{2}c^{\frac{n}{2}}\right\}$

(2) if n is odd, $\frac{1}{2n}\left\{c^n+\sum_{i=1}^{n-1}c^{(n,i)}+nc^{\frac{n+1}{2}}\right\}$,

where (n,i) is the greatest common divisor of n and i.

Proof Let $G=D_n$ be the dihedral group of order $2n$.

(1) If n is an even number, we can compute $|\mathrm{Fix}(g)|$ corresponding to each $g\in G$ in as in Table 9.

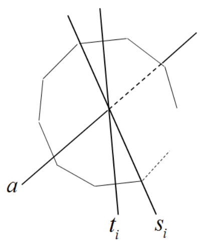


Fig. 2. Elements of G (n is even)

Table 9. $|\mathrm{Fix}(g)|$ of n-polygon group (n is even)

$g\in G$	e	$a^i\ (1\le i\le n-1)$
$\lvert\mathrm{Fix}(g)\rvert$	c^n	$c^{(n,i)}$

$s_i\ (1\le i\le \frac{n}{2})$	$t_i\ (1\le i\le \frac{n}{2})$
$c^{\frac{n+2}{2}}$	$c^{\frac{n}{2}}$

Therefore, the number of different patterns is

$$\frac{1}{2n}\left\{c^n+\sum_{i=1}^{n-1}c^{(n,i)}+\frac{n}{2}c^{\frac{n+2}{2}}+\frac{n}{2}c^{\frac{n}{2}}\right\}.$$

(2) If n is an odd number, we can compute $|\mathrm{Fix}(g)|$ corresponding to each $g\in G$ in as in Table 10.

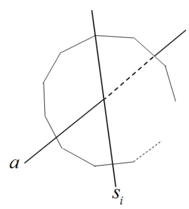


Fig. 3. Elements of G (n is odd)

Table 10. $|\mathrm{Fix}(g)|$ of n polygon group (n is odd)

$g \in G$	e	$a^i\ (1 \le i \le n-1)$	$s_i\ (1 \le i \le n)$
$\|\mathrm{Fix}(g)\|$	c^n	$c^{(n,i)}$	$c^{\frac{n+1}{2}}$

Therefore, the number of different patterns is

$$\frac{1}{2n}\left\{c^n + \sum_{i=1}^{n-1} c^{(n,i)} + nc^{\frac{n+1}{2}}\right\}. \blacksquare$$

Theorem 8 If we spread n different things in a circle, the number of different patterns is $(n-1)!$.

Proof Let $C_n = \{e, a, a^2, \cdots, a^{n-1}\}$ be the cyclic group of order n. We can compute |Fix(g)| for each $g \in C_n$ as in Table 11.

Table 11. $|\mathrm{Fix}(g)|$ of the cyclic group

g	e	a	a^2	$\cdots$	a^{n-1}
$\|\mathrm{Fix}(g)\|$	$n!$	0	0	$\cdots$	0

Hence the number of different patterns is $\frac{1}{n}\{n! + 0 + 0 + \cdots + 0\} = (n-1)!$. $\blacksquare$

Theorem 9 If we paint each face of a regular tetrahedron by duplicated choosing from different c numbers of colors, the number of different patterns is $\frac{1}{12}(c^4 + 11c^2)$.

Proof There are 12 elements of the regular tetrahedron group as follows:

- identity movement e
- 4 kinds of 120 ° rotational movement (a) on the axis passing the barycenter of the side opposite to the vertex
- 4 kinds of 240 ° rotational movement (a^2) on the axis passing the barycenter of the side opposite to the vertex
- 3 kinds of 180 ° rotational movement (α) on the axis connecting each opposite median points

Also, we get $|\mathrm{Fix}(g)|$ for each element of the regular tetrahedron group as in Table 12.

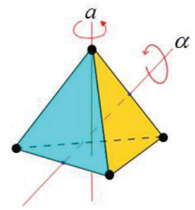


Fig. 4. Elements of the regular tetrahedron group

Table 12. $|\mathrm{Fix}(g)|$ of a regular tetrahedron

g	e	a(4types)	a^2(4types)	α(3types)
$\|\mathrm{Fix}(g)\|$	c^4	c^2	c^2	c^2

Therefore, the number of different patterns is $\frac{1}{12}(c^4 + 11c^2)$. $\blacksquare$

Theorem 10 If we paint each face of a regular hexahedron by duplicated choosing from different c numbers of colors, the number of different patterns is $\frac{1}{24}(c^6 + 3c^4 + 12c^3 + 8c^2)$.

Proof There are 24 elements of the regular hexahedron group as follows:

- identity movement e
- $3 \times 3 = 9$ kinds of $90°(a)$, $180°(a^2)$, $270°\ (a^3)$ rotational movements on the rotational axis connecting the barycenter of the opposite sides
- $2 \times 4 = 8$ kinds of $120°\ (x)$, $240°\ (x^2)$ rotational movements on the rotational axis of a straight line connecting two opposite vertex
- $1 \times 6 = 6$ kinds of $180°\ (\alpha)$ rotational movements on the rotational axis of a straight line connecting median points of opposite edges

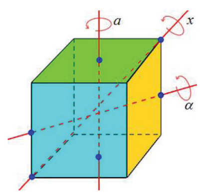


Fig. 5. Elements of the regular hexahedron group

On the other hand, $|\mathrm{Fix}(g)|$ for each element of the regular hexahedron group is as in Table 13.

Table 13. $|\mathrm{Fix}(g)|$ of a cube

g	e	a(3types)	a^2(3types)
$\lvert\mathrm{Fix}(g)\rvert$	c^6	c^3	c^4
a^3(3types)	x(4types)	x^2(4types)	α(6types)
c^3	c^2	c^2	c^3

Therefore, the number of different patterns is $\frac{1}{24}(c^6+3c^4+12c^3+8c^2)$. ▪

Theorem 11 If a chemical compound is compounded with the atoms such as Br, H, CH_3, C_2H_5 in four vertex on the axis of the middle carbon (C), and we call it organic compound, the number of different organic compounds is 36.

Proof Let's think it as a problem to compute the number of patterns for painting each vertex of a regular tetrahedron with Br, H, CH_3, C_2H_5.

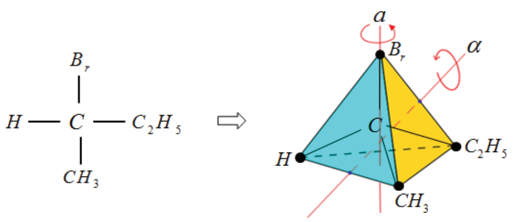


Fig. 6. Substitution of a figure for a chemical compound

We get $|\mathrm{Fix}(g)|$ for each element of the regular tetrahedron group of order 12 as in Table 14.

Table 14. $|\mathrm{Fix}(g)|$ of the organic compound

g	e	a(4types)	a^2(4types)	α(3types)
$\lvert\mathrm{Fix}(g)\rvert$	4^4	4^2	4^2	4^2

Therefore, there are $\frac{1}{12}(4^4+11\cdot 4^2)=36$ of organic compounds. ▪

Theorem 12 A cyclobutane is a hydrocarbon that has 4 carbon atoms and 2 hydrogen atoms are compounded in each carbon atom, and some of them have the structural formula described as the following picture. There are 23 kinds of compounds that can be got by changing hydrogen to nitrogen from cyclobutane.

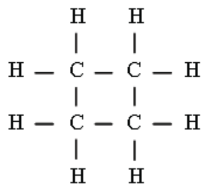


Fig. 7. Plane structure of cyclobutane

Proof Let's think it as a problem to compute the number of patterns for painting each vertex of a cube that has 4 carbon atoms by choosing from hydrogen (H) and nitrogen (N).

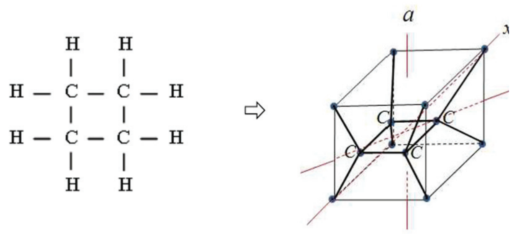


Fig. 8. Substitution of a figure for a cyclobutane

On the other hand, if we paint each vertex of a cube choosing from 2 colors, $|\mathrm{Fix}(g)|$ for each element of the regular hexahedron group of order 24 is as in Table 15.

Table 15. $|\mathrm{Fix}(g)|$ of the cyclobutane

g	e	a(3types)	a^2(3types)
$\lvert\mathrm{Fix}(g)\rvert$	2^8	2^2	2^4
a^3(3types)	x(4types)	x^2(4types)	α(6types)
2^2	2^4	2^4	2^4

Therefore, there are $\frac{1}{24}(2^8+17\cdot 2^4+6\cdot 2^2)=23$ kinds of compounds. ▪

Theorem 13 There are 138 kinds of isomers that can change hydrogen atom into nitrogen or oxygen in a compound with a molecular formula as C_3H_6 seen in the following picture:

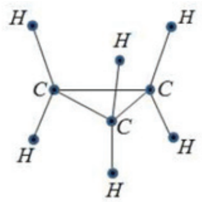


Fig. 9. Model of C_3H_6

Proof Let's think this problem as the one to compute the number of patterns for painting each vertex after choosing from atoms of hydrogen, nitrogen and oxygen with three carbon atoms inside a triangular prism as seen in the picture below.

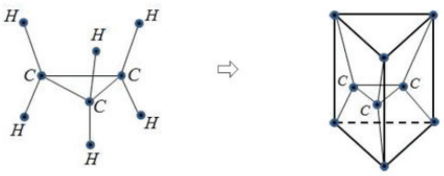


Fig. 10. Substitution of the figure for C_3H_6

On the other hand, a set that is made by the total rotational movements to make the following solid figure overlap with itself is the group $G=\{e, a, a^2, x, y, z\}$ of order 6.

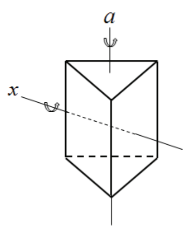


Fig. 11. Elements of G

Also, if we paint each vertex of triangular prism after choosing 3 colors, $|\mathrm{Fix}(g)|$ for each 6 elements of the group is as in Table 16.

Table 16. $|\mathrm{Fix}(g)|$ of isomers

g	e	a	a^2	x	y	z
$\lvert\mathrm{Fix}(g)\rvert$	3^6	3^2	3^2	3^3	3^3	3^3

Therefore, there are $\frac{1}{6}(3^6+2\cdot 3^2+3\cdot 3^3)=138$ kinds of compounds. ▪

4. Computation of the Number of Patterns using the Cycle Index

Let S_n be the symmetric group of degree n. For each $\pi\in S_n$, if π is the product of k_i disjoint cycles of length l_i $(i=1,2,\cdots,s)$, then the cycle type of π is defined by $x_{l_1}^{k_1}x_{l_2}^{k_2}\cdots x_{l_s}^{k_s}$ and denoted it by $\mathrm{cycle\,type}\,(\pi)$, where $\sum_{i=1}^{s}k_il_i=n$. For instance, consider the permutation $\pi=(34)(56)(789)$ in the symmetric group S_{10}. This permutation π has the cycle type $x_1^3x_2^2x_3$.

Let G be a subgroup of the symmetric group S_n. The cycle index of G is defined by $\frac{1}{|G|}\sum_{\pi\in G}\mathrm{cycle\,type}\,(\pi)$, and denoted it by $C_G(x_1, x_2, \cdots, x_n)$[9].

In solving the problem of coloring the n objects as we studied before, if we regard G as a subgroup of S_n and use a cycle index of G, we easily compute the number of patterns as the following theorem.

Theorem 14 If G has the cycle index $C_G(x_1, x_2, \cdots, x_n)$ in the coloring problems by using c colors, then the number of different patterns is $C_G(c, c, \cdots, c)$.

Proof If $\pi\in G$ has the cycle type $x_{l_1}^{k_1}x_{l_2}^{k_2}\cdots x_{l_s}^{k_s}$, then

$$\mathrm{Fix}(\pi)=c^{k_1}c^{k_2}\cdots c^{k_s}.$$

Thus the assertion is hold by Burnside's theorem. ▪

Example 1 When we paint 3×3 chessboard by duplicated choosing from different c numbers of colors, if we think an element of $G=D_4$ as that of S_9 and obtain the cycle index of G, $C_G(x_1, x_2, x_3, x_4)=\frac{1}{8}\{x_1^9+2x_1x_4^2+x_1x_2^4+4x_1^3x_2^3\}$, by Table 17.

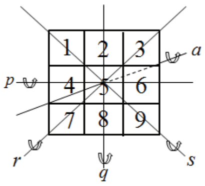


Fig. 12. Elements of D_4

Table 17. Cycle types of 3×3 chase board

elements	e	a	a^2	a^3
cycle type	x_1^9	$x_1x_4^2$	$x_1x_2^4$	$x_1x_4^2$

p	q	r	s
$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$	$x_1^3x_2^3$

Hence the number of different patterns is

$$C_G(c,c,c,c)=\frac{1}{8}\{c^9+2c\cdot c^2+c\cdot c^4+4c^3\cdot c^3\}$$
$$=\frac{1}{8}\{c^9+2c^3+c^5+4c^6\}$$

Example 2 If we paint each face of a regular hexahedron by duplicated choosing from different c numbers of colors, let's determine the number of different patterns.

From Theorem 10, there are 24 elements of the regular hexahedron group G, and each element has the cycle type as follows:

Table 18. Cycle types of the regular hexahedron for face painting

elements	e	a(3types)	a^2(3types)
cycle type	x_1^6	$x_1^2x_4$	$x_1^2x_2^2$

a^3(3types)	x(4types)	x^2(4types)	α(6types)
$x_1^2x_4$	x_3^2	x_3^2	x_2^3

Thus

$$C_G(x_1,x_2,x_3,x_4)=\frac{1}{24}\{x_1^6+6x_1^2x_4+3x_1^2x_2^2+6x_2^3+8x_3^2\},$$

and so the number of different patterns is

$$C_G(c,c,c,c)=\frac{1}{24}(c^6+3c^4+12c^3+8c^2)$$

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