[UCI]I804:24011-200000281803

2020년 2월
교육학석사(수학교육전공)학위논문

Characterizing certain flag vector pairs of 4-polytopes

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이 논문을 교육학석사(수학교육전공)학위 청구논문으로 제출함.

2019년 10월

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박나리의 교육학 석사학위 논문을 인준함.

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2019년 12월

조선대학교 교육대학원

목 차

국문초록

4차원 다면체의 특정한 플래그벡터 순서쌍의 결정에 관한 연구

박 나 리
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본 논문은 4차원 다면체의 면의 개수를 표현하는 플래그벡터에 관한 연구이다. 2018년에 Sjöberg와 Ziegler는 4차원 다면체의 플래그벡터 순서쌍 (f_0,f_{03})을 완벽하게 결정하는 연구결과를 발표하였다. Sjöberg와 Ziegler는 이 연구결과를 얻기 위해 Altshuler와 Steinberg의 최대 8개의 꼭짓점을 갖는 4차원 다면체에 관한 연구결과를 이용하였다. 이에 본 연구는 이와 같은 연구를 심도 있게 이해하고, 이를 바탕으로 기존의 연구 방법을 확장하여 4차원 다면체의 플래그벡터 순서쌍 $(f_{0,}f_{02})$, $(f_{02,}f_{03})$, $(f_{1,}f_{02})$, $(f_{1,}f_{03})$의 범위에 관한 새로운 결과를 제시하였다.

Ⅰ. Introduction and Main Results

For a d-dimensional polytope P, let $f_i = f_i(P)$ denote the number of $i-$dimensional faces of P. One of the fundamental combinatorial invariants of P is its f-vector $(f_0, f_1, \ldots, f_{d-1})$, which we are mainly concerned with in this thesis. The Euler-Poincare formula gives the well-known restriction on the f-vectors of simplicial polytopes. Another well-known restriction on the f-vectors for simplicial polytopes is the so-called Dehn-Sommerville equations. In [9], McMullen conjectured some characterization of the f-vectors of simplicial polytopes, and then it has been verified by Stanley in [11] and Billera-Lee in [4]. However, currently any complete characterization of the f-vectors of all simplicial polytopes is very much out of reach.

There is another combinatorial invariant for convex polytopes, called the flag vector, which are not relatively well known but obviously generalizes the notion of the f-vector. That is, for $S \subseteq \{0, \ldots, d-1\}$, let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \ldots \dim F_r\} = S.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0, 1, 2, \ldots d-1\}$. For instance, $f_{03}(P)$ will mean $f_{\{0,3\}}(P)$. Note that the f-vector of P is then

$(f_0, f_1, \ldots f_{d-1})$, while the flag vector of P is defined to be $(f_S)_{S \subseteq \{0, \ldots, d-1\}}$.

The set of all f-vectors of d-dimensional polytopes will be denoted by F^d. Clearly F^d is a subset of $\mathbb{Z}^d$. Let $\Pi_{i,j}(F^d)$ denote the projection of f-vector of $P \in F^d$ onto the coordinates f_i and f_j. Then $(n,m) \in \Pi_{i,j}(F^d)$ is called a *polytopal pair* if there is a d-polytope P with $f_i(P) = n$ and $f_i(P) = m$. If $(n,m) \in \Pi_{0,d-1}(F^d)$, then these pairs must satisfy the well-known upper bound theorem saying

$$m \le f_{d-1}(C_d(n)),\ n \le f_{d-1}(C_d(m)),$$

where $C_d(n)$ denotes a d-dimensional cyclic polytope with n vertices([5]). For the moment curve in $\mathbb{R}^d$ defined by

$$\alpha : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto (t, t^2, \ldots, t^d) \in \mathbb{R}^d$$

and for any $n > d$, the standard d-th cyclic polytope with n vertices, denoted by $C_d(t_1, t_2, \ldots, t_n)$, is defined as the convex hull in R^d of n different points $\alpha(t_1), \ldots, \alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$. Cyclic polytopes $C_d(n)$ are precisely those which are combinatorial equivalent to the standard cyclic polytope $C_d(t_1, t_2, \ldots t_n)$ (see [7] for more details).

In a similar vein, for any two subsets S_1 and S_2 of $\{0,1,2,\ldots,d-1\}$, a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a *flag vector pair*. More generally, for any k, not necessarily mutually disjoint, subsets $S_1, S_2, \ldots, S_k$ of $\{0,1,2,\ldots,d-1\}$, a k-tuple

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P))$$

or simply $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$, of flag numbers of P will be called a *flag vector k-tuple*.

As in the f-vectors, let us denote by $\Pi_{S_1, S_2, \ldots, S_k}$ the projection of the flag vector $(f_S)_{S \subseteq \{0, \ldots, d-1\}}$ onto its coordinates $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$. We call $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ a *polytopal flag vector k-tuple* if $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ belongs to the image of the set of all flag vectors of d-dimensional polytopes under the projection map $\Pi_{S_1, S_2, \ldots, S_k}$, that is, if there is a d-polytope P such that

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P)) = (f_{S_1}, f_{S_2}, \ldots, f_{S_k}).$$

In [6], recently Sjöberg and Ziegler showed a remarkable result that completely determines the flag vector pair (f_0, f_{03}) of any 4-dimensional polytopes. In order to obtain such results, they crucially used the work [1] of Altshuler and Steinberg on 4-polytopes up to 8 vertices. Furthermore, they used the techniques of stacking, general stacking on cyclic polytopes, facet splitting, truncating, and so on for the construction of specific 4-dimensional polytopes.

The goal of this thesis is to give some necessary conditions for other remaining vector pairs such as (f_0, f_{02}), (f_{02}, f_{03}), (f_1, f_{02}), (f_1, f_{03}) to be flag vector pairs of 4-dimensional convex polytopes.

Our main results go as follows.

Theorem 1.1 The flag vector pair $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ of a 4-polytope P satisfies the following two conditions:

(1) $f_0 \geq 6$ and for $k \in \{1,2,3,4,5,6,7,8,10,11\}$, $f_{02} \neq 3f_0(f_0 - 3) - k$.

(2) $6f_0 \leq f_{02} \leq 3f_0(f_0 - 3)$.

Recall that a convex polytope P is called *neighborly* (or *2-neighborly*) if any pair of vertices of P is connected by an edge, forming a complete graph. So any non-neighborly polytope P should have at least one pair of vertices of P which do not form an edge.

Theorem 1.2 The flag vector pair $(f_{02}, f_{03}) = (f_{02}(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the inequalities

$$2\left(3 + \sqrt{\frac{63 + 4f_{02}}{3}}\right) \leq f_{03} \leq \frac{2}{3} f_{02} .$$

Theorem 1.3 The flag vector pair $(f_1, f_{02}) = (f_1(P), f_{02}(P))$ of a 4-polytope P satisfies the inequalities

$$f_1 + 2(1 + \sqrt{1 + 8f_1}) \leq f_{02} \leq 6f_1 - 3(1 + \sqrt{1 + 8f_1}).$$

Theorem 1.4 The flag vector $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the inequalities

$$6+2\sqrt{\frac{71+4f_1+8\sqrt{1+8f_1}}{3}} \leq f_{03} \leq 4f_1 - 2(1+\sqrt{1+8f_1}).$$

It would be interesting to investigate whether or not there are some examples which achieve the lower and upper bounds given in Theorems 1.1, 1.2, 1.3, and 1.4. We also remark that some obstructions for flag vector pairs $(f_{1,}f_{04})$ of 5-polytopes have been proved in [6]. To be a little more precise, certain bounds of the flag number f_{04} of a 5-polytope have been shown in terms of a given flag number f_1. The upper bounds given in [6] are not optimal, even though the lower bounds are very sharp. On the other hand, recently very sharp and optimal upper and lower bounds for f_1 and f_{04} have been finally obtained in [8].

This thesis is organized as follows.

In Chapter 2, we collect some notation, definitions, and preliminary facts in order to prove our main theorems given in Chapter 3.

In Chapter 3, we give some necessary conditions for vector pairs such as $(f_{0,}f_{02})$, $(f_{02,}f_{03})$, $(f_{1,}f_{02})$, $(f_{1,}f_{03})$ to be flag vector pairs of 4-dimensional convex polytopes. To be more precise, in Section 3.1 we give a proof of Theorem 1.1 for vector pair $(f_{0,}f_{02})$. To do so, we use the method of a case-by-case analysis. Section 3.2 is devoted to giving a proof of Theorem 1.2 for vector pair (f_{02}, f_{03}), while Section 3.2 deals with the case of vector pair $(f_{1,} f_{02})$ and there we provide a proof of Theorem 1.3. Finally, in Section 3.4, we give some bounds for vector pair $(f_{1,} f_{03})$.

Ⅱ. Preliminaries

This chapter briefly describes some important theorems necessary for understanding the proof of our main results given in the next section. In addition, we set up some notation and definitions.

First, we begin with summarizing the well-known facts about the f-vector of convex polytopes, in particular, 4-dimensional polytopes.

Theorem 2.1 (Grünbaum, [7, Theorem 10.4.1]). The set of f-vector pairs (f_0, f_3) of 4-polytopes is equal to

$$\Pi_{0,3}(F^4) = \left\{ (f_0, f_3) \in Z^2 : 5 \leq f_0 \leq \frac{1}{2} f_3 (f_3 - 3), 5 \leq f_3 \leq \frac{1}{2} f_0 (f_0 - 3) \right\}.$$

Theorem 2.2 (Grünbaum, [7, Theorem 10.4.2]). The set of f-vector pairs (f_0, f_1) of 4-polytopes is equal to

$$\Pi_{0,1}(F^4) = \left\{ (f_0, f_1) \in Z^2 : 10 \leq 2f_0 \leq f_1 \leq \frac{1}{2} f_0 (f_0 - 1) \right\}$$
$$- \{(6,12),(7,14),(8,17),(10,20)\}.$$

Theorem 2.3 (Generalized Dehn-Sommerville equation, [3, Theorem 2.1]) Let P be a d-polytope, and let $S \subseteq \{0,1,2,\cdots,d-1\}$. Let $\{i,k\} \subseteq S \cup \{-1,d\}$ such that $i < k-1$ and $\nexists\, j \in S$ such that $i < j < k$. Then, we have

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

Theorem 2.4 (Sjöberg and Ziegler, [8, Theorem 2.5]). There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if and only if the following two conditions hold:

(1) f_0 and f_{03} are integers satisfying

$$20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0 - 3),$$

$$f_{03} \neq 2f_0(f_0 - 3) - k, \quad k \in \{1,2,3,5,6,9,13\}.$$

(2) (f_0, f_{03}) is not one of the 18 exceptional pairs

(6, 24), (6, 25), (6, 28),

(7, 28), (7, 30), (7, 31),

(7, 33), (7, 34), (7, 37), (7, 40),

(8, 33), (8, 34), (8, 37), (8, 40),

(9, 37), (9, 40), (10, 40), (10, 43).

Theorem 2.5 (Bayer, [2, Theorem 1.3 and 1.4]). The flag vector of every 4-polytope satisfies the inequalities

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \text{ and } -6f_0 + 6f_1 - f_{02} \geq 0.$$

Ⅲ. Proofs of Main Results

The aim of this chapter is to give proofs of our main Theorems 1.1, 1.2, 1.3, and 1.4.

3.1 The flag vector pair $(f_{0,} f_{02})$ for 4-polytopes

In this section, we prove a series of lemmas in order to characterize the following set

$$\Pi_{0,02}(F^4) = \left\{(f_0, f_{02}) \in \mathbb{Z}^2 \mid P \text{ is a } 4-\text{polytope}\right\}$$

We begin with the following lemma.

Lemma 3.1.1 For $d=4$ or 5, the flag vector of every d-polytope satisfies the equalities:

(1) For $d=5$, we have

$$0 = 2f_1 - f_{02} + f_{03} - f_{04}.$$

(2) For $d=4$, we have

$$f_{02} = -2f_0 + 2f_1 + f_{03}.$$

Proof. (1) For the proof, we use the Generalized Dehn-Sommerville equation ([3, Theorem 2.1] or Theorem 2.3) with $S=\{0\}, i=0$, and $k=5$. Then it is straightforward to obtain

$$\sum_{j=1}^{4}(-1)^{j-1}f_{\{0,j\}} = f_0(1-(-1)^{5-0-1})$$
$$f_{01}-f_{02}+f_{03}-f_{04}=0 \text{ and } f_{01}=2f_1$$
$$\therefore 0=2f_1-f_{02}+f_{03}-f_{04}$$

(2) Once again, we apply the Generalized Dehn-Sommerville equation ([3, Theorem 2.1] or Theorem 2.3) with $S=\{0\}, i=0, k=4$ and $f_{01}=2f_1$. Then we can obtain

$$\sum_{j=1}^{3}(-1)^{j-1}f_{\{0,j\}} = f_0(1-(-1)^{4-0-1})$$
$$f_{01}-f_{02}+f_{03}=2f_0 \text{ and } f_{01}=2f_1$$
$$\therefore f_{02}=-2f_0+2f_1+f_{03}$$

□

As an immediate consequence, we have the following result.

Corollary 3.1.2

The flag vector of every 4-polytope satisfies the inequalities

$$6f_0 \le f_{02} \le 3f_0(f_0-3),\ \ f_0 \ge 5.$$

Proof. Recall that it follows from Sjöberg and Ziegler ([8, Theorem 2.5] or Theorem 2.4) that

$$4f_0 \le f_{03} \le 2f_0(f_0-3).$$

By combining the above inequalities with the identity in Lemma 3.1.1 (2), we can obtain

$$f_{03}=2f_0-2f_1+f_{02} \le 2f_0(f_0-3)$$
$$\begin{aligned}\therefore f_{02} &\le -2f_0+2f_1+2f_0(f_0-3)\\ &\le -2f_0+f_0(f_0-1)+2f_0(f_0-3),\end{aligned}$$

where in the last inequality we used Theorem 2.2 (or Grünbaum [2, Theorem 10.4.2]). Thus, we have

$$\begin{aligned} f_{02} &\le -2f_0 + (f_0)^2 - f_0 + 2(f_0)^2 - 6f_0 \\ &= 3(f_0)^2 - 9f_0 = 3f_0(f_0-3). \end{aligned}$$

Also, it follows from $f_{03} \ge 4f_0$ and $f_1 \ge 2f_0$ that we have

$$f_{03} = 2f_0 - 2f_1 + f_{02} \ge 4f_0 \Rightarrow f_{02} \ge 2f_0 + 2f_1 \ge 6f_0.$$

□

As a consequence,

$$\Pi_{0,02}(F^4) \subseteq \{(n,m) \in \mathbb{Z}^2 \mid 6n \le m \le 3n(n-3), n \ge 5\}.$$

Note that $f_{02} = 3f_0(f_0-3)$ if and only if P is neighborly. Thus, if $f_{02} < 3f_0(f_0-3)$, then P is not neighborly,

$$\text{i.e. } f_1 < \frac{1}{2}f_0(f_0-1) = \binom{f_0}{2}$$

3.1.1 $1 \le k \le 5$ case

The aim of this subsection is to prove the following lemma.

Lemma 3.1.3 The following statement holds.

$$f_{02} \ne 3f_0(f_0-3) - k, \text{ for } k = 1,2,3,4,5.$$

Proof. We prove this lemma by contradiction. So suppose

$$f_{02} = 3f_0(f_0-3) - k$$

for some integer $1 \le k \le 5$. Then it follows from Sjöberg and Ziegler ([8, Theorem 2.5] or Theorem 2.4) that we have

$$\begin{aligned}3f_0(f_0-3)-k = f_{02} &= -2f_0+2f_1+f_{03}\\ &\le -2f_0+2f_1+2f_0(f_0-3)-4.\end{aligned}$$

Once again, by [8, Theorem 2.5] or Theorem 2.4 recall also that

$$f_{03} \neq 2f_0(f_0-3)-k$$

for $k \in \{1,2,3,5,6,9,13\}$. Thus, we have

$$\begin{aligned}2f_1 &\ge 2f_0-2f_0(f_0-3)+4+f_0(3f_0-9)-k,\ k>0\\ &= 3f_0^2-9f_0-2f_0^2+6f_0+2f_0+4-k\\ &= f_0^2-f_0+(4-k)\\ \therefore f_1 &\ge \frac{f_0(f_0-1)}{2}+\frac{4-k}{2} = \binom{f_0}{2}+\frac{4-k}{2}.\end{aligned}$$

Note also that since $f_{02}=3f_0(f_0-3)-k, k>0$, P is not neighborly. Thus

$$f_1 < \binom{f_0}{2}\ .$$

Thus, we have

$$\binom{f_0}{2}-\frac{k-4}{2} \le f_1 < \binom{f_0}{2}.$$

Therefore, if $1 \le k \le 5$, then we have

$$\binom{f_0}{2}-1 < \binom{f_0}{2}-\frac{1}{2} \le f_1 < \binom{f_0}{2}.$$

This implies that there does not exist a 4-polytope P such that

$$f_{02} = 3f_0(f_0-3)-k,\ \ k=1,2,3,4,5$$

This is a contradiction.

$\square$

3.1.2 $k=6$ or $k=7$ case

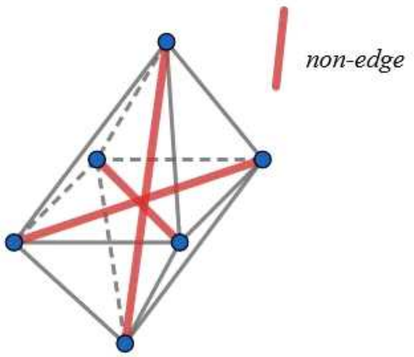
In this subsection, we exclude the case of $k=6$ and $k=7$. Note that if k=6 or 7, then we have

$$\binom{f_0}{2} - 2 < \binom{f_0}{2} - \frac{3}{2} \le f_1 < \binom{f_0}{2}.$$

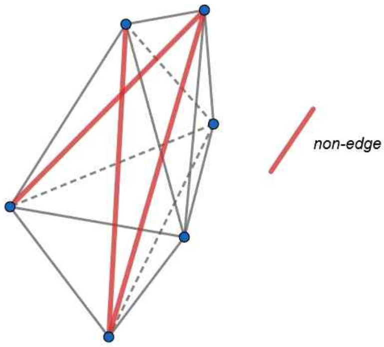
$$\therefore f_1 = \binom{f_0}{2} - 1.$$

In this case, $f_{02} = 3f_0(f_0 - 3) - 6$ or $f_{02} = 3f_0(f_0 - 3) - 7$.

In order to deal with this case, we first need to recall that if there is a pair of vertices of a polytope not forming an edge, then such an edge is called a *non-edge* (See [Figure 3.1] and [Figure 3.2]).



[Figure 3.1]



[Figure 3.2]

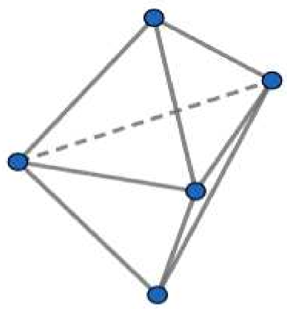
Since $f_1 = \binom{f_0}{2} - 1$ in our case, there is a unique pair v_1, v_2 of vertices of P not forming an edge. That is, there is a unique *non-edge*. Thus, let F be any facet of P which is not a simplex. Then F should contain the *unique non-edge*.

Lemma 3.1.4 F should have only five vertices.

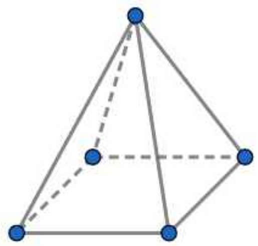
Proof. Suppose F has more than five vertices of F for which every two vertices form an edge. But there does not exist any 3-polytope satisfying such a property.

□

Lemma 3.1.5 There are only two combinatorial types of 3-polytopes with five vertices, the square pyramid or the bipyramid over a triangle (see [Figure 3.3] and [Figure 3.4]).



[Figure 3.3]
Bipyramid over a triangle



[Figure 3.4]
Square pyramid

Lemma 3.1.6

$$f_{02} = 3f_2 = 6t+9.$$

Proof. Only the bipyramid over a triangle contains exactly a non-edge. As a consequence, P is a polytope with one bipyramid facet and remaining t $(t \geq 0)$ tetrahedral facets. In particular, every 2-dimensional faces of P is a triangle. For the proof, it suffices to observe the following identity

$$f_2 = \frac{4t+6}{2} = 2t+3.$$

□

Lemma 3.1.7 The following statements hold.

(1) $f_{02} \neq 3f_0(f_0-3)-6$.

(2) $f_{02} \neq 3f_0(f_0-3)-7$.

Proof. (2) Since $f_{02} = 3f_2$, it is true that $f_{02} \equiv 0 \mod 3$. Thus, we have

$$f_{02} \neq 3f_0(f_0-3)-7.$$

(1) Since $f_{02} = 6t+9 \equiv 3 \mod 6$, we have

$$f_{02} \neq 3f_0(f_0-3)-6 \equiv 0 \mod 6.$$

□

3.1.3 $k=8$ or $k=9$ case

In this subsection, we deal with the case of $k=8$ and $k=9$. If $8 \leq k \leq 9$, then we have

$$\binom{f_0}{2} - 3 < \binom{f_0}{2} - \frac{5}{2} \le f_1 < \binom{f_0}{2}.$$

$$\therefore f_1 = \binom{f_0}{2} - 1 \text{ or } f_1 = \binom{f_0}{2} - 2.$$

By the arguments given in the previous subsections, if $f_1 = \binom{f_0}{2} - 2$, then P has only two non-edges. Let F^3 be any 3-dimensional face of P. Then we have

$$\binom{f_0(F^3)}{2} - 2 \le f_1(F^3) \le 3f_0(F^3) - 6.$$

$$\therefore f_0(F^3) < 6, \text{ and so } f_0(F^3) = 5.$$

That is, any non-tetrahedral facet is a polytope with 5 vertices: the bipyramid over a triangle or the square pyramid. Anyway, once again any 2-dimensional face is a triangle. Hence, it follows from Lemma 3.1.6 that we have

$$f_{02} = 3f_0(f_0 - 3) - 9,$$

and so at least we have $f_{02} \ne 3f_0(f_0 - 3) - 8$. In fact, it can be shown that under the condition of $f_1 = \binom{f_0}{2} - 2$ we also have

$$f_{02} \ne 3f_0(f_0 - 3) - 9.$$

To be precise, it follows from Theorem 2.5 that we have

$$-6f_0 + 6f_1 \ge f_{02}.$$

Thus, it is straightforward to obtain

$$\begin{aligned} -6f_0 + 6f_1 &= -6f_0 + 6\left(\frac{f_0(f_0 - 1)}{2} - 2\right) = 3f_0(f_0 - 3) - 12 \\ &\ge f_{02} = 3f_0(f_0 - 3) - 9, \end{aligned}$$

which is a contradiction[1].

The case of $f_1 = \binom{f_0}{2} - 1$ can be dealt with in a similar way to show that

$$f_{02} \neq 3f_0(f_0 - 3) - 8 \text{ and } f_{02} = 3f_0(f_0 - 3) - 9.$$

Note, however, that in this case it is not possible to show

$$f_{02} \neq 3f_0(f_0 - 3) - 9$$

by the arguments as above.

3.1.4 $k=10$ or $k=11$ case

Finally, we deal with the case of $k=10$ and $k=11$.

Lemma 3.1.8 The following statements hold.

(1) $f_{02} \neq 3f_0(f_0 - 3) - 10.$

(2) $f_{02} \neq 3f_0(f_0 - 3) - 11.$

Proof. If $10 \leq k \leq 11$, then we have

$$\binom{f_0}{2} - 4 < \binom{f_0}{2} - \frac{7}{2} \leq f_1 < \binom{f_0}{2}.$$

$$\therefore f_1 = \binom{f_0}{2} - 1 \text{ or } f_1 = \binom{f_0}{2} - 2 \text{ or } f_1 = \binom{f_0}{2} - 3 \ .$$

(1) If $f_1 = \binom{f_0}{2} - 1$ or $f_1 = \binom{f_0}{2} - 2$, then it follows from the previous arguments that $f_{02} \equiv 3 \bmod 6$.

$$\therefore f_{02} \neq 3f_0(f_0 - 3) - 10,$$
$$f_{02} \neq 3f_0(f_0 - 3) - 11.$$

1) This observation is due to Professor Nam Kwon Kim.

$$\therefore f_1 = \binom{f_0}{2} - 3,\ 10 \le k \le 11.$$

(2) Assume that

$$\therefore f_1 = \binom{f_0}{2} - 3,\ 10 \le k \le 11.$$

Let F^3 be any facet of P. Then, F^3 has at most 3 non-edges. Hence,

$$\binom{f_0(F^3)}{2} - 3 \le f_1(F^3) \le 3f_0(F^3) - 6.$$

$$\therefore f_0(F^3) < 7.$$

(i) Assume $f_0(F^3) = 6$ and F^3 has 12 edges and 3 non-edges. There are two such combinatorially different 3-polytopes, which are both simplicial (See [Figure 3.1] and [Figure 3.2]). Let t denote the number of tetrahedral facets of P. Then

$$f_{02} = 3f_2 = 3(2t+4) = 6t + 12 \equiv 0 \bmod 6.$$

$$\therefore f_{02} \ne 3f_0(f_0 - 3) - 10,$$
$$f_{02} \ne 3f_0(f_0 - 3) - 11\,.$$

(ii) By the previous case, P has non-tetrahedral facets, all of them with five vertices. Since there are at most 3 non-edges, we cannot have more than three non-tetrahedral facets. So we have two cases:

(a) The non-tetrahedral facets of P are 3 bipyramids over a triangle.

(b) The non-tetrahedral facets of P are two square bipyramids and one bipyramid over a triangle.

In case of (a), $f_2 = \dfrac{4t+18}{2} = 2t + 9$ (t:=number of tetrahedral facets)

$$\therefore f_{02} = 3f_2 = 3(2t+9) = 6t+27$$
$$\equiv 3 \bmod 6.$$

$$\therefore f_{02} \neq 3f_0(f_0-3) - 10,$$
$$f_{02} \neq 3f_0(f_0-3) - 11\,.$$

In case of (b), $f_2 = \dfrac{4t+8+6}{2} = 2t+7.$

$$\therefore f_{02} = 3f_2 = 6t+21$$
$$\equiv 3 \bmod 6.$$

$$\therefore f_{02} \neq 3f_0(f_0-3) - 10,$$
$$f_{02} \neq 3f_0(f_0-3) - 11\,.$$

□

To sum up all of the previous results, we can obtain the following theorem (Theorem 1.1).

Theorem 3.1.9 The flag vector pair $(f_0, f_{02}) = (f_0(P), f_{02}(P))$ of a 4-polytope P satisfies the following two conditions:

(1) $f_0 \geq 6$ and for $k \in \{1,2,3,4,5,6,7,8,10,11\}$, $f_{02} \neq 3f_0(f_0-3) - k$.

(3) $6f_0 \leq f_{02} \leq 3f_0(f_0-3)$.

3.2 The flag vector pairs (f_{02}, f_{03}) for 4-polytopes

The aim of this subsection is to give a proof of the following theorem (Theorem 1.2).

Theorem 3.2.1 The flag vector pair $(f_{02}, f_{03}) = (f_{02}(P), f_{03}(P))$ of a

4-polytope P satisfies the inequalities

$$2\left(3+\sqrt{\frac{63+4f_{02}}{3}}\right)\leq f_{03}\leq \frac{2}{3}f_{02}\ .$$

Proof. Recall first that

(1) $f_{02}=f_{03}-2(f_0-f_1)$ ($\because$ Lemma 3.1.1 (2))

(2) $-6(f_0-f_1)-f_{02}\geq 0$ ($\because$ [2, Theorem 1.3 and 1.4] or Theorem 2.5)

By (2), we can obtain

$$f_0-f_1\leq -\frac{1}{6}f_{02}.$$

Similarly, by (1) it is easy to obtain

$$\begin{aligned} f_{02}&=f_{03}-2(f_0-f_1)\\ &\geq f_{03}+\frac{1}{3}f_{02}\\ &\therefore \frac{2}{3}f_{02}\geq f_{03},\ \text{i.e., } f_{03}\leq \frac{2}{3}f_{02}. \end{aligned}$$

Recall also from Sjöberg and Ziegler [8, Theorem 2.5](or Theorem 2.4) that

(3) $f_{03}\geq 4f_0$

(4) $f_{02}\leq 3f_0(f_0-3)-9.$

Thus, it is not difficult to obtain

$$(f_0)^2-3f_0-3-\frac{1}{3}f_{02}\geq 0\ .$$

$$\therefore f_0\geq \frac{1}{2}\left(3+\sqrt{\frac{63+4f_{02}}{3}}\right).$$

$$\therefore f_{03}\geq 2\left(3+\sqrt{\frac{63+4f_{02}}{3}}\right).$$

This completes the proof of Theorem 3.2.1.

□

3.3 The flag vector pairs (f_1, f_{02}) for 4-polytopes

The aim of this subsection is to give a proof of the following theorem (Theorem 1.3).

Theorem 3.3.1 The flag vector pair $(f_1, f_{02}) = (f_1(P), f_{02}(P))$ of a 4-polytope P satisfies the inequalities

$$f_1 + 2(1+\sqrt{1+8f_1}) \le f_{02} \le 6f_1 - 3(1+\sqrt{1+8f_1}).$$

Proof. Recall first that

(1) $-6(f_0 - f_1) \ge f_{02}$ ($\because$ Bayer [2, Theorem 1.3 and 1.4] or Theorem 2.5)

(2) $2f_1 \le f_0(f_0 - 1)$ ($\because$ Grünbaum [7, Theorem 10.4.2] or Theorem 2.2)

Thus by (2) it is easy to obtain

$$(f_0)^2 - f_0 - 2f_1 \ge 0 \Rightarrow f_0 \ge \frac{1}{2}\left(1+\sqrt{1+8f_1}\right).$$

Also, it follows from (1) that

$$\begin{aligned} f_{02} &\le -6f_0 + 6f_1 \\ &\le -3(1+\sqrt{1+8f_1}) + 6f_1 \end{aligned}$$

Recall also from Sjöberg and Ziegler [8, Theorem 2.5] (or Theorem 2.4) that we have $f_{03} \ge 4f_0$. Thus it follows from

$$f_0 \ge \frac{1}{2}\left(1+\sqrt{1+8f_1}\right)$$

that we can obtain

$$f_{03} \ge 4f_0 \ge 2(1+\sqrt{1+8f_1}).$$

On the other hand, by using the identity $f_{02} = -2f_0 + 2f_1 + f_{03}$ (Lemma 3.1.1(2)) and $f_1 \ge 2f_0$, we have

$$\begin{aligned} f_{02} &= -2f_0 + 2f_1 + f_{03} \\ &\geq -f_1 + 2f_1 + 2(1+\sqrt{1+8f_1}) \\ &= 2(1+\sqrt{1+8f_1}) + f_1 \end{aligned}$$

That is, we can obtain

$$f_1 + 2(1+\sqrt{1+8f_1}) \leq f_{02}.$$

This completes the proof of Theorem 3.3.1.

□

3.4 The flag vector pairs $(f_{1,} f_{03})$ for 4-polytopes

The aim of this subsection is to give a proof of Theorem 3.4.2 (Theorem 1.4).

First, we begin with the following theorem.

Theorem 3.4.1 The flag vector $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a 4-polytope P satisfies the inequalities

$$-f_1 + 3(1+\sqrt{1+8f_1}) \leq f_{03} \leq 5f_1 - 3(1+\sqrt{1+8f_1}).$$

Proof. By the identity

$$f_{02} = -2f_0 + 2f_1 + f_{03} \text{ (Lemma 3.1.1 (2))},$$

it is straightforward to obtain

$$\begin{aligned} f_{03} &= 2f_0 - 2f_1 + f_{02} \\ &\geq 1 + \sqrt{1+8f_1} - 2f_1 + f_1 + 2(1+\sqrt{1+8f_1}) \\ &= -f_1 + 3(1+\sqrt{1+8f_1}). \end{aligned}$$

Moreover, it is also true that

$$\begin{aligned} f_{03} &= 2f_0 - 2f_1 + f_{02} \\ &\leq f_1 - 2f_1 + 6f_1 - 3(1+\sqrt{1+8f_1}) \\ &= 5f_1 - 3(1+\sqrt{1+8f_1}). \end{aligned}$$

This completes the proof of Theorem 3.4.1.

□

Actually, if we combine Theorem 1.2 with Theorem 1.3, we can improve the upper and lower bounds for f_{03} given in Theorem 3.4.1, as follows[2].

Theorem 3.4.2 The flag vector $(f_1, f_{03}) = (f_1(P), f_{03}(P))$ of a non-neighborly 4-polytope P satisfies the inequalities

$$6 + 2\sqrt{\frac{71 + 4f_1 + 8\sqrt{1+8f_1}}{3}} \leq f_{03} \leq 4f_1 - 2(1+\sqrt{1+8f_1}).$$

Proof. To show it, we crucially make use of Theorems 1.2 and 1.3. Indeed, it follows from Theorems 1.2 and 1.3 that we have

$$\begin{aligned} f_{03} \leq \frac{2}{3} f_{02} &\leq \frac{2}{3}\left(6f_1 - 3(1+\sqrt{1+8f_1})\right) \\ &= 4f_1 - 2(1+\sqrt{1+8f_1}). \end{aligned}$$

On the other hand, by using Theorems 1.2 and 1.3 once again it is also not difficult to obtain

2) This result has been motivated by the discussion with Professor Nam Kwon Kim.

$$
\begin{aligned}
f_{03} &\geq 2\left(3+\sqrt{\frac{63+4f_{02}}{3}}\right) \\
&\geq 6+2\sqrt{\frac{71+4f_1+8\sqrt{1+8f_1}}{3}}
\end{aligned}
$$

This completes the proof of Theorem 3.4.2.

$\square$

Notice that for any $f_1 \geq 10$ we have

$$4f_1-2(1+\sqrt{1+8f_1}) \geq 5f_1-3(1+\sqrt{1+8f_1})$$

and

$$6+2\sqrt{\frac{71+4f_1+8\sqrt{1+8f_1}}{3}} \geq -f_1+3(1+\sqrt{1+8f_1}).$$

Therefore, for any non-neighborly 4-polytope P Theorem 3.4.2 gives better lower and upper bounds for f_{03} in terms of f_1 than those given in Theorem 3.4.1.

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