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On the constructions of 4-polytopes for flag vector pair (f_0, f_{02})

조선대학교 교육대학원

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플래그벡터 순서쌍 (f_0, f_{02}) 을 갖는 4차원 다면체의 구성에 관한 연구

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국문초록

플래그벡터 순서쌍 (f_0, f_{02}) 을 갖는 4차원 다면체의 구성에 관한 연구

김 지 우

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d 차원 다면체의 d 보다 작은 i 차원의 면의 개수를 f_i 라 할 때, f -벡터는 $f = (f_0, f_1, \dots, f_{d-1})$ 로 정의된다. Steinitz는 1906년에 3차원의 경우 f -벡터에 관한 결정 연구를 하였고, 그 후 Grünbaum은 1967년에 4차원에 관한 꼭짓점과 선분의 개수로 이루어진 순서쌍 (f_0, f_1) 에 가능한 순서쌍을 찾았다. 또한 2018년 Kusunoki와 Murai는 5차원 다면체에 대하여 (f_0, f_1) 로 나타나는 순서쌍을 증명하였다. Sjöberg와 Ziegler는 2018년에 4차원의 경우 (f_0, f_{03}) 을 완벽하게 결정할 수 있는 연구결과를 발표하였다. 본 논문은 4차원 다면체에 대응하는 면의 개수를 표현하는 f -벡터에 관한 연구로 Sjöberg와 Ziegler의 연구를 토대로 플래그 벡터쌍 (f_0, f_{02}) 을 갖는 4차원 다면체를 구성하는 연구를 했다. 이를 위해, Stacking, Truncating, cyclic polytope 상에서 일반적인 Stacking 및 Facet Splitting 등의 기법을 사용하였다.

1. Introduction

For a d -dimensional polytope P , let $f_i = f_i(P)$ denote the number of i -dimensional faces of P , and for $S \subseteq \{0, \dots, d-1\}$, let $f_S = f_S(P)$ denote the number of chains $F_1 \subset \dots \subset F_r$ of faces of P with $\{\dim F_1, \dots, \dim F_r\} = S$. For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \dots i_k}(P)$ instead of $f_{\{i_1, i_2, \dots, i_k\}}(P)$ for any subset $\{i_1, i_2, \dots, i_k\}$ of $\{0, 1, 2, \dots, d-1\}$. For instance, $f_{02}(P)$ will mean $f_{\{0,2\}}(P)$. The f -vector of P is then $(f_0, f_1, \dots, f_{d-1})$, and the flag vector of P is $(f_S)_{S \subseteq \{0, \dots, d-1\}}$. Due to the Euler equation, the set of f -vectors lies on a hyperplane in \mathbb{R}^d , and it spans this hyperplane by Grünbaum (see [4, Section 8.1] for more details).

In [6], Sjöberg and Ziegler published their work that completely determines the flag vector pair (f_0, f_{03}) of 4-dimensional polytopes. In order to obtain such results, they crucially applied the work [1] of Altshuler and Steinberg on 4-polytopes up to 8 vertices. Furthermore, they used the techniques of stacking, general stacking on cyclic polytopes, facet splitting, truncating, and so on for the construction of specific 4-dimensional polytopes.

The goal of this thesis is to construct some explicit 4-dimensional polytopes for the flag vector pair (f_0, f_{02}) . In fact, our original motivation for this study was to completely determine the flag vector pairs (f_0, f_{02}) for 4-polytopes, which is currently out of reach.

In order to achieve our goal, we first need to know the several formulas for the change of flag vector pairs (f_0, f_{02}) after the operations such as stacking, truncating, generalized stacking on cyclic polytopes, and facet splitting. In Chapter 2, we collect some basic facts and definitions necessary for all these constructions.

In Chapter 3, we give some explicit constructions of 4-polytopes and determine the flag vector pairs (f_0, f_{02}) . More precisely, in Section 3.1 we make use of the stacking operation in order to construct examples of 4-polytopes for some possible polytope pairs (f_0, f_{02}) . In Section 3.2, instead we use the truncating operation for some similar constructions as in Section 3.1. Sections 3.3 and 3.4 are devoted to dealing with the examples which can be obtained through the operations of generalized stacking on cyclic polytopes and facet splittings.

2. Theoretical backgrounds

The aim of this chapter is to collect and to briefly explain some facts necessary for the discussion in Chapter 3. For more details, refer to [2] and [3, Chapter 1].

2.1 polytopes

Definition 1.1

A convex polytope is the convex hull of a finite set of points in \mathbb{R}^n .

Definition 1.2

A **convex polyhedron** P is an intersection of finitely many half-spaces in \mathbb{R}^n :

$$P = \{x \in \mathbb{R}^n : \langle l_i, x \rangle \geq -a_i, i = 1, \dots, m\}$$

where $l_i \in (\mathbb{R}^n)^*$, dual space of \mathbb{R}^n , are some linear functions and $a_i \in \mathbb{R}, i = 1, \dots, m$. A **(convex) polytope** is a bounded convex polyhedron.

Definition 1.3

(1) The **dimension** of a polytope is the dimension of its affine hull. Unless otherwise stated we assume that any n -dimensional polytope, or simply n -polytope, P^n is a subset in n -dimensional ambient space \mathbb{R}^n .

(2) A **supporting hyperplane** of P^n is an affine hyperplane H which intersects P^n and for which polytope is contained in one of the two closed half-spaces determined by the hyperplane.

(3) The intersection $P^n \cap H$ is then called a **face** of the polytope. We also

regard the polytope P^n itself as a face; other faces are called **proper faces**. The **boundary** ∂P^n is the union of all proper faces of P^n . Each face of an n -polytope is itself a polytope of dimension $\leq n$. 0-dimensional faces are called **vertices**, 1-dimensional faces are **edges**, and codimension one faces are **facets**.

Definition 1.4

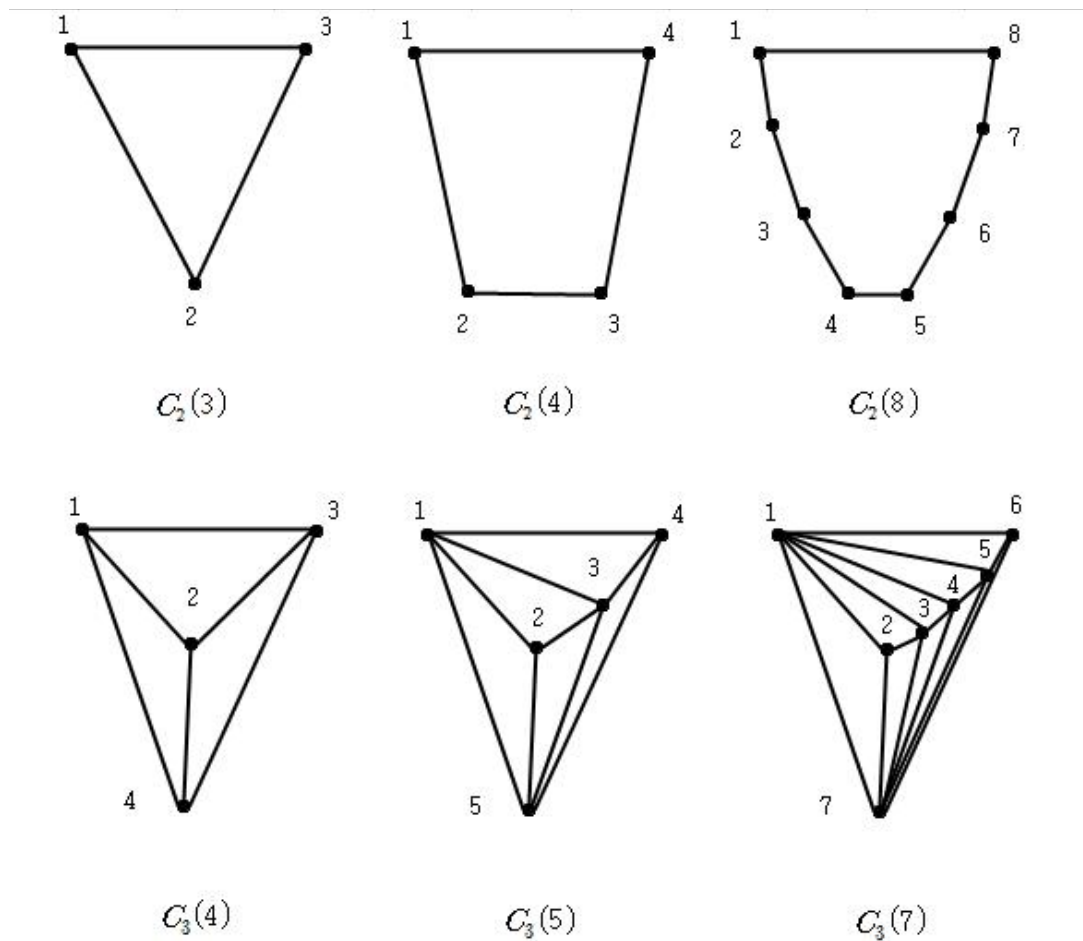
A d -polytope is said to be **neighborly** if each pair of vertices is joined by an edge.

2.2 Cyclic polytope

Define the moment curve in \mathbb{R}^n by

$$x: \mathbb{R} \rightarrow \mathbb{R}^n, t \mapsto x(t) = (t, t^2, \dots, t^n) \in \mathbb{R}^n.$$

For any $m > n$, define the **cyclic polytope** $C_n(t_1, \dots, t_m)$ as the convex hull of m distinct points $x(t_i)$, $t_1 < t_2 < \dots < t_m$, on the moment curve. It then follows from the Vandermonde determinant identity that no $(n+1)$ points on the moment curve belong to a common affine hyperplane. Hence, $C_n(t_1, \dots, t_m)$ is a simplicial n polytope. It can be shown that $C_n(t_1, \dots, t_m)$ has exactly m vertices $x(t_i)$, the combinatorial type of cyclic polytope does not depend on the specific choice of the parameters t_1, \dots, t_m , and $C_n(t_1, \dots, t_m)$ is a neighborly simplicial n -polytope. We will denote the combinatorial cyclic n -polytope with m vertices by $C_n(m)$.



[Figure 2.1] a few examples of cyclic polytopes

2.3 Upper Bound and Lower Bound theorems

The following statement, now known as the Upper Bound Conjecture(UBC), was suggested by Motzkin in 1957 in [8] and proved by P. McMullen in 1970 in [7].

Theorem 1.5 (UBC for simplicial polytopes)

For all simplicial n -polytopes P with m vertices, the cyclic polytope $C_n(m)$ has the maximal number of i -faces, $2 \leq i \leq n-1$. That is, $f_0(P) = m$ and

$$f_i(P) \leq f_i(C_n(m)) \text{ for } i = 2, \dots, n-1.$$

The equality in the above formula holds if and only if P is a neighborly polytope.

Note that, since $C_n(m)$ is neighborly,

$$f_i(C_n(m)) = \binom{m}{i+1} \text{ for } i = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1$$

Due to the Dehn-Sommerville equations, this determines the full f -vector of $C_n(m)$.

Lemma 1.6 [4]

The number of i -faces of cyclic polytope $C_n(m)$ is given by

$$f_i(C_n(m)) = \sum_{q=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{q}{n-1-i} \binom{m-n+q-1}{q} + \sum_{p=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-p}{i+1-p} \binom{m-n+p-1}{p},$$

$$i = -1, \dots, n-1$$

where we assume $\binom{p}{q} = 0$ for $p < q$.

Proof. Using the identity $\left\lfloor \frac{n}{2} \right\rfloor + 1 = n - \left\lfloor \frac{n-1}{2} \right\rfloor$ and the Dehn-Sommerville equations, we can calculate

$$\begin{aligned}
 f_i &= \sum_{q=0}^n \binom{q}{n-1-i} h_{n-q} \\
 &= \sum_{q=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{q}{n-1-i} h_q + \sum_{q=\left\lfloor \frac{n}{2} \right\rfloor+1}^n \binom{q}{n-1-i} h_{n-q} \\
 &= \sum_{q=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{q}{n-1-i} \binom{m-n+q-1}{q} + \sum_{p=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-p}{i+1-p} \binom{m-n+p-1}{p}.
 \end{aligned}$$

□

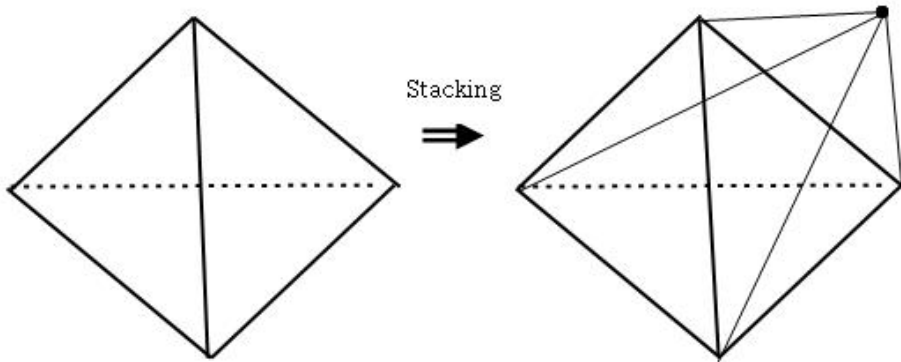
3. Main Results

As mentioned above, the aim of this chapter is to give some explicit constructions of 4-polytopes and give their flag vector pairs (f_0, f_{02}) . In order to explicitly determine the flag vector pairs (f_0, f_{02}) , we provide some interesting formulas for the change of flag vector pairs (f_0, f_{02}) after taking the operations of stacking, truncating, generalized stacking on cyclic polytopes, and facet splittings.

3.1 Stacking

The operation of stacking turns out to be essential in finding examples of polytopes for all possible polytope pairs (f_0, f_{02}) .

Let P be a 4-polytope with at least one simplex facet F , and let v be a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$ denote the convex hull of v and P (see Figure [3.1]).



[Figure 3.1] Stacking

It is easy to obtain that we have

$$f_0(Q) = f_0(P) + 1 \quad \text{and} \quad f_{03}(Q) = f_{03}(P) + 12.$$

Thus, the following lemma holds.

Lemma 3.1 Let P be a 4-polytope with at least one simplex facet F , and let v be a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$. Then the following identities hold.

$$\begin{aligned}
 f_0(Q) &= f_0(P) + 1, \\
 f_1(Q) &= f_1(P) + 4, \\
 f_{03}(Q) &= f_{03}(P) + 12
 \end{aligned}$$

The following generalized Dehn–Sommerville equations play an important role in this paper (see the paper [3, Theorem 2.1] of Bayer and Billera.

Lemma 3.2 Let P be a d -polytope and $S \subseteq \{0, 1, \dots, d-1\}$. Let $\{i, k\} \subseteq S \cup \{-1, d\}$ such that $i < k-1$ and such that there is no $j \in S$ for which $i < j < k$. Then

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

Corollary 3.3 Let P be a 4-polytope. Then the following identity holds.

$$f_{02} = -2f_0 + 2f_1 + f_{03}$$

Proof. For the proof, we apply the generalized Dehn-Sommerville equations in Lemma 3.2 to the case of $S = \{0\}$, $i = 0$, $k = 4$, and $d = 4$. In other words, by Lemma 3.2, we have

$$\sum_{j=1}^3 (-1)^{j-1} f_{0j}(P) = 2f_0(P).$$

That is, it follows that

$$\begin{aligned} & (-1)^0 f_{01} + (-1)^1 f_{02} + (-1)^2 f_{03} \\ &= f_{01} - f_{02} + f_{03} \\ &= 2f_1 - f_{02} + f_{03} \\ &= 2f_0, \end{aligned}$$

where in the second equality we used the fact that $2f_1 = f_{01}$. This implies

$$f_{02} = -2f_0 + 2f_1 + f_{03}.$$

□

By combining Corollary 3.3 with Lemma 3.1, we can obtain the following lemma.

Lemma 3.4 Let P and Q be as in Lemma 3.1. Then the following identity hold.

$$f_{02}(Q) = f_{02}(P) + 18.$$

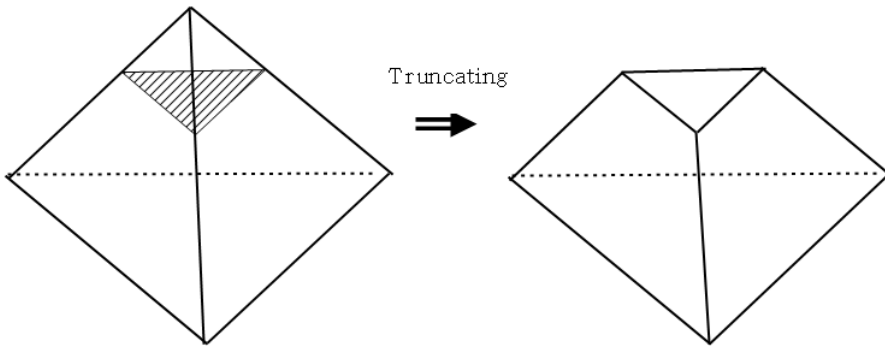
Proof. By Lemma 3.1 and Corollary 3.3, it is straightforward to obtain

$$\begin{aligned} f_{02}(Q) &= -2f_0(Q) + 2f_1(Q) + f_{03}(Q) \\ &= -2(f_0(P) + 1) + 2(f_1(P) + 4) + f_{03}(P) + 12 \\ &= -2f_0(P) - 2 + 2f_1(P) + 8 + f_{03}(P) + 12 \\ &= -2f_0(P) - 2 + 2f_1(P) + 8 + f_{02}(P) + 2f_0(P) - 2f_1(P) + 12 \\ &= f_{02}(P) + 18, \end{aligned}$$

as desired. □

3.2 Truncating

Let P be a 4-polytope with at least one simple vertex v , and let Q denote the polytope obtained by truncating the simple vertex v from P (see [Figure 3.2]).



[Figure 3.2] Truncating

Then the following identities hold.

Lemma 3.5. Let P be a 4-polytope with at least one simple vertex v , and let Q denote the polytope obtained by truncating the simple vertex v from P . Then we have

$$\begin{aligned}
 f_0(Q) &= f_0(P) + 3 \\
 f_{03}(Q) &= f_{03}(P) + 12.
 \end{aligned}$$

Proof. By its construction of truncation, the number of vertices decreases by three, while the flag number increases by 12. This completes the proof.

□

Lemma 3.6 Let P and Q be as in Lemma 3.5. Then we have

$$f_{02}(Q) = f_{02}(P) + 18.$$

Proof. By Corollary 3.3 and Lemma 3.5, it follows that

$$\begin{aligned} f_{02}(Q) &= -2f_0(Q) + 2f_1(Q) + f_{03}(Q) \\ &= -2(f_0(P) + 3) + 2(f_1(P) + 6) + (f_{03}(P) + 12) \\ &= -2f_0(P) + 2f_1(P) + f_{03}(P) + 18 \\ &= f_{02}(P) + 18, \end{aligned}$$

as desired. □

It is important to note that the polytopes obtained through the operation of stacking (resp. truncating) have a simplex facet (resp. a simple vertex) again. Therefore, we can repeat these two operations to stack vertices on simplex facets or to truncate simple vertices.

Theorem 3.7 Let P be a 4-polytope with a tetrahedral facet and a simple vertex, and let Q be the polytope obtained by taking the stacking simplex facets k times and truncating simple vertices l times from P . Then we have the following identity.

$$(f_0(Q), f_{02}(Q)) = (f_0(P) + k + 3l, f_0(P) + 18k + 18l).$$

Proof. It is immediate to obtain the identity by Lemmas 3.1, 3.4, 3.5, and 3.6. □

By Theorem 3.7, it is easy to show the following corollary.

Corollary 3.8 Let P and Q be as in Theorem 3.7. Let $m=k+l$. Then we have

$$(f_0(P)+m+2l, f_{02}(P)+18m), \quad k=m-l \geq 0, \quad 0 \leq l \leq m.$$

Example 3.9 As a concrete example, let us take P as a 4-simplex. Then, clearly $f_0=5$ and $f_{02}=20$. Let Q be the polytope obtained by taking the stacking simplex facets k times and truncating simple vertices l times from P . By Corollary 3.8, we have

$$(f_0(Q), f_{02}(Q)) = (5+m+2l, 20+18m).$$

It has been shown in [9, Lemma 2.6] that the flag vector pair (f_0, f_{02}) of a 4-polytope satisfies

$$4f_0 \leq f_{02} \leq 3f_0(f_0 - 3).$$

It is straightforward to check if the polytope Q satisfies the above inequalities. Indeed, we have

$$\begin{aligned} 3f_0(Q)(f_0(Q) - 3) &= 3(5+m+2l)(2+m+2l) \\ &= 3m^2 + 3m(4l+7) + 12l^2 + 42l + 30. \end{aligned}$$

On the other hand, since $f_{02}(Q) = 20+18m$, we have

$$\begin{aligned} 3f_0(Q)(f_0(Q) - 3) - f_{02}(Q) \\ = 3m^2 + 3m(4l+1) + 12l^2 + 42l + 10 > 0. \end{aligned}$$

For a 4-polytope with a square pyramid facet, we also have the following result.

Lemma 3.10 Let P be a 4-polytope with a square pyramid facet F , and let v be a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$.

$$f_{03}(Q) = f_{03}(P) + 16$$

Theorem 3.11

Let P be a 4-polytope with a square pyramid facet F , and let v be a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$. Then, we have

$$f_0(Q) = f_0(P) + 1,$$

$$f_{02}(Q) = f_{02}(P) + 24.$$

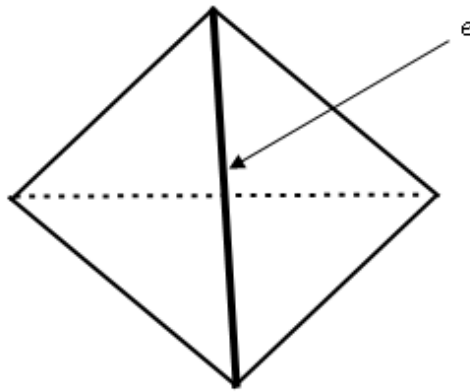
Proof. $f_{02}(Q) = -2f_0(Q) + 2f_1(Q) + f_{03}(Q)$

$$\begin{aligned} &= -2(f_0(P) + 1) + 2(f_1(P) + 5) + (f_{03}(P) + 16) \\ &= -2f_0(P) + 2f_1(P) + f_{03}(P) + 24 \\ &= -2f_0(P) + 2f_1(P) + 2f_0(P) - 2f_1(P) + f_{02}(P) + 24 \\ &= f_{02}(P) + 24 \end{aligned}$$

as desired. □

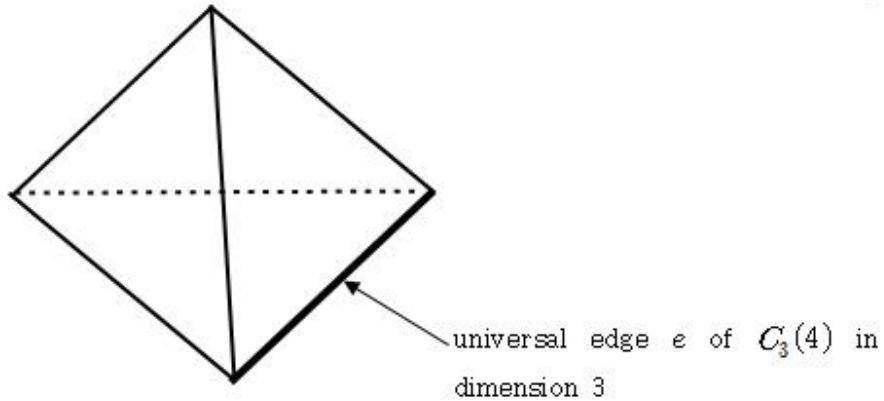
3.3 Generalized Stacking on cyclic polytopes

In this section, we want to create some more polytopes with their polytopal pair (f_0, f_{02}) by using the generalized stacking on cyclic polytopes.



[Figure 3.3] universal edge e

Note that every Cyclic 4-polytope with n vertices has edge that lie in exactly $n-2$ facets. Such edges are called a **universal edge** (see [Figure 3.3]). For example, a tetrahedron that is a cyclic polytope $C_3(4)$ has edges that lie in exactly 2 facets (see [Figure 3.4]).



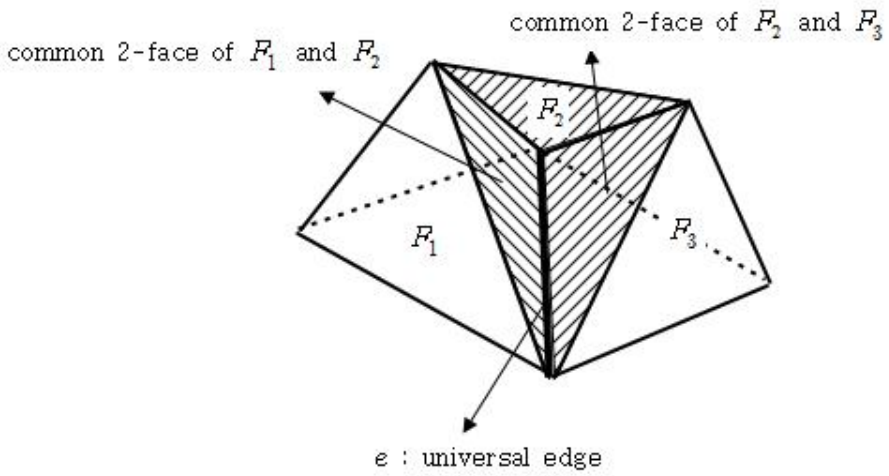
[Figure 3.4] universal edge e of $C_3(4)$

Now, for each $i=1, 2, \dots, n-3$, let $R_i(n)$ denote a polytope obtained from the cyclic polytope $C_4(n)$ with n vertices by taking the convex hull of $C_4(n)$ and a point v , where v lies beyond i facets of $C_4(n)$ which share a universal edge. Let F_1, F_2, \dots, F_i denote these i facets such that F_j and F_{j+1} meet a common 2-face for each $j=1, 2, \dots, i-1$ (see [Figure 3.5]).

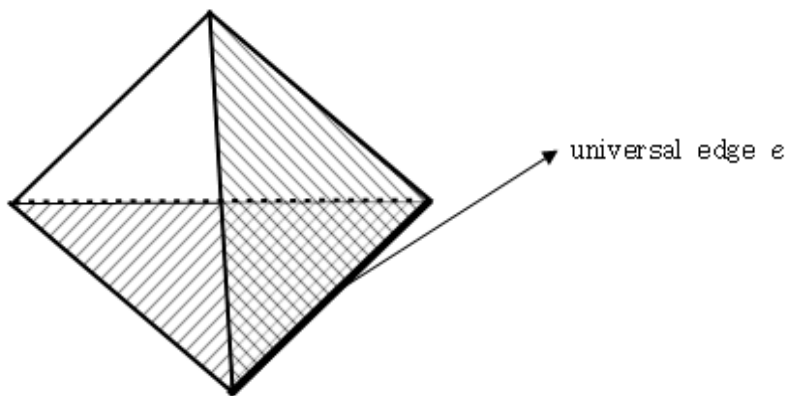
- $R_i(n)$ has one more vertex than $C_4(n)$, since $R_i(n)$ is obtained by taking the convex hull of $C_4(n)$ and v .

- All other $\frac{1}{2}n(n-3) - i$ facets of $C_4(n)$ that v lies beneath. Recall that the

total number of facets of $C_4(n)$ is equal to $\frac{1}{2}n(n-3)$.

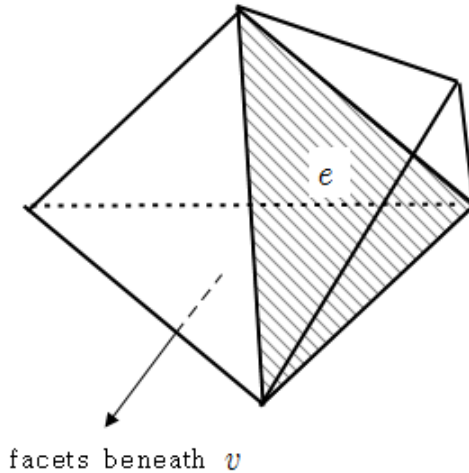


[Figure 3.5] universal edge e



[Figure 3.6] universal edge e

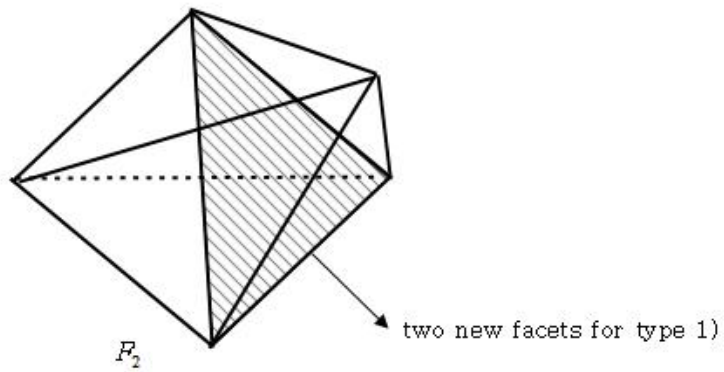
- There are facets obtained by the convex hulls of v and 2-faces of $C_4(n)$ which are contained both in a facet that v is beyond and a facet that v is beneath.



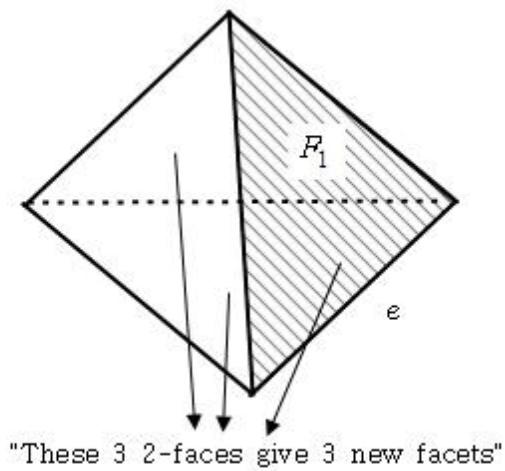
[Figure 3.7] facets beneath v

In fact, there are two types of these facets, as follows.

- 1) Two such facets for each of $(i-2)$ facets F_2, \dots, F_{j-1} which v lies beyond and which shares two 2-faces with other facets that v lies beneath (see [Figure 3.8]).
- 2) Three new facets for each of two facets F_i and F_j which v lies beyond and which share one 2-face with other facets which v lies beneath (see [Figure 3.9]).



[Figure 3.8] 1) Two such facets for each of $(i-2)$ facets



[Figure 3.9] 2) Three new facets for each of two facets

Note that all these facets are simplices. Thus, for $1 \leq i \leq n-3$ we have

$$\begin{aligned}
 f_{03}(R_i(n)) &= \left\{ \left(\frac{n(n-3)}{2} - i \right) + 2(i-2) + 2 \times 3 \right\} \\
 &= 2n(n-3) - 4i + 8i - 16 + 6 \times 4 \\
 &= 2n(n-3) + 4i + 8
 \end{aligned}$$

Therefore, we can also obtain

$$\begin{aligned}
 f_{02}(R_i(n)) &= f_{03}(R_i(n)) + 2f_1(R_i(n)) - 2f_0(R_i(n)) \\
 &= 2n(n-3) + 4i + 8 + 2f_1(R_i(n)) - 2(n+1) \\
 &= (2n - 6n + 4i + 8 + n - n + 12i + 10 - 2n - 2) \\
 &\quad (f_1(R_i(n)) = f_1(C_4(n)) + 2(i-3) + 5 + 2 \times 3) \\
 &= \binom{n}{2} + 6i - 6 + 11 \\
 &= 3n^2 - 9n + 16i + 16
 \end{aligned}$$

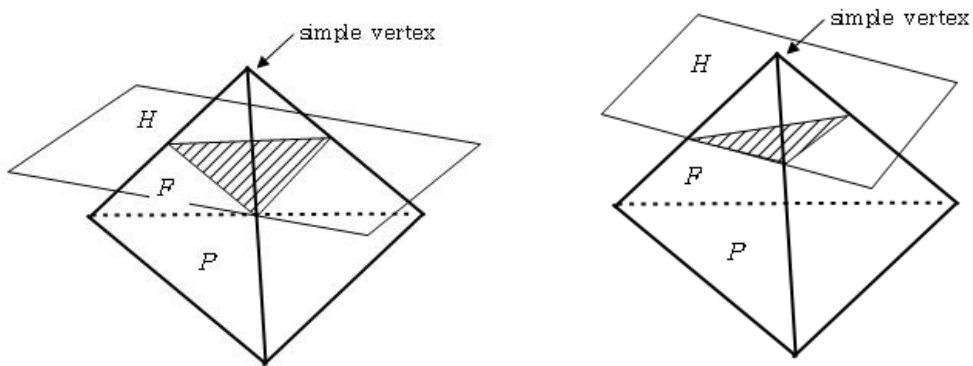
Theorem 3.12 For $1 \leq i \leq n-3$, let $R_i(n)$ denote the polytope obtained from the cyclic polytope $C_4(n)$ with n vertices by taking the convex hull of $C_4(n)$ and a point v , where v lies beyond i facets of $C_4(n)$ sharing a universal edge. Then we have

$$f_0(R_i(n)) = n + 1, \quad f_{02}(R_i(n)) = 3n^2 - 9n + 16i + 16.$$

3.4 Facet splitting

In this section, we generalize the stacking method more to look at the facet-splitting. In order to see the process more clearly, we will create a new facet in the dual polytope, instead of adding a new vertex to a polytope. The basic material of this section is largely taken from the paper [2] of Barnette.

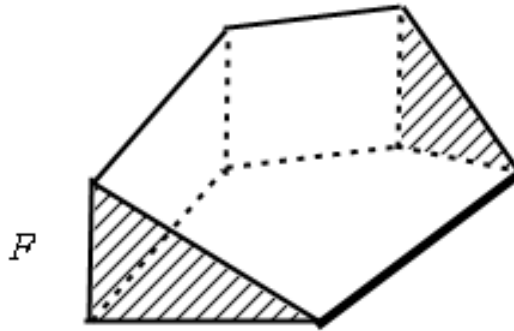
To obtain a facet-splitting, let consider a facet F of a 4-polytope P and a hyperplane H which intersect the relative interior of F in a polygon X . If the vertices of P lying on one side of H are only simple vertices, then separating the facet F into two new facets by the polygon X above (see [Figure 3.10]).



[Figure 3.10] Facet Splitting

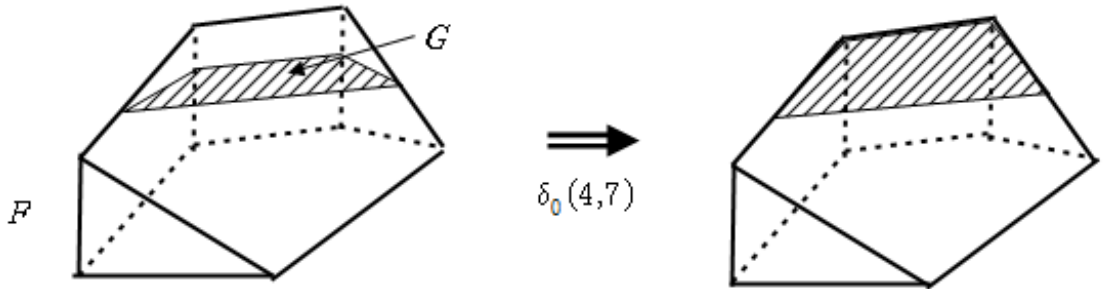
we say that P' is obtained from P by facet splitting.

More concretely, we now want to split a facet F of the dual $C_4^*(n)$ of a cyclic polytope with n facets such that each facet has $2(n-3)$ vertices. It is known that those facets are all wedges over $(n-2)$ -gons, i.e., polytopes with two triangular 2-faces, $n-5$ quadrilateral 2-faces, and two $(n-2)$ -gon meeting in an edge (see, e. g., [Figure 3.11] for $n=7$ case).



[Figure 3.11] Facet of $C_4^*(7)$

Let G be a 2-dimensional plane in the affine hull of F of $C_4^*(n)$, and let X be the intersection of F and G . Recall that all vertices of $C_4^*(n)$ are simple. So we can obtain a new polytope by taking the facet-splitting of $C_4^*(n)$ along a hyperplane H containing G such that the only vertices of $C_4^*(n)$ on one side of H are vertices of F (see [Figure 3.12]). Here we take H in such a way that G does not meet any vertices of F , and $X = G \cap F$ is an i -gon for $3 \leq i \leq n-2$. For this G , let us denote by $\delta_0(i, n)$ the polytope obtained by taking the facet-splitting.



[Figure 3.12] Facet of $C_4^*(7)$ split by i -gon

That $\delta_0(i, n)$ has one more facet and i more vertices than $C_4^*(n)$. Moreover, $\delta_0(i, n)$ has $2i$ more edges than $C_4^*(n)$. Thus, we obtain the following theorem.

Theorem 3.13 For $3 \leq i \leq n-2$, let $\delta_0(i, n)$ denote the polytope obtained by taking the facet-splitting as above. Then we have

$$\begin{aligned}
 f_0(\delta_0^*(i, n)) &= n+1, \\
 f_{02}(\delta_0^*(i, n)) &\equiv 2n^2 - 8n + 8i - 2,
 \end{aligned}$$

where $\delta_0^*(i, n)$ denotes the dual of $\delta_0(i, n)$.

Proof. By its construction, it is clear that $f_0(\delta_0^*(i, n)) = n+1$.

On the other hand, since $f_{03}(\delta_0^*(i, n)) = 2n(n-3) + 4i$, by Corollary 3.3 we

have

$$\begin{aligned} f_{02}(\delta_0^*(i,n)) &= f_{03}(\delta_0^*(i,n)) - 2f_0(\delta_0^*(i,n)) + 2f_1(\delta_0^*(i,n)) \\ &= 2n(n-3) + 4i - 2(n+1) + 2f_1(\delta_0^*(i,n)) \\ &= 2n^2 - 6n + 4i - 2n - 2 + 2f_1(\delta_0^*(i,n)) \\ &= 2n^2 - 8n + 4i - 2 + 2f_1(\delta_0^*(i,n)) \\ &= 2n^2 - 8n + 8i - 2 \end{aligned}$$

as desired.

□

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