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국문초록

6차원 다면체의 플래그벡터 순서쌍 (f_1, f_{05})의 범위에 관한 연구

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1970년과 1974년에 Grünbaum과 Barnette은 각각 4차원 다면체의 여러 가지 형태의 플래그벡터 순서쌍의 특성에 관한 결과를 발표하였다. 특히, Sjöberg 와 Ziegler는 최근 논문에서 4차원 다면체의 플래그벡터 순서쌍 (f_0, f_{03})을 완벽하게 결정하였다. 본 논문에서는 Sjöberg 와 Ziegler의 방법을 적용하여 6차원 다면체에서의 플래그벡터 순서쌍 (f_1, f_{05})의 범위에 관한 새로운 결과를 제시했다.

Ⅰ. Introduction

Our main concern of this thesis is a d-dimensional convex polytope. For simplicity, throughout this thesis a d-dimensional convex polytope will be called a d-polytope.

Let P be a d-dimensional convex polytope. For each $0 \le i \le d-1$, let $f_i(P)$ denote the number of i-dimensional faces of P. The f-vector $f(P)$ of P is defined to be

$$f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$$

(refer to [5] for more details). Similarly, for any $S \subseteq \{0,1,\ldots,d-1\}$, let $f_S = f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r$$

of faces of P with

$$\{\dim F_1, \ldots, \dim F_r\} = S.$$

The flag vector of P is defined to be

$$(f_S(P))_{S \subset \{0,1,2,\ldots,d-1\}}.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0,1,2,\ldots d-1\}$.

For any two subsets S_1 and S_2 of $\{0,1,2,...,d-1\}$ a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a flag vector pair.

More generally, for any k, not necessarily mutually disjoint, subsets $S_1, S_2, ..., S_k$ of $\{0,1,2,...,d-1\}$ a k-tuple

$$(f_{S_1}(P), f_{S_2}(P), ..., f_{S_k}(P)).$$

or simply $(f_{S_1}, f_{S_2}, ..., f_{S_k})$, of flag numbers of P will be called a flag vector k-tuple.

For example, for a 2-dimensional triangle Δ it is easy to calculate the following components of f-vector and flag vector of Δ, as follows.

$$f_0(\Delta) = 3,\ f_1(\Delta) = 3,\ f_2(\Delta) = 1,$$

and

$$f_{01}(\Delta) = 6,\ f_{02}(\Delta) = 3,\ f_{12}(\Delta) = 3.$$

As another example, if we take the polytope as a square $\square$, then the flag f-vector $f(\square)$ of $\square$ is given by

$$f(\square) = (f_0, f_1) = (4,4),$$

while the flag vector of $\square$ is given by

$$(f_{01}, f_{02}, f_{12}) = (8,4,4).$$

The f-vector and flag vector of P are one of fundamental combinatorial invariant of P, and characterizing all possible f

-vectors of convex polytopes has been one of the central problems in convex geometry. Moreover, it is easy to see that any d-polytope P satisfies

$$\frac{d}{2}f_0(P) \le f_1(P) \le \binom{f_0(P)}{2}.$$

Indeed, the first inequality follows since $f_1(P)$ equals to $\frac{1}{2}$ times the sum of degrees of the vertices of P and since each vertex of P has degree $\ge$ d.

Let F^d denote the set of all f -vectors of d-polytopes, and let $\Pi_{i,j}(F^d)$denote the projection of F^d onto the coordinates f_i and f_j. Let

$$\varepsilon^d = \{(f_0(P), f_1(P)) : P \text{ is a d} - \text{polytope}\}.$$

In [9], Steinitz completely determined all possible f -vectors of 3-polytopes, as follows.

$$\varepsilon^3 = \left\{(v,e) : \frac{3}{2}v \le e \le 3v-6\right\}.$$

By Euler's relation, this enables us to actually determine all possible f-vectors of 3-polytopes ([5]).

On the other hand, in [5] Grünbaum proved that the inequality

$$\frac{4}{2}f_0(P) \le f_1(P) \le \binom{f_0(P)}{2}.$$

characterizes ε^4, with four exceptions. More precisely, he proved the following statement.

$$\varepsilon^4 = \left\{(v,e) : 2v \le e \le \binom{v}{2}\right\} \diagdown \{(6,12),(4,14),(8,12),(10,20)\}.$$

In dimension 5, the situation is much more complicated. According to the paper [7] of Kusunoki and Murai, the set ε^5 turns out to be close to the set of integer points satisfying

$$\frac{5}{2}f_0(P) \le f_1(P) \le \binom{f_0(P)}{2},$$

but there are not only a finite list of exceptions but also an infi nite family of exceptions.

Our aim of this paper is to provide some new results about the flag vector pairs (f_1,f_{05}) of 6-polytopes, as follows.

Theorem 1.1

Let P be a 6-polytope, and let

$$A = \left[-648 + 108f_{05}(P) + 12\left\{81f_{05}(P)^2 - 972f_{05}(P) - 1200\right\}^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

Then, the flag vector pair (f_1,f_{05}) of 6-polytopes satisfies the following inequalities:

$$\frac{A}{2}+\frac{42}{A}+9 \le f_1(P) \le \frac{1}{72} f_{05}(P)(f_{05}(P)-6).$$

The question of whether or not all vector pairs (f_1,f_{05}) satisfying the inequalities given in Theorem 1.1 are flag vector pairs of 6-polytopes is unknown, and the techniques of this paper is very much out of reach to answer such a question.

This thesis is organized as follows.

In Chapter 2, we provide some preliminary material necessary for the proof of Theorem 1.1. In particular, we explain important definitions and facts such as cyclic polytopes and generalized Dehn-Sommerville equations which play an important role in the proof of our main results [1].

In Chapter 3, we give a proof of Theorem 1.1 by a series of lemmas. In this chapter, by using a similar method as in the proof of Theorem 1.1 we also give some bound for the flag vector pair $(f_{04}(P),f_{05}(P))$, as follows.

$$f_{04}(P) \le \binom{f_{05}(P)}{2}.$$

Ⅱ. Theoretical backgrounds

The aim of this chapter is to collect some definitions and to quickly review basic facts necessary for the proof of our results given in Chapter 3.

First, we begin with the definition of a cyclic polytope which plays an important role in many problems of convex geometry. To do so, let us define the moment curve in $\mathbb{R}^d$ by

$$\alpha : \mathbb{R} \to \mathbb{R}^d,\ t \mapsto (t, t^2, \ldots, t^d) \in \mathbb{R}^d$$

([4]). For any $n > d$, the standard d-th cyclic polytope with n vertices, denoted by $C_d(t_1, t_2, \ldots, t_n)$, was discovered by Caratheodory in the context of harmonic analysis ([4]). It is defined as the convex hull in R^d of n different points $\alpha(t_1), \ldots, \alpha(t_n)$ on the moment curve α such that $t_1 < t_2 < \cdots < t_n$.

Recall that the set of all the faces (including the improper faces) of a (convex) polytope P is a partially ordered set (or simply poset), when partially ordered by inclusion. Two polytopes are said to be combinatorial equivalent, or of the same combinatorial type, if they have isomorphic face posets. Cyclic polytopes $C_d(n)$ are precisely those which are combinatorial equivalent to the standard cyclic polytope $C_d(t_1, t_2, \ldots t_n)$ ([6]).

The cyclic polytopes are the simplest examples of d-dimensional neighborly polytopes, which means that is the vertex set of a face of the polytope. Therefore, we have

$$f_i(C_d(n)) = \binom{n}{i+1} \text{ for } 0 \le i \le \left\lfloor \frac{d}{2} \right\rfloor.$$

Thanks to the Dehn-Sommerville equations in [1], the above formula for $f_i(C_d(n))$ determines the full f-vector of $C_d(n)$. More precisely, the following lemma holds.

Lemma 2.1
The number of i-faces of cyclic polytope $C_d(n)$ is given by

$$f_i(C_d(n)) = \sum_{q=0}^{\left\lfloor \frac{d}{2} \right\rfloor} \binom{q}{d-1-i}\binom{n-d+q-1}{q} + \sum_{p=0}^{\left\lfloor \frac{d-1}{2} \right\rfloor} \binom{d-p}{i+1-p}\binom{n-d+p-1}{p}$$

for $i=-1,0,\ldots,d-1$. Here we used the convention that $\binom{p}{q}=0$ for $p<q$.

Proof.
See [8] for the proof of Lemma 2.1.

□

It is well known that every cyclic polytope is a simplicity polytope which means that all its facets are $(d-1)$-simplifies. One way to see this is to know that every $(d+1)$-tuple of points of the moment curve α is affined independent by the non-vanishing of the determinant of the Vandermonde matrix, so that each face of $C_d(t_1,t_2,\ldots,t_n)$ has at most d vertices.

Further, among all simplicity d-polytopes P with n vertices the cyclic polytopes $C_d(n)$ has the maximal number of i-faces for $2 \le i \le d-1$. That is, we have the following inequality ([8]).

$$f_i(P) \leq f_i(C_d(n))$$

for $i=2,3,\ldots,d-1$. Here the equality in the above inequality holds if and only if P is a neighborly polytope. In particular, since

$$f_3(C_4(n)) = \frac{n(n-3)}{2},$$

it follows that for a 4-polytope P with n vertices we have

$$f_3(P) \leq \frac{n(n-3)}{2}.$$

Similarly, for a 6-polytope P with n vertices it can be shown that

$$f_5(P) \leq f_5(C_6(n)) = \frac{n(n-4)(n-5)}{6}.$$

Ⅲ. Main Results

The aim of this chapter is to give a proof of Theorem 3.6 (or Theorem 1.1) by a series of lemmas. To do so, we begin with the proof of the following lemma.

Lemma 3.1
The flag vector pair (f_0, f_{05}) of a 6-polytope P satisfies the following inequality

$$6f_0 \le f_{05} \le f_0(f_0-4)(f_0-5).$$

Proof.
Since every vertex of a d-polytope lies in at least d facets, we have

$$6f_0(P) \le f_{05}(P).$$

On the other hand, it follows from [2] that for any d-dimensional polytope with n vertices and for any $S \subseteq \{0,1,2,\ldots,d-1\}$

$$f_s(P) \le f_s(C_d(n)),$$

where $C_d(n)$ denotes the d-dimensional cyclic polytope with n vertices. Thus, for $d=6$, we have

$$f_{05}(P) \le f_{05}(C_6(n)) = 6f_5(C_6(n)),$$

where $n = f_0(P)$. Note that the second equality in the above equation holds because $C_6(n)$ and its dual $C_6^*(n)$ are both simplicial.

Now, we calculate $f_5(C_6(n))$, as follows ([3, Lemma 1.34]).

$$\begin{aligned} f_5(C_6(n)) &= \sum_{q=0}^{\left[\frac{6}{2}\right]} \binom{q}{5-5}\binom{n-6+q-1}{q} + \sum_{p=0}^{\left[\frac{5}{2}\right]} \binom{6-r}{6-r}\binom{n-6+p-1}{p} \\ &= 2\left(\binom{n-7}{0}+\binom{n-6}{1}+\binom{n-5}{2}\right)+\binom{n-4}{3} \\ &= \frac{n(n-4)(n-5)}{6}. \end{aligned}$$

Hence, we have

$$\begin{aligned} f_{05}(P) &\le n(n-4)(n-5) \\ &= f_0(P)(f_0(P)-4)(f_0(P)-5). \end{aligned}$$

This completes the proof of Lemma 3.1. □

We also need the following lemma which is well-known.

Lemma 3.2 [10, Wikipedia]
The solution of the polynomial $ax^3+bx^2+cx+d=0$ with $a \neq 0$ is given by

$$x = \left(\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right) + \left(\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2 + \left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3 \right)^{\frac{1}{2}} \right)^{\frac{1}{3}}$$

$$+\left((-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a})-\left((-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a})^2+(\frac{c}{3a}-\frac{b^2}{9a^2})^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}-\frac{b}{3a}.$$

By Lemma 3.2, it is straightforward to obtain the following lemma.

Lemma 3.3

Let

$$A=\left(-648+108f_{05}+12(81f_{05}^2-972f_{05}-1200)^{\frac{1}{2}}\right)^{\frac{1}{3}}$$

Then, we have

$$f_0 \geq \frac{A}{6}+\frac{14}{A}+3.$$

Proof.

By Lemma 3.1, it is easy to see that the following inequality holds.

$$f_0^3-9f_0^2+20f_0-f_{05} \geq 0.$$

Note that

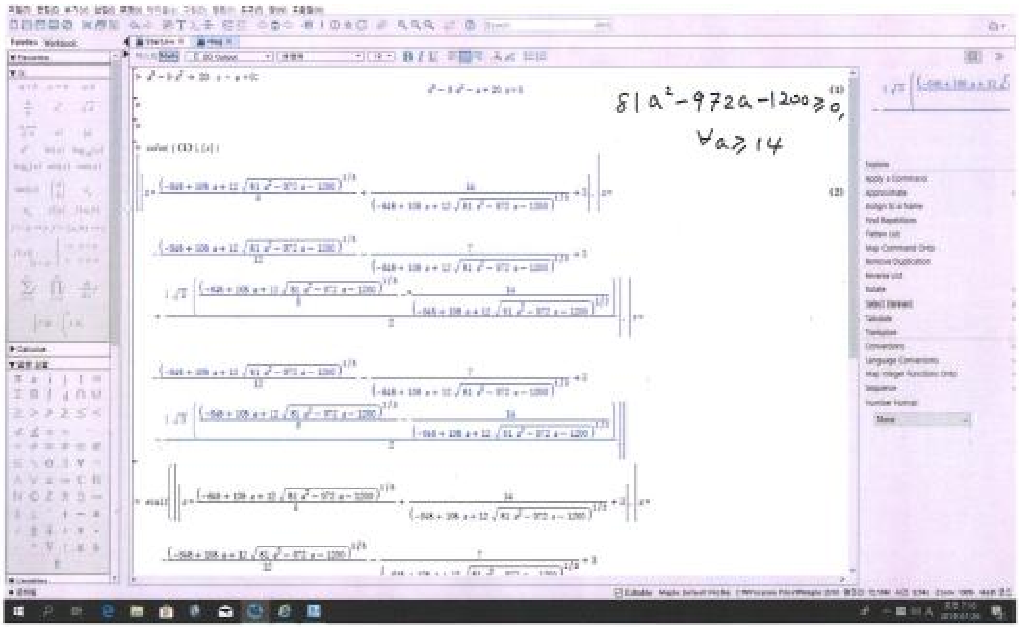
$$81t^2-972t-1200 \geq 0$$

for $t \geq 6+\frac{14\sqrt{21}}{9} \approx 13.128\cdots$. Thus, $81t^2-972t-1200 \geq 0$ for any integer $t \geq 14$.

Note also that $f_{05} \geq 6f_0 \geq 42$. Hence, by Lemma 3.2 or Maple program for cubic polynomials, it is easy to obtain the desired inequality

$$f_0 \geq \frac{A}{6} + \frac{14}{A} + 3.$$

This completes the proof of Lemma 3.3. $\square$



Proposition 3.4

The flag vector pair (f_1, f_{05}) satisfies the following inequality

$$f_1 \geq \frac{A}{2} + \frac{42}{A} + 9,$$

where A is the number given in Lemma 3.3.

Proof.

Since $f_0 \leq \frac{1}{3} f_1$, it is easy to obtain

$$\frac{1}{3} f_1 \geq f_0 \geq \frac{A}{6} + \frac{14}{A} + 3.$$

Thus, f_1 should satisfy the following inequality

$$f_1 \geq \frac{A}{2} + \frac{42}{A} + 9.$$

This completes the proof. □

Proposition 3.5

The flag vector pair (f_1, f_{05}) satisfies the following inequality

$$f_1(P) \leq \frac{f_{05}(f_{05}-6)}{72}.$$

Proof.

For the proof, recall frist

$$3f_0 \leq f_1 \leq \binom{f_0}{2}.$$

Thus, we have

$$f_0^2 - f_0 - 2f_1 \geq 0.$$

So, it follows that

$$f_0 \geq \frac{1+\sqrt{1+8f_1}}{2}.$$

On the other hand, it follows from Lemma 3.1 that we have

$$6f_0 \leq f_{05}.$$

Thus, it is easy to obtain

$$6 \times \frac{1+\sqrt{1+8f_1}}{2} \leq 6f_0 \leq f_{05}.$$

That is, we have

$$
\begin{aligned}
& 1+\sqrt{1+8f_1} \le \frac{1}{3}f_{05} \\
& \Leftrightarrow \sqrt{1+8f_1} \le \frac{1}{3}f_{05}-1 \\
& \Leftrightarrow 1+8f_1 \le \left(\frac{1}{3}f_{05}-1\right)^2 \\
& \Leftrightarrow 8f_1 \le \left(\frac{1}{3}f_{05}-1\right)^2-1.
\end{aligned}
$$

It is immediate to see that this implies the desired inequality, as follows.

$$
\begin{aligned}
f_1 &\le \frac{1}{8}\left((\frac{1}{3}f_{05}-1)^2-1\right) \\
&= \frac{1}{8}\left(\frac{1}{9}f_{05}^2-\frac{2}{3}f_{05}\right) \\
&= \frac{1}{72}f_{05}(f_{05}-6).
\end{aligned}
$$

□

By combining Propositions 3.4 and 3.5, we can obtain the following inequalities that are our main results of this thesis.

Theorem 3.6
Let P be a 6-polytope, and let

$$
A=\left(-648+108f_{05}(P)+12(81f_{05}(P)^2-972f_{05}(P)-1200)^{\frac{1}{2}}\right)^{\frac{1}{3}}.
$$

Then, the following inequalities hold.

$$
\frac{A}{2}+\frac{42}{A}+9 \le f_1(P) \le \frac{1}{72}f_{05}(P)(f_{05}(P)-6).
$$

By using some similar arguments as above, we can also obtain the following inequality.

Proposition 3.7

Let P be a 6-polytope. Then, for the flag vector pair $(f_{04}(P), f_{05}(P))$ the following inequality holds

$$f_{04}(P) \leq \binom{f_{05}(P)}{2}.$$

Proof.

Let F be any facet of P. Then clearly F is a 5-dimensional polytope.

Thus, it is easy to obtain

$$f_1(F) \leq \binom{f_0(F)}{2} = \frac{f_0(F)(f_0(F)-1)}{2}.$$

This implies

$$\begin{aligned}\sum_{F \subseteq P} f_1(F) &\leq \frac{1}{2}\sum_{F \subseteq P} f_0^2(F) - \frac{1}{2}\sum_{F \subseteq P} f_0(F) \\ &\leq \frac{1}{2}(\sum_{F \subseteq P} f_0(F))^2 - \frac{1}{2}\sum_{F \subseteq P} f_0(F),\end{aligned}$$

where for the second inequality we used the following facts

$$\sum_{i=1}^{k} x_i^2 \leq (\sum_{i=1}^{k} x_i)^2,\ x \geq 0,$$
$$\sum_{F \subseteq P} f_0(F) = f_{05}(P).$$

Therefore, it follows that

$$\begin{aligned}f_{15}(P) &\leq \frac{1}{2}f_{05}(P)^2 - \frac{1}{2}f_{05}(P) \\ &= \frac{1}{2}f_{05}(P)(f_{05}(P)-1).\end{aligned}$$

By duality, we also have

$$f_{04}(P) \leq \frac{1}{2} f_{05}(P)(f_{05}(P)-1),$$

as desired. $\square$

References

[1] M. M. Bayer and L. J. Billera, *Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets*, Invent. Math. 79 (1985), 143-157.

[2] L. J. Billera and A. Björner, *Face numbers of polytopes and complexes, Handbook of Discrete and Computational Geometry*, Chapman and Hall CRC press, Florida, 449-475, 2017.

[3] V. Buchstaber, and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, Univ. Lect Ser. Vol. 24, American Math. Soc., 2002.

[4] D. Gale, *Neighborly and cyclic polytopes, in: Proceedings of the Symposia in Pure Mathematics*, Vol.VII, American Mathematical Society, Providence, RI, 1963.

[5] B. Grünbaum, *Convex Polytopes, Grad*. Texts in Math. 221, Springer, New York, 2003.

[6] B. Grünbaum, *Convex Polytopes*, with the cooperation of Victor Klee, M.A.Perles and G.C.Shephard, Pure and Applied mathematics,16, Interscience Publishers, John Wiley & Sons, Inc., New York, 1967.

[7] T. Kusunoki and S. Murai, *The numbers of edges of 5-polytopes with a given number of vertices, to appear in*

Annals of Combinatorics; arXiv: 1702.06281v3.

[8] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika 17(1970), 179-184.

[9] H. Sjöberg and G. M. Ziegler, *Characterizing face and flag vector pairs for polytopes*, preprint (2018); arXiv:1803.04801v1.

[10] Wikipedia, www.wikidepia.org.