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Necessary conditions for flag vector pairs (f_1, f_{04}) of 5-polytopes

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국 문 초 록

5차원 다면체의 플래그벡터 순서쌍 (f_1, f_{04})의 필요조건에 관한 연구

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플래그벡터란 일반적인 차원의 다면체에 대응하는 면의 개수를 표현하는 벡터를 의미한다. 플래그벡터는 3차원의 경우에 많은 학자들에 의해 연구들이 잘 알려져 있다. 반면, 4차원 이상의 고차원 플래그벡터를 다루는 것은 중요한 문제임에도 불구하고 아직까지도 학자들에게 많이 연구되어지지 않고 있다. 이러한 이유로 본 연구에서는 5차원 플래그벡터에 관한 연구를 진행하기 위해 3, 4차원 플래그벡터의 일부에 관한 연구 결과를 이용하여 5차원 플래그벡터 순서쌍 (f_1, f_{04})의 부등식을 결론으로 도출하고자 한다. 더 나아가 앞에서 다룬 이러한 부등식들이 일반적인 경우에도 성립하는가 확인하기 위해, 스태킹과 트런케이팅 과정을 통해 플래그벡터 순서쌍 (f_1, f_{04})을 5차원 다면체에 대해 구체적으로 구성하였다. 그 결과, 아래와 같은 3가지 결과가 성립함을 보였다.

(1) P 가 5차원 다면체일 때, 플래그벡터 성분 $f_{04}(P)$의 값이 주어지면 아래와 같은 부등식이 성립하며, $f_1(P)$의 범위를 결정할 수 있다.

$$\frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right) \le f_1(P) \le \frac{1}{4}f_{04}(P)(f_{04}(P)-3)$$

(2) P 가 5차원 다면체일 때, 플래그벡터 성분 $f_1(P)$의 값이 주어지면 아래와 같은 부등식이 성립하며, $f_{04}(P)$의 범위를 결정할 수 있다.

$$\frac{1}{2}\left(3+\sqrt{9+16f_1(P)}\right) \le f_{04}(P) \le \frac{4}{5}f_1(P)^2-14f_1(P)+60$$

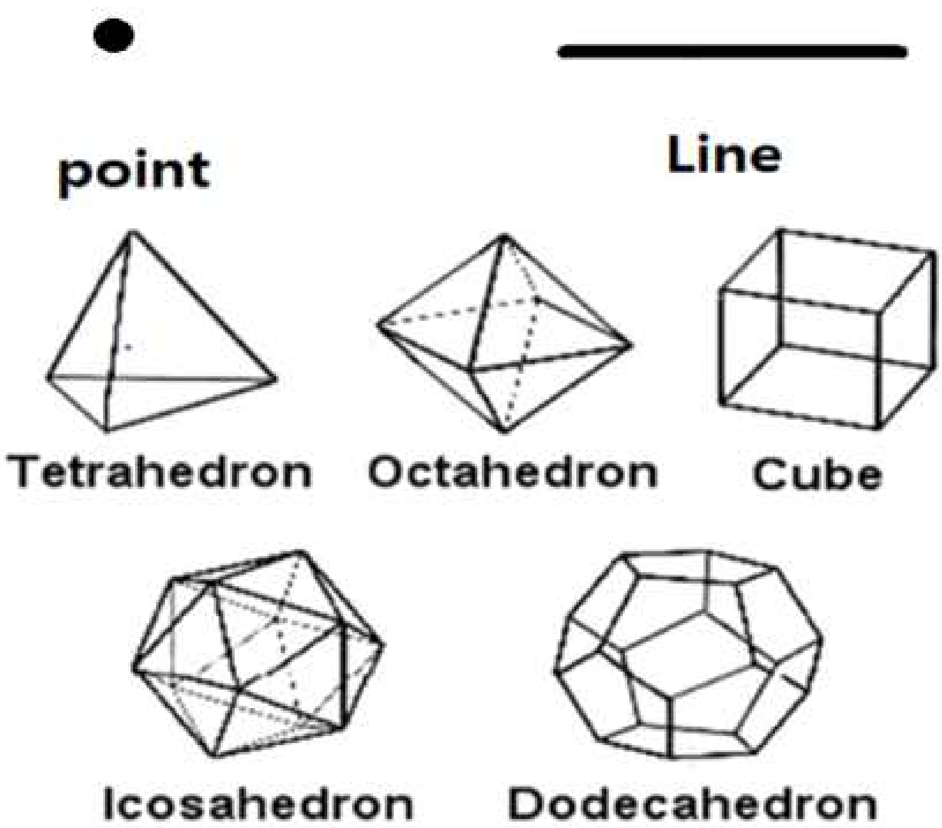
(3) 5차원 다면체가 한 개의 4차원 단순체와 단순꼭짓점을 갖고 있을 때, l 번 트런케이팅과 k 번 스태킹을 시행할 경우 다음과 같은 새로운 5차원 다면체 Q 의 플래그벡터 순서쌍 (f_1, f_{04})의 관계식을 얻을 수 있다.

$$(f_1(Q), f_{04}(Q)) = (f_1(P)+5k+10l,\ f_{04}(P)+20k+20l).$$

Chapter 1

Introduction

There are various kinds of polytopes, and some of them can be easily found in reality. It is well known that if the dimension of a polytope equals 0, then the polytope is a point, and if the dimension of a polytope equals 1, then the polytope is the line segment It is relatively easy to study polytopes of dimension less than or equal to 3, since it can be visualized in several ways. Here are some clear examples of polytopes (see [Figure 1]).



[Figure 1] Examples of polytopes

On the other hand, it is not so easy to study polytopes of dimension more than 3, and more worthwhile to study higher dimensional polytopes. In view of these, in this paper we generally deal with polytopes of dimension more than 3. Especially, in this thesis we study polytopes whose dimensions are between 3 and 5 inclusive.

Let P be a d-dimensional polytope. For each $0 \le i \le d-1$, let $f_i = f_i(P)$ be the number of i-dimensional faces of P. For a subset S of $\{0,1,2,\cdots,d-1\}$, let $f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \cdots \subset F_r$$

of faces F_i, $1 \le i \le r$, of P such that

$$\{\dim F_1, \dim F_2, \cdots, \dim F_r\} = S.$$

The $f-$vector of P is then defined to be

$$f(P) = (f_0, f_1, \cdots, f_{d-1}).$$

The F^d is the set of all f-vectors of d-dimensional polytope, and clearly $F^d \subseteq \mathbb{Z}^d$.

The **flag vector** of P is defined to be

$$(f_S)_{S \subseteq \{0,1,\cdots,d-1\}}.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \cdots i_k}(P)$ instead of $f_{\{i_1,i_2,\cdots,i_k\}}(P)$ for each $\{i_1,i_2,\cdots,i_k\} \subset \{0,1,2,\cdots,d-1\}$.

For any two subsets S_1 and S_2 of $\{0,1,2,\cdots,d-1\}$, a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a

flag vector pair. More generally, for any k, not necessary mutually disjoint, subsets $S_1, S_2, \cdots, S_k$ of $\{0,1,2,\cdots,d-1\}$ a **k-tuple**

$$\left(f_{S_1}(P), f_{S_2}(P), \cdots, f_{S_k}(P)\right),$$

or simply $\left(f_{S_1}, f_{S_2}, \cdots, f_{S_k}\right)$, of flag numbers of P will be called a **flag vector k-tuple** As in the flag vectors, let us denote by $\Pi_{S_1, S_2, \cdots S_k}$ the projection of the flag vector $(f_S(P))_{S \subseteq \{0,1,\cdots,d-1\}}$ onto its coordinates $f_{S_1}, f_{S_2}, \cdots, f_{S_k}$. We call $\left(f_{S_1}, f_{S_2}, \cdots, f_{S_k}\right)$ a **polytopal flag vector k-tuple** if

$$\left(f_{S_1}, f_{S_2}, \cdots, f_{S_k}\right)$$

belongs to the image of the set all flag vectors of d-dimensional polytopes under the projection map $\Pi_{S_1, S_2, \cdots S_k}$, that is, if there is a d-polytope P such that

$$\left(f_{S_1}(P), f_{S_2}(P), \cdots, f_{S_k}(P)\right) = \left(f_{S_1}, f_{S_2}, \cdots, f_{S_k}\right).$$

For $n \geq d$, a cyclic polytope, denoted $C(n,d)$ (or $C_d(n)$) is a convex polytope given by the convex hull of n distinct points on a rational normal curve in $\mathbb{R}^d$. Especially, (n,m) belongs to $\Pi_{0,d-1}(F^d)$, and these pairs must satisfy the U.B.T inequality.

$$m \leq f_{d-1}\left(C_d(n)\right) \text{ and } n \leq f_{d-1}\left(C_d(m)\right)$$

It implies that cyclic polytopes $C_d(n)$ have the largest possible number of faces among all convex polytopes with a given dimension and number of vertices.

The f-vectors of d-polytopes $(d \leq 3)$ have been much studied by many mathematicians such as Steinitz, Grünbaum, Barnette-Reay and Barnette, Sjöberg and Ziegler, and so on. While the f-vector set F^3 of 3-polytopes was completely determined by Steinitz in 1906 (see

[7] for more details), any complete determination of all possible f-vector of d-polytopes for $d \geq 4$ is still illusive.

As some partial results, for $d=4$ the projections of the f-vector set $F^4 \subseteq \mathbb{Z}^4$ onto two of the four coordinates have been determined in 1967-1974 by Grünbaum, Barnette-Reay and Barnette in [1] and [2]. Moreover, Sjöberg and Ziegler in [6] determined all possible values of the pairs (f_0, f_{03}) of flag face numbers of 4-polytopes and Kusunoki and Murai in [5] characterized all possible (f_0, f_1) pairs of the f-vectors of 5-polytopes.

However, in spite of the importance of higher dimensional cases of d-polytope ($d \geq 4$), the problem of completely determining their flag vectors is still unknown. For this reason, in this paper we try to determine new results about some obstructions of flag vector pairs (f_1, f_{04}) of 5-polytopes. For this, a few techniques such as general stacking, facet splitting of stacking, truncating, cyclic polytopse will be crucially used for making new d-polytopes.

This paper is organized as follows. In Chapter 2, by using the methods of stacking and truncating we study the d-polytopes, and show some obstructions of the flag vector pairs (f_1, f_{02}) for $d=3$ as well as (f_0, f_{03}) for $d=4$. In Chapters 3 and 4, we give some proofs of the following inequalities for flag vector pairs (f_1, f_{04}) of 5-polytopes hold, as follows.

(1) For a given flag number $f_{04}(P)$, we have

$$\frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right) \leq f_1(P) \leq \frac{1}{4}f_{04}(P)(f_{04}(P)-3).$$

(2) For a given flag number $f_1(P)$, we have

$$\frac{1}{2}\left(3+\sqrt{9+16f_1(P)}\right)\leq f_{04}(P)\leq\frac{4}{5}f_1(P)^2-14f_1(P)+60$$

Note that the upper and lower bounds of the flag vector pairs (f_1,f_{04}) given in (1) and (2) above very sharp, since there is an explicit example, such as a 5-simplex with $(f_1,f_{04})=(15,30)$, which satisfies the equalities in (1) and (2).

(3) Let P be a 5-polytope with a 4-simplex and a simple vertex. By truncating simple vertices l times, and stacking vertices on a simplex facet k times repeatedly, we can obtain a new 5-polytope Q satisfying the following identity.

$$\left(f_1(Q),f_{04}(Q)\right)=\left(f_1(P)+5k+10l,\ f_{04}(P)+20k+20l\right)$$

It is easy to check if these examples satisfy the inequalities in the main results.

Chapter 2

Stacking and truncating

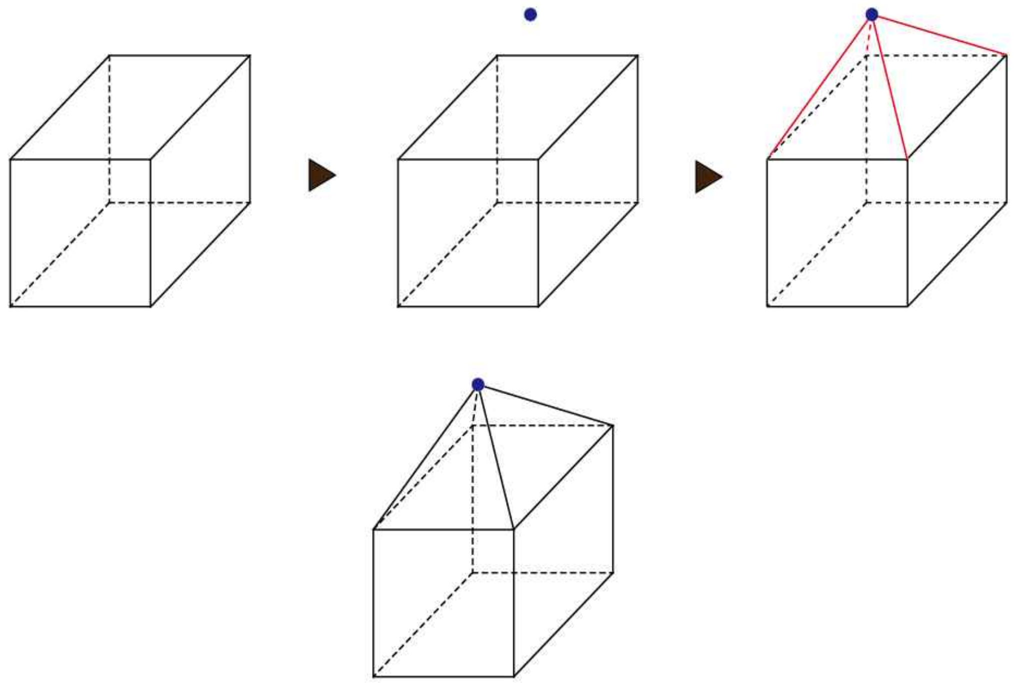
The aim of this chapter is to set up some basic operations such as stacking and truncating for our main results in Chapters 3 and 4.

What we mean by **stacking** is an operation to obtain a new polytope formed as the smallest convex set containing a given polytope and one more vertex. To be more precise, let P be a d-polytope with a facet F and a point v beyond F and beneath all other facets. The operation of obtaining a new d-polytope $Q = \mathrm{conv}(P \cup \{v\})$ is called a stacking.

On the other hand, let P be a d-polytope with a vertex v, and let H be a hyperplane intersecting the interior of P such that on one side of H the only vertex of P is v. What we mean by **truncating** at a vertex v is an operation of obtaining a new polytope by cutting off the side of H that contains v (see from [Figure 2] to [Figure 9] for more details about stacking and truncating).

2.1 Flag vectors of 3-polytopes

Now, let us show some explicit examples to illustrate the procedure of stacking polytopes. In [Figure 2], each step shows the procedure of stacking a cube.



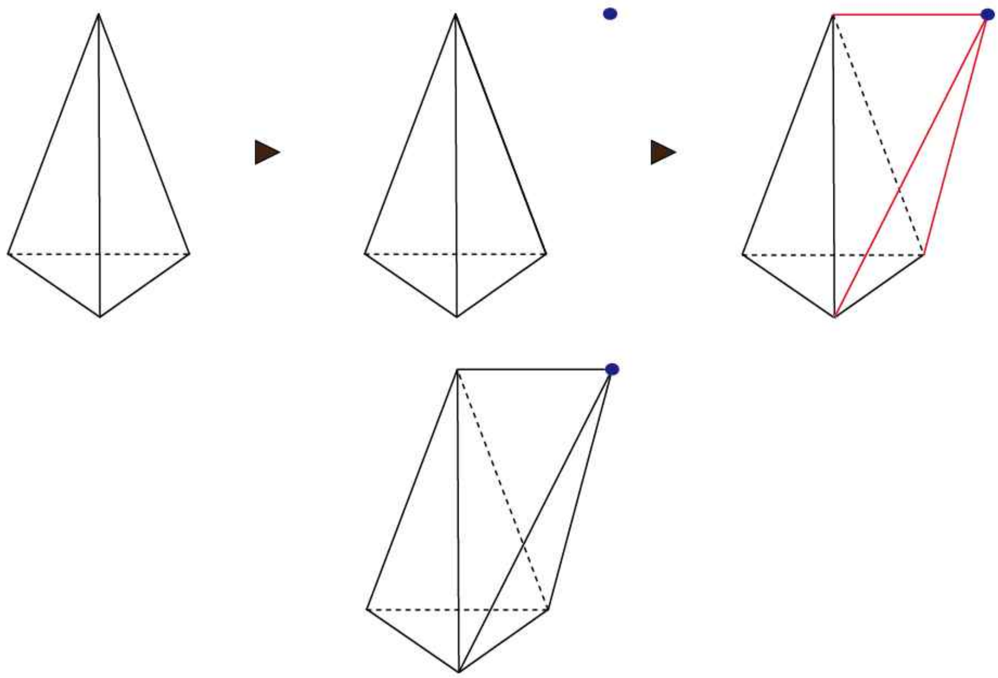
[Figure 2] Stacking of a cube

Let P be a cube and let Q be the polytope obtained by stacking operation of P. By comparing P and Q, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q) = f_0(P) + 1, \\ f_1(Q) = f_1(P) + 4, \\ f_2(Q) = f_2(P) + 3. \end{cases}$$

Here we obtain $f_2(Q) = f_2(P) + 3$ when we take the stacking operation, since four more facets are created, but one facet disappears.

The figure in [Figure 3] sequentially shows each step of stacking a tetrahedron.

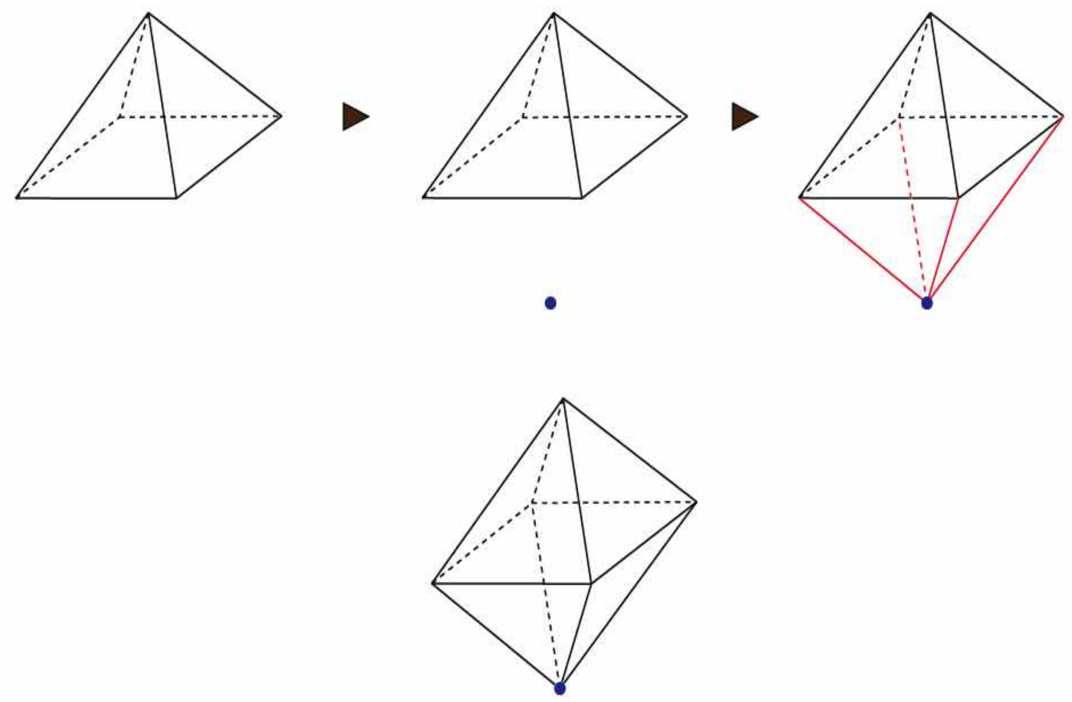


[Figure 3] Stacking of a tetrahedron

Likewise, let P be a tetrahedron and let Q be the polytope obtained by stacking operation of P. As above, by comparing P and Q, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q)=f_0(P)+1, \\ f_1(Q)=f_1(P)+3, \\ f_2(Q)=f_2(P)+2. \end{cases}$$

The figure shown in [Figure 4] sequentially shows each step of stacking a pyramid.



[Figure 4] Stacking of a pyramid

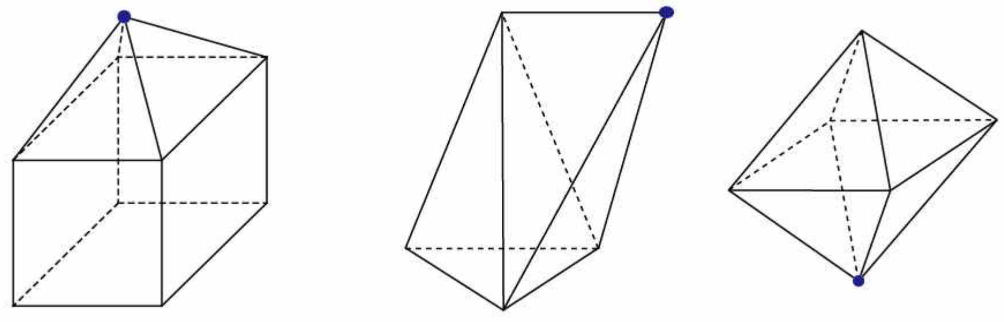
Finally, let P be a pyramid and let Q be the polytope obtained by stacking operation of P over a square facet. By comparing P and Q, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q) = f_0(P) + 1, \\ f_1(Q) = f_1(P) + 4, \\ f_2(Q) = f_2(P) + 3. \end{cases}$$

On the other hand, if we take the stacking operation of P over a triangle facet, then we have the following equations.

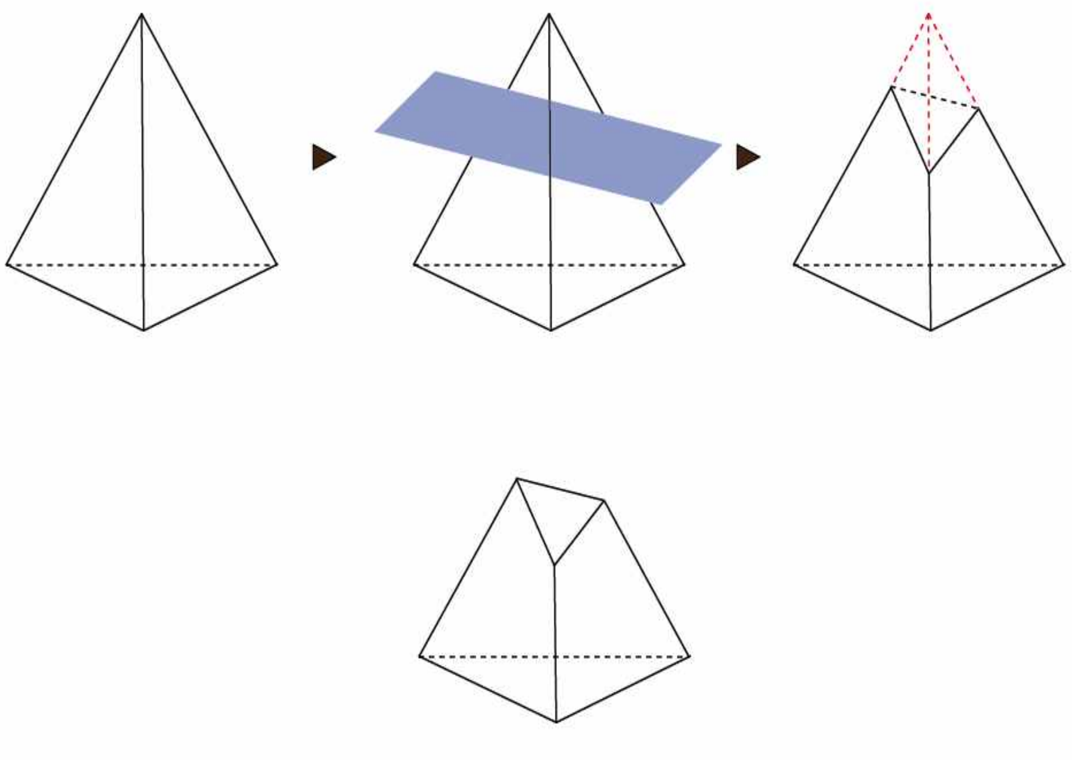
$$\begin{cases} f_0(Q) = f_0(P) + 1, \\ f_1(Q) = f_1(P) + 3, \\ f_2(Q) = f_2(P) + 2. \end{cases}$$

To sum up, the figures in [Figure 5] show the results of polytopes obtained by stacking cube, tetrahedron, and pyramid.



[Figure 5] Polytopes obtained by taking the stacking

Next, let us show some examples to illustrate the procedure of truncating polytopes. In [Figure 6], each step shows the procedure of truncating of a tetrahedron.



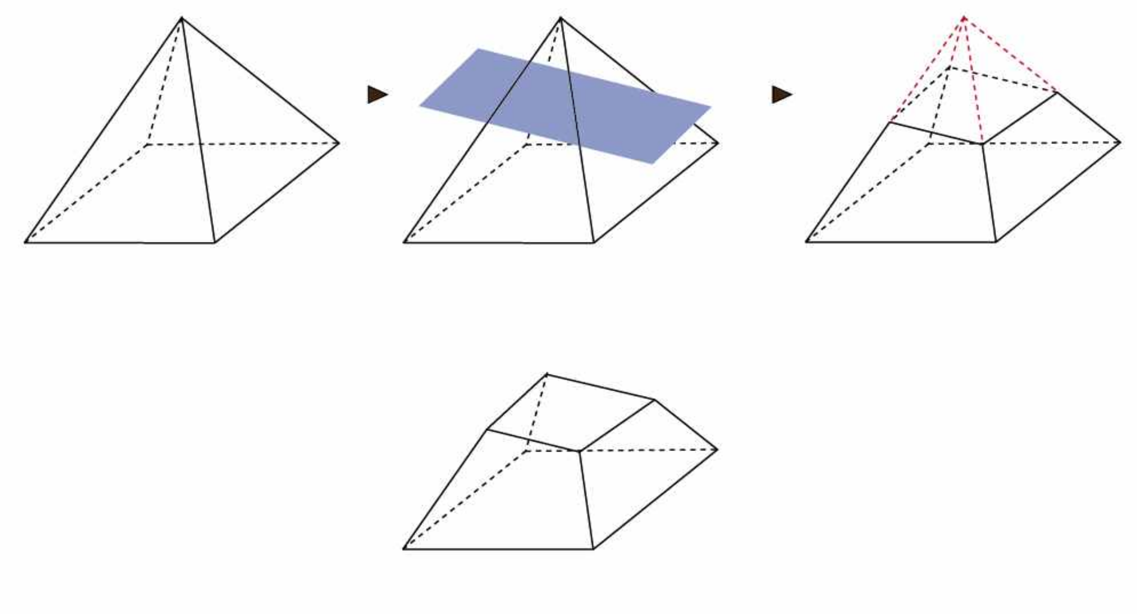
[Figure 6] Truncating of a tetrahedron

For this, let P be a tetrahedron and let Q be the polytope obtained by truncating operation of P. Then, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q) = f_0(P) + 2, \\ f_1(Q) = f_1(P) + 3, \\ f_2(Q) = f_2(P) + 1. \end{cases}$$

Here, we have obtained the above equation $f_0(Q) = f_0(P) + 2$ by using the fact that, by the operation of truncating, three more facets are created, but one facet disappears.

Next, in the example of [Figure 7], each step shows the procedure of truncating of a pyramid.

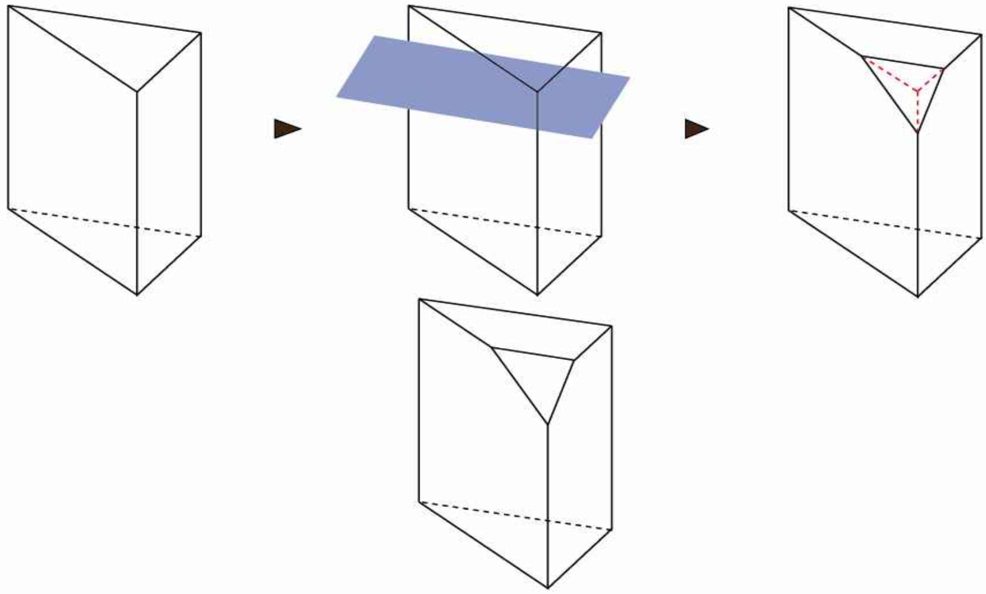


[Figure 7] Truncating of a pyramid

For this, let P be a pyramid and let Q be the polytope obtained by truncating operation of P. Once again, by comparing P and Q, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q) = f_0(P) + 3, \\ f_1(Q) = f_1(P) + 4, \\ f_2(Q) = f_2(P) + 1. \end{cases}$$

Now, in the example of [Figure 8], each step shows the procedure of truncating of a triangular prism.

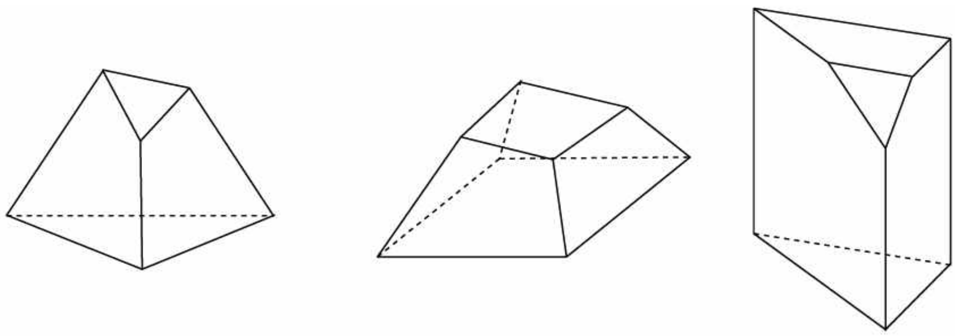


[Figure 8] Truncating of a triangular prism

This time, let P be a triangular prism and let Q be the polytope obtained by truncating operation of P. Then, we can directly show that the following equations hold true.

$$\begin{cases} f_0(Q) = f_0(P) + 2, \\ f_1(Q) = f_1(P) + 3, \\ f_2(Q) = f_2(P) + 1. \end{cases}$$

As a summary of the discussions above, the figures in [Figure 9] show the results of polytopes obtained by truncating tetrahedron, pyramid, and triangular prism.



[Figure 9] Polytopes obtained by taking the truncating

Now, we want to check the change of the flag vector pairs (f_1, f_{02}) after the staking and truncating operations, in detail.

First, in case of stacking of a cube as in [Figure 2] we easily check that if we let P be a cube and let Q be the polytope obtained by stacking operation of P, then we can directly show that the following equations hold true.

$$\begin{cases} f_1(P) &= 12, \\ f_{02}(P) &= f_0(P) \times 3 = 8 \times 3 = 24. \end{cases}$$

Here we obtained $f_{02}(P) = 24$, since at each vertex exactly 3 facets meet and $f_0(P)$ is equal to 8. It is easy to see that the following equations hold true.

$$\begin{cases} f_1(Q) &= f_1(P) + 4 = 16, \\ f_{02}(Q) &= (4 \times 5) + (3 \times 4) = 32. \end{cases}$$

Next, for the case of truncating the triangular prism as in [Figure 8] we easily check that if we let P be a triangular prism and let Q be the polytope obtained by truncating operation of P, then the following equations hold true.

$$\begin{cases} f_1(P) &= 9, \\ f_{02}(P) &= f_0(P) \times 3 = 6 \times 3 = 18. \end{cases}$$

As in the previous case, we can easily calculate the flag vector pair $(f_1(Q), f_{02}(Q))$. Indeed, clearly $f_1(Q) = f_1(P) + 3$. Further, since $f_0(Q)$ is equal to 8 and at each vertex exactly three meet, we should have

$$\begin{cases} f_1(Q) &= 12, \\ f_{02}(Q) &= f_0(Q) \times 3 = 8 \times 3 = 24. \end{cases}$$

2.2 Flag vector pairs (f_0, f_{03}) of 4-polytopes

In this chapter, we explain the stacking and truncating operation for 4-polytope. To do so, let P be a 4-polytope having at least one simplex facet F, and let v be a point beyond F and beneath all other facets of P. Let $Q=\mathrm{conv}(\{v\}\cup P)$, i.e., let Q be the convex hull of P and v. Then, as in the previous cases of 3-polytopes, we can calculate the flag vector pair $(f_0(Q), f_{03}(Q))$, as follows.

$$\begin{cases} f_0(Q) = f_0(P)+1, \\ f_{03}(Q) = f_{03}(P)+12. \end{cases}$$

Let Q be a polytope obtained by truncating a simple vertex from a polytope P. Then we calculate the flag vector pair$(f_0(Q), f_{03}(Q))$, as follows.

$$\begin{cases} f_0(Q) = f_0(P)+3, \\ f_{03}(Q) = f_{03}(P)+12. \end{cases}$$

Note that the polytope obtained by stacking and truncating a polytope with a simplex facet has a simple vertex and a simplex facet, once again. This implies that we can repeatedly stack vertices over simplex facets and truncate simple vertices. By truncating simple vertices and stacking vertices on simplex facets inductively, starting from a polytope with $(f_0(Q), f_{03}(Q))$ with tetrahedral facet and simple vertex, we obtain new polytopes with

$$(f_0+2m+n,\ f_{03}+12n) \text{ for } n \geq 0, 0 \leq m \leq n.$$

On the other hand, given a polytope P with a pyramid facet F, let v be a point beyond F and beneath all other facets of P. As before, let $Q=\mathrm{conv}(\{v\}\cup P)$. Then, we have

$$\begin{cases} f_0(Q) = f_0(P) + 1, \\ f_{03}(Q) = f_{03}(P) + 16. \end{cases}$$

See [6, Section 2.3] for more details.

Chapter 3

Some obstructions of flag vector pairs (f_1, f_{04}) of 5-polytopes

In this chapter, we show some inequalities satisfied by the flag vector pairs (f_1, f_{04}) of 5-polytope. To do so, we begin with a recent result of Grünbaum in [1].

Lemma 3.1. ([1, Theorem 10.4.1.])
The set of flag vector pairs (f_0, f_3) of 4-polytopes is equal to

$$\Pi_{0,3}(F^4) = \left\{ \begin{array}{r} (f_0, f_3) \in \mathbb{Z}^2 : 5 \le f_0 \le \frac{1}{2} f_3(f_3 - 3), \\ 5 \le f_3 \le \frac{1}{2} f_0(f_0 - 3) \end{array} \right\}.$$

Lemma 3.2.
The flag vector pair $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies the following inequalities.

$$f_1(P) \le \frac{1}{4} f_{04}(P)(f_{04}(P) - 3)$$

Proof.
Let F^4 be any facet of a 5-polytope P. Then, it follows from [5, Theorem 10.4.1.] of Grünbaum that we have

$$f_3(F^4) \le \frac{1}{2} f_0(F^4)(f_0(F^4) - 3) = \frac{1}{2} f_0^2(F^4) - \frac{3}{2} f_0(F^4)$$

$$\therefore \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_3(F^4) \le \frac{1}{2} \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_0^2(F^4) - \frac{3}{2} \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_0(F^4).$$

Since $\sum_{i=1}^{k} x_i^2 \leq (\sum_{i=1}^{k} x_i)^2$ for any non-negative $x_i\,(1 \leq i \leq k)$, it follows from by the above inequality we have

$$f_{34}(P) = \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_3(F^4) \leq \frac{1}{2}\left(\left(\sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_0(F^4)\right)^2 - 3\left(\sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_0(F^4)\right)\right)$$

$$= \frac{1}{2} f_{04}^2(P) - \frac{3}{2} f_{04}(P).$$

By considering the dual polytope P^* of P, we can obtain

$$2f_1(P^*) = f_{01}(P^*) \leq \frac{1}{2} f_{04}(P^*)(f_{04}(P^*) - 3).$$

Since P is an arbitrary polytope, so it its dual P^*. Therefore, we can obtain

$$f_1(P) \leq \frac{1}{4} f_{04}(P)(f_{04}(P) - 3),$$

as desired. This completes the proof.

□

Lemma 3.3. (Generalized U.B.T and L.B.T equation [4, Lemma 1.34.])

The number of i-face of cyclic polytope $C_n(m)$ (or any neighborly n-polytope with m vertices) is given by

$$f_i(C^n(m)) = \sum_{q=0}^{\left[\frac{n}{2}\right]} \binom{q}{n-1-i}\binom{m-n+q-1}{q} + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \binom{n-p}{i+1-p}\binom{m-n+p-1}{p},$$

for $i = -1, \cdots, n-1$

where we assume

$$\binom{p}{q} = 0 \text{ for } p < q$$

and $[a]$ denotes the Gauss symbol (greatest integer function) of a rational number a.

Lemma 3.4.

The flag vector $(f_0(P), f_{04}(P))$ of a 5-polytope P satisfies the following inequalities.

$$5f_0(P) \leq f_{04}(P) \leq 5(f_0(P)-3)(f_0(P)-4)$$

Here the second inequality becomes the equality if and only if P is neighborly.

Proof.

For the proof, note first that every vertex of a d-polytope meets at least d facets. Thus we have $5f_0(P) \leq f_{04}(P)$, where the equality holds if and only if P is a simple polytope.

On the other hand, it follows from that for any d-dimensional polytope Q with n vertices (i.e.,$n = f_0(Q)$) and for any subset $S \subset \{0, \cdots, d-1\}$ we have

$$f_s \leq f_s(C_d(n))$$

where $C_d(n)$ denotes the d-dimensional cyclic polytope with $n = f_0(Q)$ vertices. Hence, for $d=5$ we have

$$f_{04} \leq f_{04}(C_5(n)) = 5f_4(C_5^*(n)).$$

Here, the second equality holds because $C_5(n)$ and its dual $C_5^*(n)$ are both simplicial, and the first inequality becomes an equality if and only if P is neighborly.

Now, we calculate by using the formula of Lemma 3.3.

$$f_4(C_5(n)) = \sum_{q=0}^{\left[\frac{5}{2}\right]} \binom{q}{4-4}\binom{n-5+q-1}{q} + \sum_{p=0}^{\left[\frac{4}{2}\right]} \binom{5-p}{5-p}\binom{n-5+p-1}{p}$$

$$= 2\times\sum_{q=0}^{2}\binom{n+q-6}{q} = 2\times\left(\binom{n-6}{0}+\binom{n-5}{1}+\binom{n-4}{2}\right)$$

$$= 2(1+(n-5)+\frac{1}{2}(n-4)(n-5)) = 2((n-4)(1+\frac{n-5}{2}))$$

$$= (n-4)(n-3).$$

$$\therefore\ f_{04} \le 5f_4(C_5(n)) = 5(f_0-3)(f_0-4).$$

This completes the proof.

□

Lemma 3.5. ([5, Theorem 1.2.])

Let $L=\left\{\left(v, \left\lfloor \frac{5}{2}v+1 \right\rfloor\right) : \ v \ge 7\right\}$ and $G=\{(8,20),(9,25),(13,35)\}$.

Then,

$$\epsilon^5 = \left\{(v,e) : \frac{5}{2}v \le e \le \binom{v}{2}\right\}\setminus(L\cup G).$$

Here, $\epsilon^5 = \{(f_0(P), f_1(P)) : P \text{ is a } 5-\text{polytope}\}$, and $\lfloor a \rfloor$ denotes the integer part of a rational number a.

Lemma 3.6.

The flag vector $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies the following inequalities.

$$f_1(P) \ge \frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right).$$

Proof.

Let P be a 5-polytope. By Lemma 3.4, we have

$$f_{04} \le 5(f_0-3)(f_0-4)\,,\; f_0^2-7f_0+12 \ge \frac{1}{5}f_{04}\,.$$

Thus, we have

$$f_0^2(P)-7f_0(P)+12-\frac{1}{5}f_{04}(P) \ge 0\,.$$

Since $f_0(P)$ is greater than or equal to 6, it is easy to obtain

$$f_0(P) \ge \frac{1}{2}\left(7+\sqrt{49-48+\frac{4}{5}f_{04}(P)}\right) = \frac{1}{2}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right)\,.$$

Recall now that by Lemma 3.5, we have

$$\frac{2}{5}f_1(P) \ge f_0(P).$$

Thus we obtain $\frac{2}{5}f_1(P) \ge \frac{1}{2}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right)$, which implies

$$f_1(P) \ge \frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right)\,.$$

This completes the proof.

□

By Lemmas 3.2 and 3.6, we can show the following Theorem 3.7.

Theorem 3.7.

Given a flag number $f_{04}(P)$ of a 5-polytope P, $f_1(P)$ satisfies the following inequalities.

$$\frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right)\leq f_1(P)\leq \frac{1}{4}f_{04}(P)(f_{04}(P)-3)$$

Lemma 3.8. (Generalized Dehn-Sommerville equation [3, Theorem. 2.1.])

Let P be a d-polytope, and let $S\subseteq \{0,1,2,\cdots,d-1\}$.

Let $\{i,k\}\subseteq S\sqcup\{-1,d\}$ such that $i<k-1$ and such that there is no $j\in S$ such that $i<j<k$. Then, we have

$$\sum_{j=i+1}^{k-1}(-1)^{j-i-1}f_{s\cup\{j\}}(P)=f_s(P)(1-(-1)^{k-i-1}).$$

Example 3.1.

① Assume that $d=5, S=\{0\}, i=0, k=5$. Then, we have

$$0=f_{01}-f_{02}+f_{03}-f_{04}=2f_1-f_{02}+f_{03}-f_{04}.$$

② Assume that $d=4, S=\{0\}, i=0, k=4$. Similarly, we have

$$f_{02}=-2f_0+2f_1+f_{03}.$$

Lemma 3.9. ([1, Theorem 10.4.2.])

The set of flag vector (f_0, f_1) of 4-polytopes is equal to

$$\Pi_{0,1}(F^4)=\left\{(f_0,f_1)\in\mathbb{Z}^2:10\leq 2f_0\leq f_1\leq\frac{1}{2}f_0(f_0-1)\right\}$$
$$\setminus\{(6,12),(7,14),(8,17),(10,20)\}.$$

Lemma 3.10.

The flag vector 3 -tuple $(f_1(P), f_{02}(P), f_{04}(P))$ of a 5-polytope P satisfies the following satisfies the following inequality.

$$2f_1(P)-f_{02}(P)+f_{04}(P)\leq 0$$

Proof.

Let P be a 5-polytope. Let F^4 be any facet of a 5-polytope P. By Lemma 3.9, we have

$$f_1(F^4) \geq 2f_0(F^4) \geq 10.$$

Thus, it is easy to obtain

$$f_{14}(P) = \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_1(F^4) \geq 2 \sum_{\substack{F^4 \subset P \\ \dim F^4 = 4}} f_0(F^4) = 2f_{04}(P).$$

By applying the duality, it is true that $f_{03}(P) \geq 2f_{04}(P)$. It also follows from Lemma 3.8 and Example 3.1 that we have

$$\begin{aligned} f_{04}(P) &= f_{01}(P) - f_{02}(P) + f_{03}(P) \\ &= 2f_1(P) - f_{02}(P) + f_{03}(P) \\ &\geq 2f_1(P) - f_{02}(P) + 2f_{04}(P). \\ \therefore\ 0 &\geq 2f_1(P) - f_{02}(P) + f_{04}(P). \end{aligned}$$

This completes the proof.

□

Lemma 3.11.

The flag vector $(f_1(P), f_{02}(P))$ of a 5-polytope P satisfies the following inequalities.

$$f_{02}(P) \leq f_{02}(C_5(n)) = 3f_2(C_5(n)) \ \leq 6(f_0^2(P) - 6f_0(P) + 10)$$

Proof.

Let P be a 5-polytope. As in the proof of by Lemma 3.4, by applying the upper bound theorem we can obtain

$$f_{02}(P) \le f_{02}(C_5(n)) = 3f_2(C_5(n)),$$

where $f_0(P) = n$ and the fact that $C_5(n)$ is a simplicial polytope was used in the last equality.

On the other hand, by using the formula of Lemma 3.3

$$f_i(C^n(m)) = \sum_{q=0}^{\left[\frac{n}{2}\right]} \binom{q}{n-1-i}\binom{m-n+q-1}{q} + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \binom{n-p}{i+1-p}\binom{m-n+p-1}{p},$$

$$\text{for } i = -1, \cdots, n-1$$

it is straightforward to compute

$$f_2(C_5(n)) = \sum_{q=0}^{\left[\frac{5}{2}\right]} \binom{q}{5-1-2}\binom{n-5+q-1}{q} + \sum_{p=0}^{\left[\frac{5-1}{2}\right]} \binom{5-p}{2+1-p}\binom{n-5+p-1}{p}$$

$$= \sum_{q=0}^{2} \binom{q}{2}\binom{n+q-6}{q} + \sum_{p=0}^{2} \binom{5-p}{3-p}\binom{n+p-6}{p}$$

$$= \binom{n-4}{2} + 10 \times \binom{n-6}{0} + 6 \times \binom{n-5}{1} + 3 \times \binom{n-4}{2}$$

$$= \frac{1}{2}(n-4)(n-5) + 10 + 6(n-5) + \frac{3}{2}(n-4)(n-5)$$

$$= 2(n-4)(n-5) + 10 + 6(n-5)$$

$$= 2(n^2 - 6n + 10)$$

$$\therefore\ f_{02}(P) \le 6(f_0^2(P) - 6f_0(P) + 10)$$

Here we used the convention that $\binom{q}{i} = 0$ for $q < i$.

This completes the proof.

$\square$

Lemma 3.12.

The flag vector $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies the following inequalities.

$$f_{04}(P) \leq \frac{1}{25}(24f_1^2(P) - 410f_1(P) + 1{,}500)$$

Proof.

Let P be a 5-polytope. Then we have

$$\begin{aligned}
f_{04}(P) &\leq -2f_1(P) + f_{02}(P) \ (\because \text{Lemma } 3.10) \\
&\leq -2f_1(P) + 6(f_0^2(P) - 6f_0(P) + 10) \ (\because \text{Lemma } 3.11) \\
&\leq -2f_1(P) + 6\left(\frac{4}{25}f_1^2(P) - 6\times\frac{2}{5}f_1(P) + 10\right) \ (\because \text{Lemma } 3.5) \\
&= -2f_1(P) + \left(\frac{24}{25}f_1^2(P) - \frac{72}{5}f_1(P) + 60\right) \\
&= \frac{24}{25}f_1^2(P) - \frac{82}{5}f_1(P) + 60 \\
&= \frac{1}{25}\left(24f_1^2(P) - 410f_1(P) + 1{,}500\right)
\end{aligned}$$

$$\therefore\ f_{04}(P) \leq \frac{1}{25}(24f_1^2(P) - 410f_1(P) + 1{,}500).$$

□

Lemma 3.13.

The flag vector $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies the following inequalities.

$$f_{04}(P) \geq \frac{1}{2}\left(3 + \sqrt{9 + 16f_1(P)}\right)$$

Proof.

Let P be a 5-polytope. By Lemma 3.2,

$$f_1(P) \leq \frac{1}{4} f_{04}(P)(f_{04}(P)-3).$$

$$\text{i.e., } 4f_1(P) \leq f_{04}(P)(f_{04}(P)-3)$$

That is, we have

$$f_{04}^2(P) - 3f_{04}(P) - 4f_1(P) \geq 0$$

$$\therefore\ f_{04} \geq \frac{1}{2}\left(3+\sqrt{9+16f_1(P)}\right)$$

This completes the proof.

□

By Lemmas 3.12 and 3.13, we can show the following Theorem 3.14.

Theorem 3.14.

Given a flag number $f_1(P)$ of a 5-polytope P, $f_{04}(P)$ satisfies the following inequalities.

$$\frac{1}{2}\left(3+\sqrt{9+16f_1(P)}\right) \leq f_{04}(P) \leq \frac{1}{25}(24f_1^2(P) - 410f_1(P) + 1{,}500).$$

In fact, it turns out that the upper bound of $f_{04}(P)$ given in Theorem 3.14 can be improved further by using the inequality in Theorem 3.7.

Theorem 3.15.

Given a flag number $f_1(P)$ of a 5-polytope P, $f_{04}(P)$ satisfies the following inequalities.

$$\frac{1}{2}\left(3+\sqrt{9+16f_1(P)}\right)\le f_{04}(P)\le \frac{4}{5}f_1^2(P)-14f_1(P)+60$$

Proof.

It suffices to prove the upper bound of $f_{04}(P)$. To do so, first recall that the following inequality from Theorem 3.7 holds.

$$\frac{5}{4}\left(7+\sqrt{1+\frac{4}{5}f_{04}(P)}\right)\le f_1(P)$$

Thus, by solving the above inequality for $f_{04}(P)$ we can easily obtain

$$f_{04}(P)\le \frac{4}{5}f_1^2(P)-14f_1(P)+60\,.$$

Note that

$$\frac{4}{5}f_1^2(P)-14f_1(P)+60\le \frac{1}{25}(24f_1^2(P)-410f_1(P)+1{,}500)$$

with equality if and only if $f_1(P)\ge 15$.

This completes the proof.

□

Chapter 4

Constructions of 5-polytopes with (f_1, f_{04}) using stacking and truncating

The aim of this chapter is to provide some examples of 5-polytopes whose flag vector pairs (f_1, f_{04}) satisfy the inequalities Theorems 3.7 and 3.14 given in Chapter 3.

In order to construct such examples, we use the well-known operations such as stacking and truncating. In many instances, these operations turn out to be essential in finding new examples of polytopes for possible polytopal pairs. To begin with, we have the following lemma.

Lemma 4.1.

Let P be a 5-polytope with at least one simplex facet F, and let v be a point beyond F and beneath all other facets of P. Let Q be the 5-polytope obtained by stacking the vertex v over P, i.e, $Q = \mathrm{conv}(\{v\} \cup P)$. Then, we have the following identities.

$$\begin{cases} f_1(Q) = f_1(P) + 5, \\ f_{04}(Q) = f_{04}(P) + 20 (= 5 \times {}_5C_4 - 5). \end{cases}$$

Proof.

By the way of the construction of Q, it suffices to show the last identity. To see it, note first that F is a 4-simplex with five vertices. If we apply the stacking operation to P with such a vertex v over F, then it is easy to see that the flag number $f_{04}(P)$ increase by $5 \times {}_5C_4$ and decreases by 5. Thus, we have $f_{04}(Q) = f_{04}(P) + 20$.

This completes the proof.

□

Lemma 4.2.

Let P be a 5-polytope with at least on simple vertex v, and let R be the 5-polytope obtained by truncating the vertex v from P. Then, we have the following identities.

$$\begin{cases} f_1(R) = f_1(P) + 10(= {}_5C_2), \\ f_{04}(R) = f_{04}(P) + 20(= 5 \times {}_5C_1 - 5). \end{cases}$$

Proof.

By the way of the construction of R, once again it suffices to prove the last identity. To prove it, note first that by the truncating operation we have 5 new vertices and these five new vertices are all simple. Thus, the flag number f_{04} increase by 5×5 and decreases by 5 coming from the old vertex v. This implies $f_{04}(R) = f_{04}(P) + 20$.

This completes the proof.

□

Let P be a 5-polytope with a 4-simplex and a simple vertex. By truncating simple vertices l times and a stacking vertices on 4 -simplex facets k times repeatedly, we can obtain a new 5-polytope Q with the flag vector pair

$$(f_1(Q), f_{04}(Q)) = (f_1(P) + 5k + 10l,\ f_{04}(P) + 20k + 20l). \quad (k, l \geq 0)$$

Let $n = k + l$. Then, $l = n - k$ and $0 \leq k \leq n$, $l \geq 0$, we have

$$\begin{aligned} (f_1(Q), f_{04}(Q)) &= (f_1(P) + 5k + 10l,\ f_{04}(P) + 20k + 20l) \\ &= (f_1(P) + 10(k+l) - 5k,\ f_{04}(P) + 20(k+l)) \\ &= (f_1(P) + 10n - 5k,\ f_{04}(P) + 20n). \end{aligned}$$

As a special case, let P be a 5-simplex. Then, the flag vector pair $(f_1(P), f_{04}(P))$ is equal to $(15, 30)$. Thus, we can obtain the flag vector pair

$$\left(f_1(Q), f_{04}(Q)\right) = (15+5k+10l, 30+20k+20l)$$
$$= (10n-5k+15, 30+20n). \quad (n \geq 0, 0 \leq k \leq n)$$

One may directly check if the flag vector pair $(f_1(Q), f_{04}(Q))$ satisfies the inequalities Theorems 3.7 and 3.15 in Chapter 3.

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