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On the simplicial actions of real toric spaces

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K는 꼭지점의 집합 $\{1,2,...,m\}$의 $n-1$차원 단체이고 G를 K에 단체작용을 하는 m개의 원소로 이루어진 집합의 치환군 S_m의 유한 부분군이라 할 때, 선형사상 $\Lambda : Z_2^m \to Z_2^n$의 핵 $\ker\Lambda$가 G에 대하여 불변이라 가정하자. 이 때, G는 $\ker\Lambda$의 실모멘트-앵글 다양체 RZ_K의 몫공간인 실토릭 다양체 $RZ_K/\ker\Lambda$에 작용한다. 이 논문에서는 이러한 실토릭 다양체 $RZ_K/\ker\Lambda$에 단체 작용을 하는 유한군 G에 대한 구조 정리를 고찰하였다. 즉, 이 경우에는 G가 위수가 2인 원소를 항상 포함하고 있음을 증명하여 G의 위수가 항상 짝수라는 결과를 자세히 알아보았다.

Chapter 1
Introduction

Our main concern in this thesis is the real moment-angle complex and its real toric space which have recently attracted much attention from many people working on toric topology (see [3], [4], and [7] for more details). Recall that the real toric variety is given by the fixed point set of a real toric variety under the involution defined by the complex conjugation. The real toric space is a notion which is a generalization of a real toric variety in algebraic geometry. By definition, any finite subgroup G of the permutation group acting simplicially on a simplicial complex K naturally acts on the real moment-angle complex associated to K, and it further induces an action on the real toric space under a certain invariance condition of G.

Our primary aim in this thesis is to survey the results in the paper [10] for the structure of the group G acting simplicially on the toric space. As a consequence, we can see a certain structure theorem of such a finite group G acting simplicially on the real toric space. In addition, in Chapter 2 we also quickly review some related material in [3], [4], and [6] necessary for explaining main results given in Chapters 3 and 4.

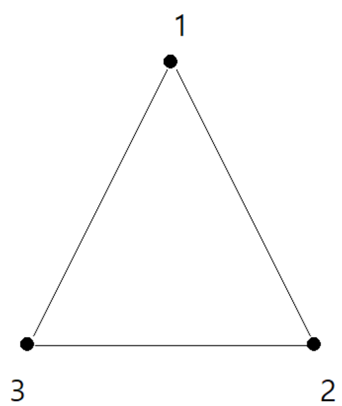
In order to explain our main results more precisely, let K be a simplicial complex on the vertex set $[m]:=\{1,2,\cdots,m\}$, and let D^1 and S^0 denote the closed interval $[-1,1]$ and its boundary $\partial D^1=\{-1,1\}$, respectively. Roughly speaking, the real moment-angle complex RZ_K of K is defined as

$$RZ_K := \left(\underline{D^1}, \underline{S^0}\right)^K$$
$$= \bigcup_{\sigma \in K} \left\{ (x_1, x_2, \cdots, x_m) \in (D^1)^m | x_i \in S^0 \text{ for } i \notin \sigma \right\}.$$

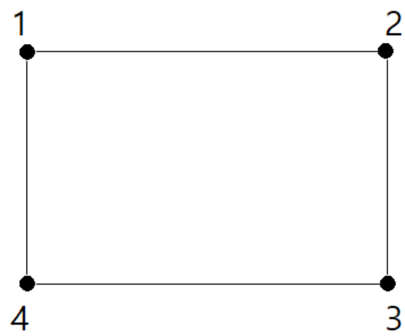
Now, let Z_2 denote the finite group $\{0,1\}$ under the natural addition. Then Z_2^m acts on $(D^1)^m$ diagonally by the sign, and in turn it induces the action on the real moment-angle complex RZ_K. Let $\Lambda : Z_2^m \to Z_2^n$ be a linear map with $m > n$. Then the *real toric space* associated to the pair (K, Λ) is defined by the quotient space $RZ_K/\ker \Lambda$. It is a well known fact that the action of $\ker \Lambda$ on RZ_K is free if and only if Λ satisfies a certain regularity condition. To be more precise, let us denote by $\lambda(i)$ the i-th column of Λ. Then we say that Λ satisfies a regularity condition if

(1.1) $\lambda(i_1), \lambda(i_2), \cdots, \lambda(i_l)$ are linearly independent, whenever $\{i_1, i_2, \ldots, i_l\} \in K$.

Now, let G be a finite subgroup of the permutation group S_m on m letters. We say that G acts simplicially on K if G acts on the vertex set $[m]$ of K as a subgroup of S_m in such a way that it preserves the simplices of K. For example, let K_3 be the 3-gon (or triangle) on the vertex set [3], and let S_3 denote the permutation group acting on K_3 on three vertices. Then S_3 acts simplicially on K_3.

$K_3 =$


On the other hand, let K_4 be the 4-gon (or rectangle) on the vertex set [4], and let S_4 denote the permutation group acting on K_4 by permuting four vertices.

K_4=


Then the action of S_4 does not act simplicially on K_4. This is because an element $g=(23)\in S_4$ permutes two vertices 2 and 3, fixing two other vertices 1 and 4. So the edge $\{1,2\}$ maps to $\{1,3\}$ that is not an edge of K_4.

Clearly by its construction any simplicial action of G induces an action on the real moment-angle complex RZ_K. Moreover, such a G-action on RZ_K induces an action on the quotient space $RZ_K/\ker\Lambda$, whenever the kernel $\ker\Lambda$ of Λ is invariant under G in that for any $z=(z_1, z_2, \cdots, z_{g(m)})\in\ker\Lambda$ and $g\in G$, we have

$$g\bullet z := (z_{g(1)}, z_{g(2)}, \cdots, z_{g(m)})\in\ker\Lambda .$$

With these understood, the main result of this survey paper from [10] is to show that the following theorem for the structure of the finite group G acting simplicially on the real toric space holds.

Theorem 1.1 Let K be a simplicial complex of dimension $n-1$ on the vertex set $[m]$ with $m>n$, and let G act simplicially on K as a subgroup of the permutation group S_m on m letters. Assume that the kernel $\ker \Lambda$ of a linear map $\Lambda : Z_2^m \to Z_2^n$ is invariant under G. Then the following statements hold:

(1) The action of G on K induces an action on the real toric space.

(2) G always contains an element of order 2, and thus the order of G should be even.

The first statement (1) of Theorem 1.1 is the restatement of a result given in [6, Theorem 2.3], while the second statement (2) of Theorem 1.1 is that of the main result in [10].

One wide and well-known class of examples supporting the validity of Theorem 1.1 can be provided with the real toric associated to the Weyl chambers of classical groups, In these cases, the simplicial complexes are the Coxeter complexes of type R and the Weyl groups play the role of the finite groups G which preserve the kernel of a characteristic map (see [6, Chapter 3]). It is interesting to notice that all of the Weyl groups of classical groups always have an even order (see [9]).

As an immediate consequence of Theorem 1.1, we can now state the following corollary.

Corollary 1.2 Let K be a simplicial complex of dimension $n-1$ on the vertex set $[m]$ with $m > n$, and let G act simplicially on K as a subgroup of the permutation group S_m on m letters. If the order of G is odd, then G cannot act simplicially on K so that the kernel $\ker \Lambda$ of a linear map $\Lambda : Z_2^m \to Z_2^n$ is invariant under G.

We organize this thesis, as follows.

In Chapter 2, we briefly summarize some notations and basic facts necessary for the proof of Theorem 1.1 given in Chapter 4.

Especially, we mainly collect basic facts regarding the moment-angle complexes as well as real moment-angle complexes. Refer to an excellent paper [7] of Davis and Januszkiewicz and books [3], [4] of Buchstaber and Panov for more details.

Chapter 3 is devoted to giving a proof of a key lemma which plays an important role in the proof of Theorem 1.1.

In Chapter 4, we give a proof of Theorem 1.1. To do so, we first establish a non-trivial homomorphism Ψ from the finite group G to the homomorphism group $\mathrm{Hom}(\ker\Lambda, \ker\Lambda)$. With this homomorphism Φ in place, the proof of Theorem 1.1 immediately follows, as we can see in Chapter 4.

Finally, in Chapter 5 we close this thesis by giving some nontrivial examples supporting the validity of the main result in this thesis.

Chapter 2
Real moment-angle complexes and real toric spaces

The aim of this chapter is to quickly review some material regarding moment-angle complexes and real toric spaces which we are mainly concerned with. Refer to [1], [2], [3], and [4] for more details on real moment-angle complexes as well as moment-angle complexes.

2.1 Moment-angle complexes

Let $F=\{F_1, \ldots, F_m\}$ be the set of facets of a polytope P^n. For each facet $F_i \in F$, denote by T_{F_i} the one-dimensional coordinate subgroup of $T^F \cong T^m$ corresponding to F_i. Then we assign to every face E the coordinate subtorus

$$T_E = \prod_{F_i \supset E} T_{F_i} \subset T^F.$$

Note that $\dim T_E = \operatorname{codim} E$. Recall that for every point $q \in P^n$ we denoted by $E(q)$ the unique face containing q in the relative interior.

Definition 2.1. For any combinatorial simple polytope P^n, introduce the identification space

$$Z_P = \left(T^F \times P^n\right)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p=q$ and $t_1 t_2^{-1} \in T_{E(q)}$. This space is called a moment-angle manifold associated to the polytope P^n.

Then the free action of T^m on $T^F \times P^n$ descends to an action on Z_P, with quotient P^n. Let $\rho: Z_P \to P^n$ be the orbit map. The action of T^m on Z_P is free over the interior of P^n, while each vertex $v \in P^n$ represents the orbit $\rho^{-1}(v)$ with maximal isotropy subgroup of dimension n.

Lemma 2.2. The space Z_P is a smooth manifold of dimension $m+n$.

Proof: See [3, Lemma 6.2].

The following fact ([3, Proposition 6.4]) also holds to be true.

Lemma 2.3. Let P_1, P_2 be two polytopes. If $P = P_1 \times P_2$, then the moment-angle manifold Z_P is diffeomorphic to $Z_{P_1} \times Z_{P_2}$. Moreover, if F is a face of P, then Z_F is a submanifold of Z_P.

For more interesting properties on moment-angle complexes, please refer to [1], [2], [3], and [4].

Now we give some examples of moment-angle complexes.

Example 2.4.

(1) Let $P^n = \triangle^n$ (the n-simplex). Then Z_P is homeomorphic to the $(2n+1)$−sphere S^{2n+1}.

(2) The cubical complex $C(\triangle^n)$ constructed in [3, Construction 4.5] consists of $(n+1)$ cubes C_v^n. In this case, each subset $B_v = \rho^{-1}(C_v^n)$ is homeomorphic to $(D^2)^n \times S^1$. In particular, for $n=1$ we obtain the representation of the 3-sphere S^3 as a union of two solid toric $D^2 \times S^1$ and $S^1 \times D^2$, glued by the identity diffeomorphism of their boundaries.

It is true that we can extend the definition of a moment-angle manifold associated to a given polytope to that associated to a simplicial complex. There are several equivalent ways to do so. In this thesis, we adopt one of them.

Indeed, let K be a simplicial complex of dimension $n-1$ with the vertex set $[m]=\{1,2,\ldots,m\}$, and let D^2 and S^1 denote the unit disk in the complex plane C and its boundary $\partial D^2 = S^1$, respectively. For each $\sigma \in K$, let $(D^2,S^1)^{\sigma}$ be a subspace of $(D^2)^m$ whose i-th component $(D^2,S^1)_i^{\sigma}$ is given by

$$(D^2,S^1)_i^{\sigma} = \begin{cases} D^2, \, i \in \sigma, \\ S^1, \, i \notin \sigma. \end{cases}$$

Then the moment-angle complex Z_K of K is defined to be

$$\begin{aligned} Z_K &:= (\underline{D}^2,\underline{S}^1)^K = \bigcup_{\sigma \in K} (\underline{D}^2,\underline{S}^1)^{\sigma} \\ &= \bigcup_{\sigma \in K} \left\{ (x_1,x_2,\cdots,x_m) \in (D^2)^m \,|\, x_i \in S^1 \text{ for } i \notin \sigma \right\}. \end{aligned}$$

It can be shown that if K is a simplicial $(n-1)$-sphere, then Z_K is an $(m+n)$-dimensional (closed) manifold. However, Z_K is not a topological manifold, but just a simplicial complex, if K is not a simplicial sphere.

Next, we provide some examples of moment-angle complexes, as follows.

Example 2.5.
(1) Let K be a simplicial complex on the vertex set $[2]$ so that $K=\{\{1\},\{2\},\varnothing\}$.

$K=$ ● 1 ● 2

Then we have

$$Z_K = D^2 \times S^1 \bigcup S^1 \times D^2 = S^3.$$

(2) More generally, let $K = \partial\Delta$ denote the boundary of the $(m-1)$-simplex Δ. Then it is easy to see that there is a homeomorphism between Z_K and S^{2m-1}. Moreover, the diagonal subgroup $\Delta(T^m) = S^1$ of the m-dimensional torus T^m acts freely on $Z_K = S^{2m-1}$. So it is interesting to observe that we can obtain the complex projective space CP^m by taking the quotient of S^{2m-1} under the action of $\Delta(T^m)$.

2.2 Real moment-angle complexes

Let K be a simplicial complex on the vertex set [m], and let D^1 and S^0 denote the closed interval $[-1,1]$ and its boundary $\partial D^1 = \{-1,1\}$, respectively. Let $(D^1,S^0)^\sigma$ be a subspace of $(D^1)^m$ whose i-th component $(D^1,S^0)_i^\sigma$ is given by

$$(D^1,S^0)_i^\sigma = \begin{cases} D^1, & i \in \sigma, \\ S^0, & i \notin \sigma. \end{cases}$$

Then the real moment-angle complex RZ_K of K is defined to be

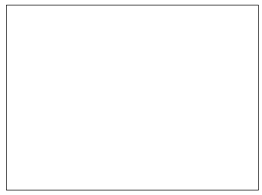
$$\begin{aligned} RZ_K &:= (\underline{D}^1, \underline{S}^0)^k = \bigcup_{\sigma \in K} (\underline{D}^1, \underline{S}^0)^\sigma \\ &= \bigcup_{\sigma \in K} \left\{ (x_1, x_2, \cdots, x_m) \in (D^1)^m \mid x_i \in S^0 \text{ for } i \notin \sigma \right\}. \end{aligned}$$

As in the case of moment-angle complexes, it can be shown that if K is a simplicial $(n-1)$-sphere, then RZ_K is an n-dimensional (closed) manifold. However, RZ_K is not a topological manifold, but just a simplicial complex, if K is not a simplicial sphere.

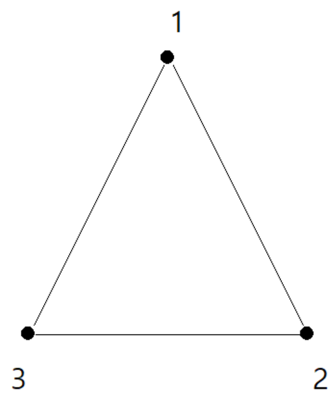
Example 2.6.

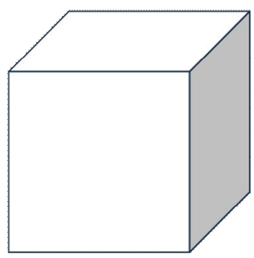
(1) Let K be a simplicial complex on the vertex set $[2]$ so that $K=\{\{1\},\{2\},\varnothing\}$. Then we have

$K=$ 

$RZ_K=D^1\times S^0\bigcup S^0\times D^1=$  $=S^1$.

(2) Let K be a 3-gon on the vertex set [3] so that $K=\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{3,1\}\}$. Then we have

$K=$ 

$RZ_K=D^1\times D^1\times S^0\bigcup S^0\times D^1\times D^1\bigcup D^1\times S^0\times D^1=$  $=S^2$.

2.3 Real toric spaces

Let Z_2 denote the vector space $\{0,1\}$ over Z_2 under the natural scalar multiplication, and let $\Lambda: Z_2^m \to Z_2^n$ be simply a linear map. Note that Z_2^m acts on $(D^1)^m$ diagonally, as follows.

(2.1)
$$Z_2^m \times (D^1)^m \to (D^1)^m$$
$$((z_1,\cdots,z_m),\,(x_1,\cdots,x_m)) \mapsto ((-1)^{z_1}x_1,\cdots,(-1)^{z_m}x_m),$$

so that in turn it induces an action on the real moment-angle complex RZ_K. Clearly any subgroup of Z_2^m can be given by the kernel $\ker\Lambda$ of a linear map $\Lambda: Z_2^m \to Z_2^n$ for $m > n$, and the quotient space $RZ_K/\ker\Lambda$ is called the real toric space associated to the pair (K,Λ). It turns out that the action of $\ker\Lambda$ on RZ_K is free if and only if Λ satisfies the regularity condition as below.

Lemma 2.7. The action of $\ker\Lambda$ on RZ_K is free if and only if the condition (1.1) is satisfied.

Proof: ($\Leftarrow$) Let $p=(x_1,x_2,\ldots,x_m)$ be a fixed point of RZ_K under the action of $g=(g_1,g_2,\ldots,g_m)\in\ker\Lambda$. Then it follows from (2.1) that we have either $g_i=0$ for all $i\in[m]$ or $x_i=0$ for all $i\in[m]$. Now let $\sigma\in K$ be a simplex such that $p\in(D^1,S^0)^\sigma$ and let Λ_σ denote the submatrix of Λ consisting of columns corresponding to σ. Let g_σ denote the subvector of g corresponding to σ. Then, since $g\in\ker\Lambda$, we have
$$\Lambda(g)=\Lambda_\sigma g_\sigma+\Lambda_{[m]-\sigma}g_{[m]-\sigma}=0.$$
The fact that Z_2 acts freely on S^0 implies that we have $g_i=0$ for all $i\in[m]-\sigma$. Thus, we should have $\Lambda_\sigma g_\sigma=0$. This means that $g_\sigma=0$ and so we have $g=0$, as desired.

($\Rightarrow$) Conversely, if (1.1) does not hold, then for a simplex $\sigma\in K$

there should be a point $p=(x_1,x_2,\ldots,x_m)$ such that $x_i=0$, if $i\in\sigma$ and $x_i=0$, otherwise. Thus p is fixed under the action of $\ker\Lambda$.

This completes the proof of Lemma 2.7. $\square$

Chapter 3
A key lemma

A real toric space is a generalization of a more specific space which is defined as the quotient space $RZ_K/\ker\Lambda$ for a characteristic map Λ. Recall that a linear map $\Lambda : Z_2^m \to Z_2^n$ is called a characteristic map if it satisfies a certain regularity condition. That is, when we write the linear map Λ as an $n\times m$-matrix

$$\Lambda = (\lambda(1)\lambda(2)\cdots\lambda(m-1)\lambda(m))_{n\times m}, \tag{3.1}$$

$\lambda(i_1), \lambda(i_2), \cdots, \lambda(i_k)$ are linearly independent over Z_2 for any simplex $\{i_1, i_2, \cdots, i_k\}$ in K. Here each $\lambda(i)$ is regarded as a column vector of size n, and the matrix in (3.1) is called the characteristic matrix.

For a characteristic map Λ, the quotient space $RZ_K/\ker\Lambda$ is called a small cover (resp. real topological toric manifold) if K is a polytopal sphere (resp. star-shaped sphere). Refer to [7] and [8] for more details. Note also that any small cover is in turn a generalization of a real toric variety which is given by the fixed point set of a toric variety under the natural involution defined by the complex conjugation.

Now, as before let G be a finite subgroup of the permutation group S_m on m letters. Recall that G acts simplicially on K if G acts on the vertex set $[m]$ of K as a subgroup of S_m in such a way that it preserves the simplices of K. It is easy to see that any simplicial action of G induces an action on the real moment-angle complex RZ_K. Moreover, it induces an action on the homology group

$$H_*(RZ_K;Q) \cong \bigoplus_{S\in \mathrm{Row}(\Lambda)} \widetilde{H}_{*-1}(K_S;Q),$$

where $\mathrm{Row}(\Lambda) \subset Z_2^m$ denotes the row space of Λ and K_S denote the full subcomplex of K with its vertex set $S \subseteq [m]$. Here, we identify any element S of $\mathrm{Row}(\Lambda)$ with an element I_S of $[m]$ in the natural way. That is, for $S = (s_1, s_2, \cdots, s_m) \in \mathrm{Row}(\Lambda)$, we set

$$I_S = \{ i \in [m] \| s_i \neq 0 \}.$$

The following lemma and its proof plays an important role in the proof of our main Theorem 1.1 (see [6, Theorem 2.3]).

Lemma 3.1. Assume that $\ker \Lambda$ is invariant under G in the sense that for any $z = (z_1, z_2, \cdots, z_m) \in \ker$ and $g \in G$, we have

$$g \bullet z = (z_{g(1)}, z_{g(2)}, \cdots, z_{g(m)}) \in \ker \Lambda.$$

Then the action of G on RZ_K induces an action on the real toric space $RZ_K / \ker \Lambda$.

Proof: For the proof, let x and y be any two elements of RZ_K such that $x = z \bullet y$ for some $z = (z_1, z_2, \cdots, z_m) \in \ker \Lambda$. Then by (2.1) we have

$$\begin{aligned} g \bullet x = g \bullet (z \bullet y) &= g \bullet ((-1)^{z_1} y, \cdots, (-1)^{z_m} y_m) \\ &= ((-1)^{z_{g(1)}} y_{g(1)}, \cdots, (-1)^{z_{g(m)}} y_{g(m)}) \\ &= ((-1)^{z'_1} y_{g(1)}, \cdots, (-1)^{z'_m} y_{g(m)}), \end{aligned} \tag{3.2}$$

where $z' = z_{g(i)}$ for each $i \in [m]$. By assumption, note that

$$z' := (z'_1, z'_2, \cdots, z'_m)$$

is an element of $\ker \Lambda$.

On the other hand, it is also easy to obtain

(3.3)
$$z' \bullet (g \bullet y) = z' \bullet (y_{g(1)}, y_{g(2)}, \cdots, y_{g(m)})$$
$$= ((-1)^{z'_1} y_{g(1)}, (-1)^{z'_2} y_{g(2)}, \cdots, (-1)^{z'_m} y_{g(m)}).$$

Thus it follows from (3.2) and (3.3) that we have $g \bullet x = z' \bullet (g \bullet x)$. This implies that $g \bullet x$ and $g \bullet y$ represent the same element in $RZ_K/\ker \Lambda$.

Now, it is straightforward to show that there is a well-defined action of G on $RZ_K/\ker \Lambda$. This completes the proof of Lemma 3.1. $\square$

Chapter 4
Main result : Proof of Theorem 1.1

The aim of this chapter is to give a proof of Theorem 1.1, essentially following the paper [10]. To do so, we first need to prove the following proposition.

Proposition 4.1 Let K be a simplicial complex of dimension $n-1$ with the vertex set $[m]$ and $m>n$, and let G act simplicially on K. Assume that $\ker\Lambda$ is invariant under G. Then there is a group homomorphism

$$\Psi : G \to \mathrm{Hom}(\ker\Lambda, \ker\Lambda).$$

In particular, this implies that the invariance of G on $\ker\Lambda$ induces an action on $\ker\Lambda$.

Proof: Since K is a simplicial complex on the vertex set $[m]$, it follows from definition that every singleton $\{i\}$ is an element of K for each $i\in[m]$. In particular, if we let $\sigma=\{1\}\in K$, then we have

$$(\underline{D}^1, \underline{S}^0)^{\sigma} = D^1\times S^0\times\cdots\times S^0.$$

Thus we have an element

$$1=(1,1,\cdots,1)\in(\underline{D}^1, \underline{S}^0)^{\sigma}\subset RZ_K.$$

Let G be the finite subgroup of the permutation group S_m on m letters, as before. Then we have $g\cdot 1=1$ for all $g\in G$. Hence the equivalence class $[1]$ in $RZ_K/\ker\Lambda$ is fixed under the action of G on $RZ_K/\ker\Lambda$. Note that every component of 1 is not zero.

Next, we want to construct a homomorphism

$$\Psi : G \to \mathrm{Hom}(\ker \Lambda, \ker \Lambda),$$

as follows. To do so, note first that from the proof of Lemma 3.1 for any $z \in \ker \Lambda$ and $g \in G$ there is a unique element $z'_g \in \ker \Lambda$ such that

$$g \bullet (z \bullet 1) = z'_g \bullet (g \bullet 1) = z'_g \bullet 1,$$

where $z'_g(i)$ is given by $z_{g(i)}$ for each $i \in [m], i.e., z'_g = g \bullet z$. Since $\ker \Lambda$ is assumed to be invariant under G, for each $g \in G$ we can thus define

$$\Psi(g) : \ker \Lambda \to \ker \Lambda,\ z \mapsto z'_g = g \bullet z.$$

Using the map Ψ, we now define a map

$$\Psi : G \to \mathrm{Hom}(\ker \Lambda, \ker \Lambda),\ g \mapsto \Psi(g).$$

It is easy to see that Ψ is indeed a homomorphism we want. To be precise, for any two elements $g_1, g_2 \in G$ we have

(4.1) $$(g_1 g_2) \bullet (z \bullet 1) = z'_{g_1 g_2}(g_1 g_2 \bullet 1) = z'_{g_1 g_2} \bullet 1.$$

on the other hand, it is also true that

(4.2) $$\begin{aligned} ((g_1 g_2) \bullet)(z \bullet 1) &= g_1(g_2 \bullet (z \bullet 1)) \\ &= g_1(z'_{g_2} \bullet (g_2 \bullet 1)) = g_1(z'_{g_2} \bullet 1) \\ &= (z'_{g_2})' \bullet z'_{g_1} \bullet 1. \end{aligned}$$

Note that we have

$$(4.3) \qquad ({z'}_{g_2})'_{g_1} = g_1 \bullet {z'}_{g_2} = g_1 \bullet (g_2 \bullet z).$$

By (4.1), (4.2), and (4.3), we have

$$(4.4) \qquad {z'}_{g_1 g_2} = ({z'}_{g_2})'_{g_1} = g_1 \bullet (g_2 \bullet z).$$

This means that the map Ψ is a homomorphism. To be precise, by (4.4) we have

$$\begin{aligned} \Psi(g_1 g_2)(z) &= \Psi(g_1 g_2)(z) = {z'}_{g_1 g_2} = g_1 \bullet (g_2 \bullet z) \\ &= g_1 \bullet (\Psi(g_2)(z)) = g_1 \bullet (\Psi(g_2)(z)) \\ &= \Psi(g_1) \circ \Psi(g_2)(z), \ \ z \in \ker \Lambda. \end{aligned}$$

That is, we have

$$\Psi(g_1 g_2) = \Psi(g_1) \circ \Psi(g_2),$$

i.e., Ψ is a homomorphism. This completes the proof of Proposition 4.1. □

By using the invariance of G on $\ker \Lambda$, it is also possible to directly show that there is a well-defined action of G on $\ker \Lambda$. It is well-known that this will then induce a group homomorphism Ψ as in the proof of Proposition 4.1. In view of the proof of Proposition 4.1, the action of G on $\ker \Lambda$ satisfies the property: for each $z \in \ker \Lambda$,

$$\begin{aligned} g_1 \bullet (g_2 \bullet z) &= (z_{g_2(g_1(1))}, z_{g_2(g_1(2))}, \cdots z_{g_2(g_1(m))}) \\ &= (z_{g_2 g_1(1)}, z_{g_2 g_1(2)}, \cdots z_{g_2 g_1(m)}) \\ &= (g_2 g_1) \bullet z, \qquad g_1, g_2 \in G. \end{aligned}$$

Finally, we are ready to prove Theorem 1.1 that goes as follows.

Proof of Theorem 1.1: By Proposition 4.1, there is a group homomorphism

$$\Psi : G \to \mathrm{Hom}(\ker \Lambda, \ker \Lambda).$$

Thus, we have an isomorphism

$$G/\ker \Psi \cong \mathrm{Im}\,\Psi \subset \mathrm{Hom}(\ker \Lambda, \ker \Lambda).$$

Note that $\ker \Lambda$ contains a subspace of Z_2^m isomorphic to Z_2^{m-n} and so $\ker \Lambda$ is isomorphic to Z_2^l for some $l \geq m-n$. Hence Hom $(\ker \Lambda, \ker \Lambda)$ is isomorphic to

$$Z_2^l \otimes Z_2^l \cong Z_2^{l^2}.$$

Since the map $\Psi : G \to \mathrm{Hom}(\ker \Lambda, \ker \Lambda)$ is non-trivial by definition, there should be an element of G whose image under Ψ is non-trivial. That is, since the order of $\mathrm{Hom}(\ker \Lambda, \ker \Lambda)$ is 2^{l^2}, it follows from the theorem of Lagrange that the order of $\mathrm{Im}\Psi$ is divisible by 2. Since we have

$$|G| = |G/\ker\Psi||\ker\Psi| = |\mathrm{Im}\Psi||\ker\Psi|,$$

we see that the order of G should be also divisible by 2. Therefore, there exists an element of G whose order is equal to two, completing the proof of Theorem 1.1 $\square$

Chapter 5
Examples

In this chapter, we close this chapter with some simple examples, taken from [6], including real toric varieties associated to the Weyl chambers of classical groups given in Chapter 1, which illustrates our main result.

Example 5.1. Let K_4 be the 4-gon on the vertex set [4], and let G be the cyclic group of order 4 acting on K_4 cyclically on four vertices. Let Λ be the characteristic map whose associated matrix is given by

$$\begin{pmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{pmatrix}.$$

Then it is easy to see that the kernel $\ker\Lambda$ is invariant under the action of G. The real toric space associated to the pair (K_4,Λ) is actually the 2-dimensional torus $T^2 = S^1 \times S^1$ and the induced action of G on T^2 is generated by

$$g : T^2 \rightarrow T^2,\ (x,y) \mapsto (-y,x).$$

At any rate, the order of G is four which is clearly even and contains an element g^2 of order two.

Example 5.2. The Weyl group W_{A_n} of the classical groups of type A_n is the symmetric group S_{n+1}, and let K_{A_n} denote the dual of the permutohedron of order $n+1$. Here the vertex set V_{A_n} can be described by the S_{n+1}-equivariant bijection between V_{A_n} and

$2^{[n+1]} - \{[n+1], \varnothing\}$. Under this bijection, it is known that k subsets $J_1, J_2, \ldots, J_k$ of $[n+1]$ form a simplex of K_{A_n} if and only of they form nested chain of subsets up to permutations. We can also describe the characteristic map Λ_{A_n} by using the basis consisting of $e_k := (1k)w_1$, where $(1k)$ means the transposition in S_{n+1} and $w_1 \in V_{A_n}$. Then it can be shown as in [6, 3.1] that W_{A_n} acts simplicially on the real toric space $X^R_{A_n}$. Note that the order of W_{A_n} is $(n+1)!$ which is clearly even, as desired.

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