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이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

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황금률의 교육학 석사학위 논문을 인준함.

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$P_6(J)$의 실유사토릭 다양체에 관한 연구

황 금 률
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본 논문에서는 정육각형 P_6에 대한 실유사토릭 다양체가 결정됨을 보이고, 그 결과를 바탕으로 P_6의 단체 쐐기 복합체 $P_6(J)$의 실유사토릭 다양체를 결정하였다.
좀 더 구체적으로, 정육각형 P_6의 회전대칭과 기저 바꿈을 통하여 실유사토릭 다양체에 대응하는 7개의 특성행렬을 결정하였고, 단체 쐐기 작용을 이용하여 얻어진 $P_6(J)$의 실유사토릭 다양체에 대응하는 10개의 특성행렬이 결정됨을 증명하였다.

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Chapter 1

Introduction

In toric geometry, there is a well-known one-to-one correspondence between toric varieties and complete fans or underlying simplicial complexes, up to certain equivalence relations (refer to [6] and [10]). This means that in order to study toric varieties it suffices to study their corresponding fans or underlying simplicial complexes which are combinatorial and so somewhat more tractable. Our primary goal in this thesis is to apply this strategy to classify real quasi-toric varieties or more specifically real quasi-toric manifolds over certain simplicial complexes.

In order to explain our results more precisely, we first need to set up some notation and basic definitions, as follows. To do so, recall that a *fan* in the vector space $\mathbb{R}^m$ is a collection of strongly convex rational cones such that every face of cones and every intersection of a finite number of cones are also in the fan. In addition, a fan is called *complete* if the union of all cones covers the whole vector space $\mathbb{R}^m$, while a fan is called *non-singular* if one-dimensional faces of each cone are unimodular in the lattice $\mathbb{Z}^m$ embedded in $\mathbb{R}^m$. On the other hand, a fan is called *simplicial* if one-dimensional faces of each cone are linearly independent in $\mathbb{R}^m$.

It is possible to think of a complete non-singular fan Σ as a pair (K_Σ, λ)

consisting of a underlying simplicial complex K_Σ and a characteristic map λ, where λ is a map from the vertex set of K to the lattice $\mathbb{Z}^m$ obtained by assigning a primitive integral vector to each vertex of K. Also, K_Σ (or Σ) is called *polytopal* if there is an embedding of the geometric realization $|K_\Sigma|$ of K into $\mathbb{R}^n$ such that $|K_\Sigma|$ is given by the boundary of the simplicial dual polytope P^* of a simple convex polytope P. If, in addition, P^* contains the origin and Σ is given by the positive hulls of proper faces of P^*, then K_Σ (or Σ) is said to be *strongly polytopal*. It is well known that the toric variety associated to a strongly polytopal fan is projective. By abuse of terminology, in this thesis we will just say that the corresponding fan or underlying simplicial complex is projective. Recall also that a simplicial complex K is *fan-like* if there is a complete fan Σ whose underlying simplicial complex K_Σ is exactly same as K.

In the papers [7] and [8], Davis and Januszkiewicz and Hattori and Masuda generalized the notion of a toric manifold to several categories of manifolds equipped with torus actions. Among other things, a *torus manifold* of dimension $2n$, first introduced in [8], is defined to be a closed orientable manifold which admits an effective T^n-action with the non-empty fixed point set. Here, when S^1 denotes the unit circle of complex numbers in $\mathbb{C}$, T^n means the product $(S^1)^n$ of n copies of S^1. By definition, any toric manifold is clearly a torus manifold. On the other hand, a quasi-toric manifold of dimension $2n$, first introduced in [7], is defined to be a closed smooth manifold with an effective T^n-action satisfying the following two conditions:

- The torus action is locally standard in that it is locally isomorphic to the standard action of T^n on $\mathbb{R}^{2n}$.

- The orbit space is homeomorphic to a simple convex polytope of dimension n.

It is obvious to see that every quasi-toric manifold is a torus manifold. There exist other general notions of a toric manifold such as the topological toric manifold which will not be dealt with in this thesis (see [9] for more details).

There are several general ways to construct a quasi-toric manifold. One of them we want to present in this thesis goes as follows. That is, given a simple convex polytope P, let $\mathcal{F}$ denote the collection of all facets $F_1, F_2, \ldots, F_m$ of P, and let

$$\lambda : \mathcal{F} \to \mathbb{Z}^n$$

be a characteristic function on $\mathcal{F}$ such that

(1) $\lambda(F_i)$ is a primitive vector for each $i \in [m] := \{1, 2, \ldots, m\}$, and

(2) for a non-empty $P_I := \cap_{i \in I} F_i$ for $I \subset [m]$, $\lambda(F_i)$'s are linearly independent over $\mathbb{Q}$.

For a non-empty P_I, we can form an abelian subgroup T_I^n of T^n generated by $\lambda(F_i)$'s for $i \in I$. Then one can construct a manifold $X(P, \lambda)$ by using the quotient space

$$X(P, \lambda) = (P \times T^n)/\sim .$$

Here, the equivalence relation $\sim$ on the product space $P \times T^n$ is given by

$$(x, t) \sim (y, s) \text{ if and only if } x = y \text{ and } t^{-1}s \in T_I^n,$$

where I is a subset of $[m]$ such that P_I is the minimal face of P containing $x = y$. The manifold $X(P, \lambda)$ is usually called a *quasi-toric manifold*, and, in general, X is just an orbifold. Further, it admits a T^n-action induced from the natural T^n-action on the second factor of $P \times T^n$ whose orbit space is P itself. Hence there is a quotient map

$$\pi : X(P, \lambda) \to P = X(P, \lambda)/T^n.$$

For the sake of simplicity, we shall also use the notation X for $X(P, \lambda)$ if there is no confusion. One typical example of a toric manifold can be provided by the natural action of T^n on the complex projective space $\mathbb{CP}^n$ associated to the n-simplex Δ^n. See [3], [4], and [7] for more details.

Since the dual P^* of P is a simplicial polytope, the characteristic map λ induces a complete non-singular characteristic map λ on $K := \partial P^*$ that is a polytopal sphere, so that we can obtain a pair (K, λ).

Instead of S^1 and T^n, one may repeat the above construction with $\mathbb{Z}_2 = \{0, 1\}$ and

$$\mathbb{Z}_2^n = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$$

to obtain a *real quasi-toric manifold* $X(P, \lambda_{\mathbb{R}})$ of dimension n for a characteristic function $\lambda_{\mathbb{R}} : \mathcal{F} \to \mathbb{Z}_2^n$. However, note that the image $\lambda(F_i)$ of a characteristic function $\lambda_{\mathbb{R}}$ is always primitive and that every linearly independent vectors in $\mathbb{Z}_2^n$ is a part of a basis of $\mathbb{Z}_2^n$. So the quotient space $X(P, \lambda_{\mathbb{R}})$ with the quotient map $\pi : X(P, \lambda_{\mathbb{R}}) \to P$ is always smooth. As in the case of $X(P, \lambda)$, $X(P, \lambda_{\mathbb{R}})$ has a $\mathbb{Z}_2^n$-fixed point if and only if P has a vertex. As in the case of $X(P, \lambda)$, we shall also use the notation $X_{\mathbb{R}}$ for $X(P, \lambda_{\mathbb{R}})$ if there is no confusion.

One example of a real quasi-toric manifold can be given by the natural action of $\mathbb{Z}_2^n$ on the real projective space $\mathbb{RP}^n$ associated to the n-simplex Δ^n. When P is a simple convex polytope, $X_{\mathbb{R}}$ is very often called a *small cover* in the literature (see [7]).

There is a well-known operation, called a *simplicial wedge operation*, from abstract simplicial complexes with n vertices to another abstract simplicial complexes with $n+1$ vertices (see [1] and [2]). That is, for a simplicial complex K with n vertices and any sequence $J = (j_1, j_2, \ldots, j_n)$ of positive integers we can construct a new simplicial complex $K(J)$ with $d(J) = j_1 + j_2 + \cdots + j_n$ vertices, called a *simplicial wedge complexes*. Such a simplicial wedge complex $K(J)$ is obtained inductively by starting from K and applying the simplicial wedge operation to one of the vertices of K. More specifically, as above let K be a fan-like simplicial complex in $\mathbb{R}^m$ with vertices $w_1, w_2, \ldots, w_n$, and let $v = w_1$ be a vertex of K. Then we can obtain the simplicial wedge complex $\mathrm{wedge}_v(K) := K(2, 1, 1, \ldots, 1)$ obtained by applying the simplicial wedge operation to K at v.

Our main concern in this thesis is to investigate quasi-toric or real quasi-toric manifolds over the simplicial wedge complexes $K(J)$. It is well-known that every simple polytope of dimension n with not more than $n+3$ facets has a corresponding standard Gale diagram on $\mathbb{R}^2$ (see [5, Section 6] for more details). It is also true that two simple polytopes of dimension n with $n+3$ facets are combinatorially equivalent to each other if and only if their standard Gale diagrams coincide after some orthogonal transformation of $\mathbb{R}^2$ onto itself. This means that we can classify simple polytopes with $n+3$ facets in terms of their standard Gale diagrams in $\mathbb{R}^2$. This is exactly the way we take in this thesis to quasi-toric manifolds, and first we consider the regular 6-gon P_6 for the standard Gale diagram on $\mathbb{R}^2$. As a consequence, first we can obtain the following result.

Theorem 1.1. *Up to rotational symmetry of P_6 and basis changes of $\mathbb{Z}_2^2$, any complete non-singular fans over P_6 can be determined by one of the following, not necessarily exclusive, seven characteristic matrices:*

(1)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & a \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(2)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & a & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(3)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & a \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(4)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & a & 1 & 0 & 1 \end{pmatrix}.$$

(5)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & a & 1 & 1 & 1 \end{pmatrix}.$$

(6)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & a & 1 \end{pmatrix}.$$

(7)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 1 & a+1 & a \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Here, a is an arbitrary element of $\mathbb{Z}_2 = \{0, 1\}$.

As an immediate consequence of Theorem 1.1, we have the following classification of real quasi-toric manifolds over a simple convex polytope whose dual boundary simplicial complex is P_6.

Corollary 1.2. *Any real quasi-toric manifold over a simple convex polytope whose dual boundary simplicial complex is P_6 can be completely determined by one of the seven characteristic matrices appearing in Theorem 1.1.*

Next we want to explicitly determine all possible complete non-singular fans (or equivalently, real quasi-toric manifolds) corresponding to simplicial wedge complexes $P_6(J)$ of the regular 6-gon P_6. That is, our main theorem is

Theorem 1.3. *Up to rotational symmetry of P_6 and basis changes of $\mathbb{Z}_2^3$, any complete non-singular fans over $P_6(2,1,1,1,1,1)$ can be determined by one of the following, not necessarily exclusive, ten characteristic matrices:*

(1)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n & 0 & m \end{pmatrix}.$$

(2)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & a & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n & 0 & m \end{pmatrix}.$$

(3)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n & 0 & m \end{pmatrix}.$$

(4)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & a & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n & 0 & m \end{pmatrix}.$$

(5)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(6)
$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(7)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & a & 1 \\ 1 & 1 & 0 & 0 & n & 0 & m \end{pmatrix}.$$

(8)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(9)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & a & a \end{pmatrix}.$$

(10)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & a & a \end{pmatrix}.$$

Here, $a, n,$ and m are all arbitrary elements of $\mathbb{Z}_2$.

As an immediate consequence of Theorem 1.3, we have the following classification of real quasi-toric manifolds over a simple convex polytope whose dual boundary simplicial complex is $P_6(2,1,1,1,1,1)$.

Corollary 1.4. *Any real quasi-toric manifold over a simple convex polytope whose dual boundary simplicial complex is $P_6(2,1,1,1,1,1)$ can be completely determined by one of the ten characteristic matrices appearing in Theorem 1.3.*

We organize this paper, as follows. In Chapter 2, we briefly review basic facts regarding simplicial wedge complexes.

In Chapter 3, we give proofs of Theorems 1.1 and 1.3 by the detailed case-by-case analysis of the characteristic matrices over the regular 6-gon P_6 and its simplicial wedge complex $P_6(2,1,1,1,1,1)$.

Chapter 2

Simplicial wedge complexes

The aim of this chapter is to briefly collect basic facts necessary for the proof of main results given in Chapter 3.

A *simplicial complex* K on a finite set V is a collection of subsets of V satisfying

(1) if $v \in V$, then $\{v\} \in K$,

(2) if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

Each element $\sigma \in K$ is called a face of K. $V = [m]$ is a set of vertices.The dimension of σ is defined by $\dim(\sigma)$=$|\sigma| - 1$. The *dimension* of K is defined by $\dim(K) = \max\{\dim(\sigma) |\ \sigma \in K\}$.

There is a useful way to construct new simplicial complexes from a given simplicial complex. We briefly present the construction here. Let K be a simplicial complex of dimension $n-1$ on vertices $V = [m] = \{1, 2, \ldots, m\}$. A subset $\tau \subset V$ is called a *non-face* of K if it is not a face of K. A non-face τ is *minimal* if any proper subset of τ is a face of K. Note that a simplicial complex is determined by its minimal non-faces.

In the setting above, let $J = (j_1, \ldots, j_m)$ be a sequence of positive integers. Denote by $K(J)$ the simplicial complex on vertices

$$\{1_1, \ldots, 1_{j1}, 2_1, \ldots, 2_{j2}, \ldots, m_1, \ldots, m_{jm}\}$$

with minimal non-faces

$$\{(i_1)_1, \ldots, (i_1)_{j_{i_1}}, (i_2)_1, \ldots, (i_2)_{j_{i_2}}, \ldots, (i_k)_1, \ldots, (i_k)_{j_{i_k}}\}$$

for each minimal non-face $\{i_1, \ldots, i_k\}$ of K.

There is another way to construct $K(J)$ called the *simplicial wedge construction*. Recall that for a face σ of a simplicial complex K, the *link* of σ in K is the subcomplex

$$Lk_K\sigma := \{\tau \in K| \ \sigma \cup \tau \in K, \ \sigma \cap \tau = \varnothing\}$$

and the *join* of two disjoint simplicial complexes K_1 and K_2 is defined by

$$K_1 \star K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in K_1, \ \sigma_2 \in K_2\}$$

Let K be a simplicial complex with vertex set [m] and fix a vertex i in K. Let I denote a 1-simplex whose vertices are i_1 and i_2 and let ∂I denote the boundary complex of I consisting of two vertices i_1 and i_2. Now, let us define a new simplicial complex on $m+1$ vertices, called the *simplicial wedge* of K at i, denoted by $\mathrm{wedge}_i(K)$, by

$$\mathrm{wedge}_i(K) = (I \star Lk_k\{i\}) \cup (\partial I \star (K \setminus \{i\})),$$

where $K \setminus \{i\}$ is the induced subcomplex with $m-1$ vertices except i. The operation itself is called the *simlplicial wedge operation* or the *(simplicial) wedging*.

It is an easy observation to show that $\mathrm{wedge}_i(K) = K(J)$ where $J = (1, \ldots, 1, 2, 1, \ldots, 1)$ is the $m-$tuple with 2 as the $i-$th entry. By consecutive application of this construction starting from $J = (1, \ldots, 1)$, we can produce

$K(J)$ for any J. Although there is some ambiguity to proceed from $J = (j_1, \dots, j_m)$ to $J' = (j_1, \dots, j_{j-1}, j_i + 1, j_{i+1}, \dots, j_m)$ if $j_i \geq 2$, we have no problem since any choice of the vertex yields the same minimal non-faces of the resulting complex $\text{wedge}_v(K(J)) = K(J')$ keeping in mind the original definition of $K(J)$. In conclusion, one can obtain a simplicial complex $K(J)$ by successive simplicial wedge constructions starting from K, independent of order of wedgings.

Example : Simplicial wedge complexes

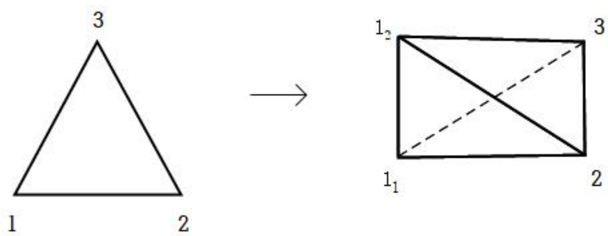

Figure 2.1: $P_3 \to P_3(2, 1, 1)$

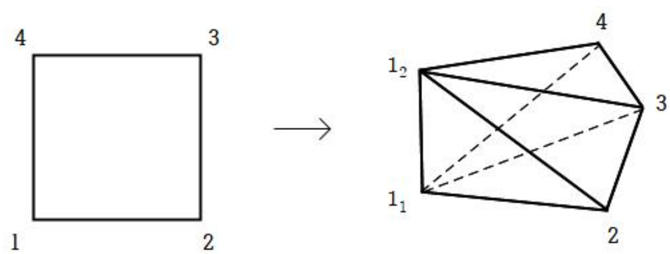

Figure 2.2: $P_4 \to P_4(2, 1, 1, 1)$

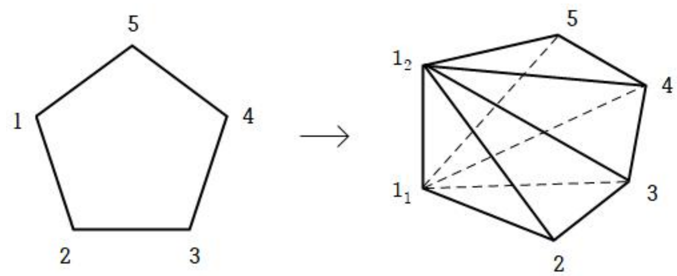

Figure 2.3: $P_5 \to P_5(2,1,1,1,1)$

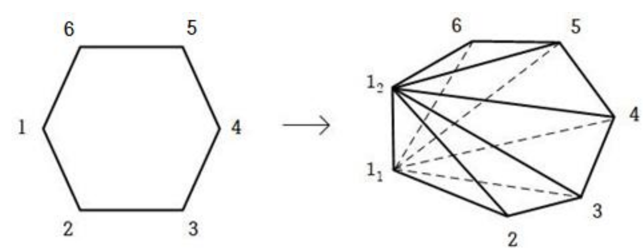

Figure 2.4: $P_6 \to P_6(2,1,1,1,1,1)$

Chapter 3

Main results: Proofs of Theorems 1.1 and 1.3

The aim of this chapter is to provide the proofs of Theorems 1.1 and 1.3 in detail.

3.1 Proof of Theorem 1.1

We begin with the proof of Theorem 1.1. To do so, up to a basis change of $\mathbb{Z}_2^2$ we may assume without loss of generality that the characteristic matrix λ has the following form. That is,

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & b & d & f \\ 0 & 1 & a & c & e & 1 \end{pmatrix}.$$

By the positiveness of the characteristic matrix λ , the following system of equations

$$\begin{cases} c + ab = 1 \\ be + cd = 1 \\ d + ef = 1 \end{cases}$$

holds. Then we divide the proof into several cases, as follows.

1. $a = 0$. In this case, the system of equations becomes

$$\begin{cases} c = 1 \\ be + d = 1 \\ d + ef = 1. \end{cases}$$

Thus,

$$be + ef = 0,$$

$$e(b + f) = 0.$$

(1) $e = 1, \ b + f = 0$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & f \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(2) $e = 0, \ b + f = 1$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & f & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(3) $e = 0, \ b + f = 0$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & f \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

2. $b = 0$. Then the system of equations is reduced to

$$\begin{cases} c = 1 \\ d = 1 \\ ef = 0. \end{cases}$$

(1) $e = 0, \ f = 1$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & a & 1 & 0 & 1 \end{pmatrix}.$$

(2) $f = 0, \ e = 1$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & a & 1 & 1 & 1 \end{pmatrix}.$$

(3) $e = 0,\ f = 0$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & a & 1 \end{pmatrix}.$$

3. $a \neq 0, b \neq 0$. Then, as above we have the following system of equations

$$\begin{cases} c = 0 \\ e + cd = 1 \\ d + ef = 1. \end{cases}$$

(1) $e = 1,\ d + f = 1$

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 1 & f+1 & f \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

To sum up, we can obtain the following theorem.

Theorem 3.1. *Up to rotational symmetry of P_6 and basis changes of $\mathbb{Z}_2^2$, any complete non-singular fans over P_6 can be determined by one of the following, not necessarily exclusive, seven characteristic matrices:*

(1)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & f \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(2)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & f & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(3)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & f \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(4)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & a & 1 & 0 & 1 \end{pmatrix}.$$

(5)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & a & 1 & 1 & 1 \end{pmatrix}.$$

(6)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & a & 1 \end{pmatrix}.$$

(7)

$$\lambda = \begin{pmatrix} 1 & 0 & 1 & 1 & f+1 & f \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Here, a and f denote an arbitrary element of $\mathbb{Z}_2 = \{0, 1\}$.

3.2 Proof of Theorem 1.3

Next we start the proof of Theorem 1.3. To do so, for each d with $1 \le d \le 7$ let $\lambda_d = (v_1, v_2, \ldots, v_6)$ be one of the seven cases as in Theorem 3.1. We then perform a wedge operation on the facet 1 and rename facets by $1_1, 1_2, 3, 4, 5,$ and 6. Let λ be a characteristic matrix for the wedged polytope $\text{wedge}_1 P_6 = P_6(2, 1, 1, 1, 1, 1)$. Note that the facet $\{1_1, i, i+1\}, i \in \mathbb{Z}_6$, of $P_6(2, 1, 1, 1, 1, 1)$ corresponds to a vertex of the boundary of the dual of $P_6(2, 1, 1, 1, 1, 1)$, where we choose i so that neither i nor $i+1$ is same as 1.

Now we choose a basis of $\mathbb{Z}_2^3$ such that

- $\lambda(1_1) = (0, 0, 1)^T$,
- $\lambda(2) = (v_2^T, 0)^T$, and
- $\lambda(3) = (v_3^T, 0)^T$

Then the matrix λ for $P_6(2, 1, 1, 1, 1, 1)$ should be of the following form:

$$\lambda = \left(\begin{array}{ccccccc} 1_1 & 1_2 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 0 & & & & & & \\ \hline 1 & n_1 & 0 & 0 & n_4 & n_5 & n_6 \end{array}\right)_{4\times 7},$$

where n_j is the unknown and j is not equal to i. It is easy to see that at least we can obtain $n_1 = 1$. This is because we have

$$\det(\lambda(1_2)\lambda(2)\lambda(3)) = \det(\lambda(1_1)\lambda(2)\lambda(3)) = 1$$

by the positiveness of λ. As a consequence, we have the following form for the characteristic matrix:

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 1 & 0 & 0 & n_4 & n_5 & n_6 \end{pmatrix}_{3\times 6}.$$

Starting from the above preliminary calculation, we can now determine λ for each seven characteristic matrices λ_d as in Theorem 3.1.

In case of (1) in Theorem 3.1, observe first that λ should be of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & f \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & n_5 & n_6 \end{pmatrix}.$$

Since

$$\det(\lambda(1_2)\lambda(4)\lambda(5)) = \det(\lambda(1_1)\lambda(4)\lambda(5)) = 1$$

and

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & n_4 & n_5 \end{vmatrix} = 1 + n_5 = 1$$

by the positiveness of λ, we have $n_5 = 0$. Therefore, the characteristic matrix for the case (1) in Theorem 3.1 is of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & f \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

Similarly, in the case of (2) in Theorem 3.1 λ should be of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & f & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & n_5 & n_6 \end{pmatrix}.$$

Thus, it is easy to see that the characteristic matrix for the case (2) in Theorem 3.1 is of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & f & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

The cases of (3) and (4) in Theorem 3.1 can be dealt with in a similar way. So these cases will be left to the reader.

For the case of (5) in Theorem 3.1, as above λ should be of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & n_4 & n_5 & n_6 \end{pmatrix}.$$

Note then that the following elementary facts hold:

- $$\det(\lambda(1_2)\lambda(3)\lambda(4)) = \det(\lambda(1_1)\lambda(3)\lambda(4)) = 1.$$

- $$\begin{vmatrix} 1 & 1 & 0 \\ 0 & a & 1 \\ 1 & 0 & n_4 \end{vmatrix} = 1 + n_4 a = 1.$$

- $$\det(\lambda(1_2)\lambda(4)\lambda(5)) = \det(\lambda(1_1)\lambda(4)\lambda(5)) = 1.$$

- $$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & n_4 & n_5 \end{vmatrix} = 1 + n_4 + n_5 = 1.$$

- $$\det(\lambda(1_2)\lambda(5)\lambda(6)) = \det(\lambda(1_1)\lambda(5)\lambda(6)) = 1,$$

  and
  $$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & n_5 & n_6 \end{vmatrix} = 1 + n_5 + n_6 = 1$$

  by the positiveness of λ.

It is now straightforward to obtain the following equations

$$n_4 a = 0, \quad n_4 + n_5 = 0, \quad n_5 + n_6 = 0.$$

If $n_5 = 0$, then we have $n_6 = 0$ and $n_4 = 0$. On the other hand, if $n_5 = 1$, then we have $n_6 = 1$, $n_4 = 1$, and $a = 0$. Therefore, λ for the case (5) in Theorem 3.1 should be of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Similarly, in the case of (6) in Theorem 3.1 we can show that λ is of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & a & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}$$

or

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Finally, for the case of (7) in Theorem 3.1, as in the above cases we may assume that λ is of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & f+1 & f \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & n_4 & n_5 & n_6 \end{pmatrix}.$$

Since

$$\det(\lambda(1_2)\lambda(3)\lambda(4)) = \det(\lambda(1_1)\lambda(3)\lambda(4)) = 1$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & n_4 \end{vmatrix} = 1 + n_4 = 1$$

by the positiveness of λ, we can obtain $n_4 = 0$. Note that the following facts hold:

- $$\det(\lambda(1_2)\lambda(5)\lambda(6)) = \det(\lambda(1_1)\lambda(5)\lambda(6)) = 1.$$

- $$\begin{vmatrix} 1 & f+1 & f \\ 0 & 1 & 1 \\ 1 & n_5 & n_6 \end{vmatrix} = \begin{vmatrix} 0 & f+1 & f \\ 0 & 1 & 1 \\ 1 & n_5 & n_6 \end{vmatrix} = 1.$$

- $$(f+1+f) + n_5 + n_6 = 1 = f+1+f = 1$$

  by the positiveness of λ.

Since $2f = 0$, it follows from $n_5 + n_6 = 0$ that we can set $m = n_5 = n_6$. Therefore, we can see that λ for the case (7) in Theorem 3.1 is of the form

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & m & m \end{pmatrix}$$

or

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & m & m \end{pmatrix}.$$

Consequently, we can summarize the above calculations, as follow.

Theorem 3.2. *Up to rotational symmetry of P_6 and basis changes of $\mathbb{Z}_2^3$, any complete non-singular fans over $P_6(2,1,1,1,1,1)$ can be determined by one of the following ten characteristic matrices:*

(1)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & f \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

(2)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & f & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

(3)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & f \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

(4)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & a & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

(5)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(6)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & a & 1 \\ 1 & 1 & 0 & 0 & n_4 & 0 & n_6 \end{pmatrix}.$$

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(7)

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & m & m \end{pmatrix}.$$

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & m & m \end{pmatrix}.$$

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