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On the B-rigidity of certain 4-dimensional polytopes

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4차원 폴리토프의 B-견고성에 대한 연구

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포고렐로프 류(Pogorelov class) P란 단순 3-차원 폴리토프(polytope)로 플레그(flag)이면서 4-벨트를 갖고 있지 않은 것들의 모임이다. Buchstaber, Erokhovets, Masuda, Panov 그리고 Park는 포고렐로프 류에 대한 B-견고성을 증명하였다.

본 논문에서는 3차원 포고렐로프 류를 일반화하여 4-차원 일반화된 포고렐로프 류를 정의하였고 4-차원 일반화된 포고렐로프 경우에도 적절한 조건 하에서 B-견고성이 성립함을 증명하였다. 좀 더 구체적으로, 먼저 일반화된 플레그와 6-벨트를 갖는 성질이 B-견고성을 만족함을 보였으며 4차원 폴리토프가 4-벨트와 6-벨트를 갖지 않고,

$$\widetilde{H}^1(\mathrm{K}_J) = 0 \quad \text{단, } |J| = 7$$

을 만족하는 일반화된 포고렐로프 류도 B-견고성을 만족함을 모멘트-앵글 복합체(moment-angle complex)의 코호몰로지의 계산을 통해 보였다.

Chapter 1

Introduction

Torus actions on topological spaces are classical and have very rich theory. In particular, the algebro-geometric part of the theory is usually called toric geometry and studies the geometry of toric varieties which has been greatly developed by Danilov, Oda, Fulton, and Ewald (see [7], [11], [10], and [9]). One special feature of toric geometry is that the orbit space of the torus action carries a very rich combinatorial structure. In other words, in many cases studying the combinatorics of the orbit space provides the most efficient and useful way to understand the topology of a toric space itself. Moreover, this procedure can be completely reversed. It means that the equivariant topology of a torus action very often makes us to understand and prove many combinatorial results of the orbit space.

A toric variety of complex dimension n is a normal algebraic variety with an algebraic action of torus $(\mathbb{C}^*)^n$ with one dense orbit which extends to the variety itself. In [8], Davis and Januszkiewicz introduced a topological analogue, called the *quasi-toric manifold*, of a non-singular toric variety in algebraic geometry, and the geometry and topology of quasi-toric manifolds has become one of the most interesting topics in toric topology. An n-dimensional convex polytope is called *simple* if the number of codimension-one faces (or facets) meeting at each vertex is exactly equal to n. Roughly speaking, a quasi-toric manifold of dimension $2n$ is a smooth closed manifold with a locally standard $(S^1)^n$-action whose orbit space is a simple convex polytope of dimension n. In [8], there is also one important $(S^1)^m$-space $\mathcal{Z}_K$, called the *moment-angle complex*, for each simplicial complex on the vertex set $[m] := \{1, 2, \ldots, m\}$.

A family of closed manifolds is called *cohomologically rigid* if a cohomology ring isomorphism implies a diffeomorphism for any two manifolds in the family. Recently, in [4] Buchstaber, Erokhovets, Masuda, and Park show the cohomological rigidity for 6-dimensional manifolds defined by 3-dimensional polytopes. To be more precise, they consider the class of 3-dimensional combinatorial simple polytopes, different from the 3-simplex, whose facets do not form 3-belts as well as 4-belts. Those 3-dimensional class is called the *Pogorelov class*, and it turns out that they are simple polytope with only 5-gonal and 6-gonal facets.

The main aim of this thesis is to extend the results of Buchstaber, Erokhovets, Masuda, and Park to certain simple polytopes of dimension 4. To do so, we first introduce a notion of the generalized Pogorelov class consisting of certain 4-dimensional simple polytopes which is analogous to that of 3-dimensional simple polytopes. Let P be a convex polytope of dimension n. P is called a *flag polytope* if every collection of its pairwise intersecting facets has a nonempty intersection. A *k-belt* in P is a cyclic sequence

$$\mathcal{B}_k = (F_{i_1}, \cdots, F_{i_k})$$

of $k \geq 3$ facets in which pairs of consecutive facets (including $\{F_{i_k}, F_{i_1}\}$) are adjacent, other pairs of facets do not intersect, and no three facets have a common vertex.

The Pogorelov class $\mathcal{P}$ consists of simple 3-dimensional polytopes which are flag and do not have 4-belts of facets. It is not difficult to see that, in particular, any polytopes in $\mathcal{P}$ do not have triangular and quadrangular facets. On the other hand, the generalized Pogorelov class first introduced in this thesis is the collection of simple 4-polytopes P which are generalized flag, and do not contain generalized 6-belts, and $\widetilde{H}^1(\mathcal{K}_J) = 0$ with $|J|$. Here $\mathcal{K}$ or $\mathcal{K}_P$ (resp. $\mathcal{K}_J$) denotes the dual complex (resp. full subcomplex) of a simple convex polytope P. In particular, above two conditions translate to the absence of 3-belts and 4-belts. Moreover, it is easy to see that any convex polytope in the Pogorelov class belongs to the generalized Pogoelov class.

With these understood, our main result is

Theorem 1.1. *Let P be a simple 4-polytope such that*

(1) *it does not have any generalized 4-belts,*

(2) *it does not have any generalized* 6*-belts, and*

(3) $\widetilde{H}^1(\mathcal{K}_J) = 0$ *for* $|J| = 7$.

Assume that there is a ring isomorphism

$$H^*(\mathcal{Z}_{\mathcal{K}_P}) \;\cong\; H^*(\mathcal{Z}_{\mathcal{K}_{P'}})$$

for some other simple 4*-polytope* P' *Then* P' *also satisfies the above three conditions* (1)*,* (2)*, and* (3)*.*

We organize this thesis, as follows. In Chapter 2, we give precise definitions of the generalized flag polytope and generalized k-belt in a simple polytope. As mentioned above, a generalized analogue of Pogorelov class $\mathcal{P}$ is the collection of simple 4-polytope P which are generalized flag, and do not contain generalized 6-belt, and satisfies $\widetilde{H}^1(\mathcal{K}_J) = 0$ with $|J| = 7$.

In Chapter 3, we review some definitions and basic properties of Tor-algebra and Koszul algebra. There are isomorphisms

$$H^*(\mathcal{Z}_{\mathcal{K}}) \cong \mathrm{Tor}_{\mathbb{K}[v_1,\cdots,v_m]}(\mathbb{K}[\mathcal{K}], \mathbb{K})$$

as multi-graded commutative algebras, and $H^*(R^*(\mathcal{K}))$ is also isomorphic to $H^*(\mathcal{Z}_{\mathcal{K}})$ as cohomology algebras. We explain all of these facts precisely in Chapter 3.

Finally, in Chapter 4 we state and prove our main results regarding the B-rigidity of certain 4-dimensional polytopes.

Chapter 2

Generalized Pogorelov classes

The primary aim of this chapter is to introduce the notion of the generalized Pogorelov classes which plays an important role throughout all of this thesis.

First, we briefly explain the definition of convex polyhedron P first. A convex polyhedron P is an intersection of finitely many half-spaces in some $\mathbb{R}^n$:

$$P = \{x \in \mathbb{R}^n : \langle l_i, x\rangle \geq -a_i, \ i = 1, \cdots, m\},$$

where $l_i \in (\mathbb{R}^n)^*$ are some linear functions and $a_i \in \mathbb{R}$, $i = 1, \cdots, m$. A *(convex) polytope* is a bounded convex polyhedron. A set of $m > n$ hyperplanes $\langle l_i, x\rangle = -a_i$, $l_i \in (\mathbb{R}^n)^*$, $x \in \mathbb{R}^n$, $a_i \in \mathbb{R}$, $i = 1, \cdots, m$ is in *general position* if no point belongs to more than n hyperplanes. That is, there are exactly n facets meeting at each vertex of P^n. Such polytopes are called *simple*. Note that each face of a simple polytope is again a simple polytope.

Next, we give the definition of a flag polytope. Refer to [5] and [6] for more details.

Definition 2.1. A simple polytope P is called *flag* if every collection of its pairwise intersecting facets has a non-empty intersection.

For example, when the facets are all one-dimensional case, note that the boundaries $\partial\triangle^2$ and ∂I^2 of the 2-simplex $\triangle^2$ and the square I^2, respectively, are flag complexes. On the other hand, it is easy to see that $\triangle^2$ is not a flag complex, while I^2 is a flag complex. Note also that the n-simplex $\triangle^n$ $(n \geq 3)$ is not a flag complex.

Figure 2.1: Flag polytopes.

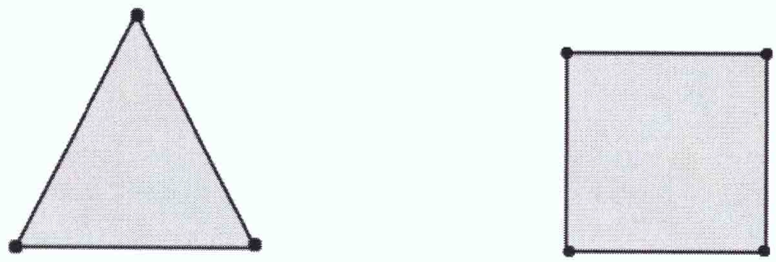

Figure 2.2: Examples of non-flag or flag polytopes.

For the purposes of this thesis, we need the notion of a generalized flag polytope, as follows.

Definition 2.2. A simple polytope P is called a *generalized flag polytope* if every collection of its triply intersecting facets has a non-empty intersection.

A 3-simplex $\triangle^3$ is not a generalized flag ploytope, so that every generalized flag polytope is not always flag. However, it follows from its definition that every flag polytope is always a generalized flag one. Note also that the square I^2 is a generalized flag polytope as well as a

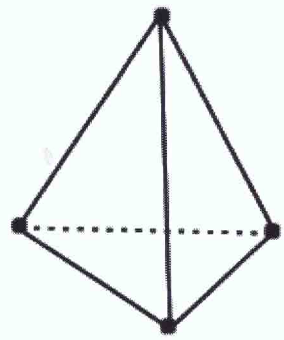

Figure 2.3: An example of a non-flag polytope.

flag one by the very definition. On the other hand, the 2-simplex $\triangle^2$ is a generalized flag, but not simply flag, polytope by Definitions 2.1 and 2.2.

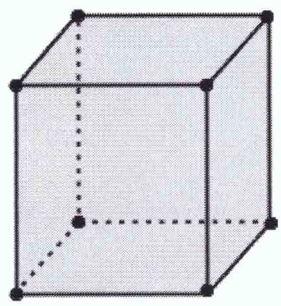

Figure 2.4: A generalized flag polytope.

It is not difficult to see that the 3-cube I^3 is a generalized flag polytope. More generally, the n-dimensional cube I^n $(n \geq 1)$ is a generalized flag polytope.

Recall that a k-belt in a simple polytope is a cyclic sequence

$$\mathcal{B}_K = (F_{i_1}, \cdots, F_{i_k}), \quad k \geq 3$$

facets in which pairs of consecutive facets including $\{F_{i_k}, F_{i_1}\}$ are adjacent, other pairs of facets do not intersect, and no three facets have a common vertex. We also need the notion of a generalized k-belt, as follows.

Definition 2.3. A *generalized k-belt* in a simple polytope is a cyclic sequence $(F_{i_1}, \cdots, F_{i_k})$ of $k \geq 4$ facets such that

(1) every triple of three consecutive facets is adjacent,

(2) other triples of facets do not intersect, and

(3) no four facets have a common vertex.

Next, we have the following lemma.

Lemma 2.4. *A simple 4-polytope $P \neq \Delta^4$ is a generalized flag polytope if and only if it does not contain a generalized 4-belt.*

Proof. (⇒) suppose that P contains a generalized 4-belt, that is, there exists an a cyclic sequence (F_1, F_2, F_3, F_4) of facets such that

(1) F_i, F_{i+1}, and F_{i+2} are adjacent for i mod 4,

(2) other triples of facets do not intersect, and

(3) $F_1 \cap F_2 \cap F_3 \cap F_4$ has no common vertex, that is, $F_1 \cap F_2 \cap F_3 \cap F_4 = \varnothing$.

Then $\{F_1, F_2, F_3, F_4\}$ is a collection of facets with triply intersecting facets such that $F_1 \cap F_2 \cap F_3 \cap F_4 = \varnothing$. This immediately implies that P is not a generalized flag polytope.

(⇐) Suppose that P is not a generalized flag polytope. Then there exists a collection of triply intersecting facets whose intersection is empty, that is, there exists an $\{F_1. F_2, \cdots, F_m\}$ of facets such that for any distinct $i, j, k \in [m] = \{1, 2, \cdots, m\}$

$$F_i \cap F_j \cap F_k = \varnothing \text{ and } \bigcap_{i=1}^{m} F_i = \varnothing.$$

Note that $m \geq 4$ and all facets of a simple convex polytope are transverse. Thus, $F_i \cap F_j \cap F_k$ ($\neq \varnothing$) consists of common edges.

Assume first that $m \geq 5$. If some collection $\{F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}\}$ has an empty intersection, then $\{F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}\}$ will be a generalized 4-belt we want to find. On the other hand, if any collection $\{F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}\}$ happens to have a non-empty intersection, then there should be another facet F_j different from all of $F_{i_1}, F_{i_2}, F_{i_3}$, and F_{i_4} such that the collection

$$\{F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}, F_j\}$$

has an empty intersection. This is because our simple convex polytope P is not equal to Δ^4. Hence, we may assume without loss of generality that $m = 4$. This implies that $\{F_1, F_2, F_3, F_4\}$ forms a generalized 4-belt for P. This completes the proof of Lemma 2.4. □

Recall that the Pogorelov class consists of simple 3-dimensional polytopes which are flag and do not have 4-belts of facets. Now, we generalize the notion of a generalized Pogorelov class, as follows.

Definition 2.5. A *generalized Pogorelov class* $\mathcal{P}$ is the collection of simple 4-polytopes P which are generalized flag, and do not contain generalized 6-belts, and $\widetilde{H}^1(\mathcal{K}_J) = 0$ with $|J| = 7$.

Note that this definition is equivalent to the condition that a simple 4-polytope $P \neq \Delta^4$ does not contain any generalized 4-belts and 6-belts.

As in the usual Pogorelov class, we have the following lemma.

Lemma 2.6. *For a simple 4-polytope $P \in \mathcal{P}$, there are no $\triangle^3$ or I^3 facets in P, where $\triangle^3$ denotes the 3-simplex and I^3 denotes the cube of dimension 3.*

Proof. Suppose that P has $\triangle^3$ as a facet. Then, there is a collection $\{F_1, F_2, F_3, F_4\}$ of facets around $\triangle^3$ with triply intersecting facets such that $\bigcap_{i=1}^4 F_i = \varnothing$ (see 2.5). Hence, P is not a generalized flag. This is a contradiction.

On the other hand, suppose that P has I^3 as a facet. Then, there is a generalized 6-belt, that is, there is a cyclic sequence $(F_1, F_2, \cdots, F_6)$ satisfying three conditions in the definition of a generalized 6-belt (see 2.6). But clearly this is a contradiction, completing the proof of Lemma 2.6. □

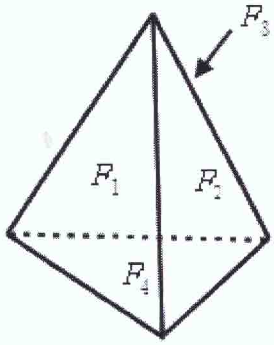

Figure 2.5: $\triangle^3$

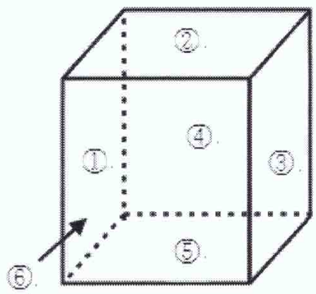

Figure 2.6: I^3

The following lemma also plays a role in this thesis.

Lemma 2.7. *For any three distinct facets F_i, F_j, and F_k of a simple 4-polytope $P \in \mathcal{P}$, there exists an $x \notin F_i \cup F_j \cup F_k$.*

Proof. Take any facet F_l such that $F_l \neq F_i, F_j, F_k$. Then, F_l has at most one 2-dimensional common face with F_i, F_j, and F_k, respectively. But, it follows from Lemma 2.7 that there exist at least 7 2-dimensional faces in F_l. Thus, there exists at least one 2-dimensional face of F_l which does not lie in $F_i \cup F_j \cup F_k$. This clearly implies that there exists an $x \in F_l$ such that $x \notin F_i \cup F_j \cup F_k$. This completes the proof. $\square$

As a consequence, we have the following lemma.

Lemma 2.8. *Let P be a simple 4-polytope. Then, P is a generalized flag if and only if each facet F of P is surrounded by a generalized k-belt.*

Proof. ($\Rightarrow$) Suppose that P is a generalized flag polytope. Let F be any facet of P, and let $\mathcal{B} = (F_{j_1}, \cdots, F_{j_k})$ be a cyclic sequence of facets of P adjacent to a facet F such that three consecutive facets intersect. Clearly, such a sequence $\mathcal{B}$ exists, since P is simple and so exactly four facets meet at each vertex. Note that $k \geq 4$ and that if $k = 4$, then $P \simeq \triangle^4$, since $F_{i_1} \cap \cdots \cap F_{i_4} \neq \varnothing$ and is a vertex of P. But this is a contradiction. Recall that $\triangle^4$ is not a generalized flag polytope. Now, if $k \geq 5$, then it follows from the proof of Lemma 2.6 that $k \geq 7$. Furthermore, by the way of the choice any four of $\mathcal{B}$ does not intersect. Thus, $\mathcal{B}$ is a generalized k-belt.

($\Leftarrow$) We first the case that $P \simeq \triangle^4$. Then P is not a generalized flag polytope and no face of $\triangle^4$ is surrounded by a k-belt ($k = 4$, in fact), since all four surrounding facets should meet as a vertex and so they cannot be a 4-belt. Thus, in this case ($\Leftarrow$) holds to be true.

Next, assume that $P \not\simeq \triangle^4$ and that P is not a generalized flag polytope. Then there should be a generalized 4-belt

$$(F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4}).$$

Let $F := F_{i_1}$. Then, we see that F_{i_2}, F_{i_3} and F_{i_4} are all adjacent to F by the definition of 4-belt. But, since $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} = \varnothing$, F_{i_2}, F_{i_3}, and F_{i_4} are not consecutive, this implies that we cannot surround F by using a k-belt. This is because in order to surround F by consecutive facets, we would have

$$F \cap F_{j_1} \cap F_{j_2} \cap F_{j_3} \cap F_{j_4} \neq \varnothing.$$

for some facets F_{j_1}, F_{j_2}, F_{j_3}, and F_{j_4} of P. So P would not be simple. But this is a contradiction. $\square$

Finally, we need the following lemma for the proof of our main Theorem 1.1.

Lemma 2.9. *For any facet F_i of a generalized flag 4-polytope P, there is another facet F_j such that $F_i \cap F_j = \varnothing$.*

Proof. By the above Lemma 2.8, each facet F_i is surrounded by a k-belt $\mathcal{B}_k$. Note that $\partial P - \mathcal{B}_k$ consists of three dimensional components. So we can always choose any F_j in the interior of the components of $\partial P - \mathcal{B}_k$ is possible, which will prove the lemma. □

Chapter 3

Cohomology of moment-angle complexes

In this chapter, we quickly review the cohomology of moment-angle complexes. Here we consider the cohomology with coefficients in $\mathbb{K}$. Let $\Lambda[u_1, \cdots, u_m]$ denote the exterior algebra on m generators over $\mathbb{K}$ satisfying the relations

$$u_i^2 = 0,\ u_i u_j = -u_j u_i.$$

The Koszul complex, or the Koszul algebra, of the face ring $\mathbb{K}[\mathcal{K}]$ is defined to be the differential $\mathbb{Z} \oplus \mathbb{Z}^n$-graded algebra

$$(\Lambda[u_1, \cdots, u_m] \oplus \mathbb{K}[\mathcal{K}],\ d),$$

where the multi-degrees of u_i and v_i are

$$(-1, 2e_i) = -e_0 + 2e_i, \text{ and } (0, 2e_i) = 0 \cdot e_0 + 2e_i,$$

respectively, and

$$du_i = v_i,\ dv_i = 0.$$

Then, the cohomology of $(\Lambda[u_1, \cdots, u_m] \oplus \mathbb{K}[\mathcal{K}],\ d)$ is called the Tor-algebra, denoted $\mathrm{Tor}_{\mathbb{K}[v_1, \cdots, v_m]}(\mathbb{K}[\mathcal{K}],\ \mathbb{K})$. Notice that there is also a $\mathbb{Z} \oplus \mathbb{Z}^m$-grading on

$$\mathrm{Tor}_{\mathbb{K}[v_1, \cdots, v_m]}(\mathbb{K}[\mathcal{K}],\ \mathbb{K})$$

inherited from $\Lambda[u_1, \cdots, u_m] \oplus \mathbb{K}[\mathcal{K}]$. It is known that

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \mathrm{Tor}_{\mathbb{K}[v_1, \cdots, v_m]}(\mathbb{K}[\mathcal{K}],\ \mathbb{K})$$

as multi-graded commutative algebras, and

$$H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}) \cong \mathrm{Tor}^{-i,2J}_{\mathbb{K}[v_1,\cdots,v_m]}(\mathbb{K}[\mathcal{K}],\ \mathbb{K}),$$

where $J = (j_1,\cdots, j_m) \in \mathbb{Z}^m$ and $|J| = j_1 + \cdots + j_m$. Furthermore, we have

$$H^l(\mathcal{Z}_{\mathcal{K}}) \ \cong \ \bigoplus_{-i+2|J|=l} H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}).$$

Let

$$R^*(\mathcal{K}) = \Lambda[u_1,\cdots,u_m] \ \otimes \ \mathbb{K}[\mathcal{K}]/\langle v_i^2 = u_i v_i = 0,\ 1 \le j \le m\rangle,$$

where $\mathcal{K}$ is a simplicial complex as the vertex set $[m] = \{1, 2, \cdots, m\}$. Then, $R^*(\mathcal{K})$ is a finite-dimensional vector space over $\mathbb{K}$, even though $\Lambda[u_1,\cdots,u_m] \ \otimes \ \mathbb{K}[\mathcal{K}]$ is infinite-dimensional over $\mathbb{K}$. Note that there is an isomorphism of cohomology algebras

$$H^*(R^*(\mathcal{K})) \ \cong \ H^*(\mathcal{Z}_{\mathcal{K}}).$$

For $J = \{J_1,\cdots,j_k\} \ \subseteq \ [m]$, we denote by v_J the square-free monomial

$$v_{j1},\cdots,v_{jk} \ \in \ \mathbb{K}[v_1,\cdots,v_m].$$

Similarly, we let $u_J = u_{j1},\cdots,u_{jk} \ \in \ \Lambda[u_1,\cdots,u_m]$. We use

$$u_J v_I \ := \ u_J \ \otimes \ v_I$$

in $\Lambda[u_1,\cdots,u_m] \otimes \mathbb{K}[\mathcal{K}]$. Then, it follows from the definition of $R^*(\mathcal{K})$ that $R^*(\mathcal{K})$ has a finite $\mathbb{K}$-basis of square-free monomials $u_J v_I$, where $J \ \subseteq \ [m]$, $I \in \mathcal{K}$, and $J \cap I = \varnothing$. For $J \subset [m]$, $\mathcal{K}_J = \{I \in \mathcal{K} | I \subset J\}$ is called the *full subcomplex* of $\mathcal{K}$. It is known that

$$\widetilde{H}^{|J|-i-1}(\mathcal{K}_J) \ \cong \ \mathrm{Tor}^{-i,2J}_{\mathbb{K}[v_1,\cdots,v_m]}(\mathbb{K}[\mathcal{K}],\ \mathbb{K})$$

and that

$$\widetilde{H}^{-i,2J}(\mathcal{Z}_{\mathcal{K}}) \ \cong \ \widetilde{H}^{|J|-i-1}(\mathcal{K}_J)$$

Thus we can obtain

$$H^l(\mathcal{Z}_{\mathcal{K}}) \ \cong \ \bigoplus_{J\subseteq[m]} \widetilde{H}^{l-|J|-1}(\mathcal{K}_J),$$

as k-modules. Here we have $l = -i+2|J|$. In fact, a simple computation shows

$$\begin{aligned} H^l(\mathcal{Z}_\mathcal{K}) &= \bigoplus_{-i+2|J|=l} \widetilde{H}^{-i,2J}(\mathcal{Z}_\mathcal{K}) \\ &= \bigoplus_{-i+2|J|=l} \widetilde{H}^{|J|-i-1}(\mathcal{K}_J) \\ &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{|J|+(l-2|J|)-1}(\mathcal{K}_J) \\ &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{l-|J|-1}(\mathcal{K}_J). \end{aligned}$$

As a consequence, we have

$$H^*(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathcal{K}_J),$$

as a ring isomorphism, and the ring structure is given by

$$\begin{aligned} &\widetilde{H}^{k-|I|-1}(\mathcal{K}_I) \otimes H^{l-|J|-1}(\mathcal{K}_J) \\ &\longrightarrow \widetilde{H}^{k+l-|I|-|J|-1}(\mathcal{K}_{I\cup J}) \text{ for } I\cap J = \varnothing, \end{aligned}$$

which is induced from the simplicial inclusion

$$\mathcal{K}_{I\cup J} \longleftrightarrow \mathcal{K}_I * \mathcal{K}_J.$$

Now, we recall the following general fact which appears in [5, Proposition 2.20].

Proposition 3.1. *$H^3(\mathcal{Z}_\mathcal{K})$ is freely generated by the cohomology classes $[u_iv_j]$ for pairs (i,j), $i\neq j$ such that $\{i,j\}\notin\mathcal{K}$. Moreover, if $\mathcal{K}=\mathcal{K}_P$ for a simple polytope P, then $[u_iv_j]$ corresponds to pairs of non-adjacent facets F_i and F_j.*

For an illustration of Proposition 3.1, we give an example, as follows.

Example 3.2. Let $\mathcal{K}$ be the union of two segments. Then nontrivial integral cohomology groups of $\mathcal{Z}_\mathcal{K}$ are given with a basis represented by algebra $R^*(\mathcal{K})$. Then we can find that

$$H^3(\mathcal{Z}_\mathcal{K}) \cong \bigoplus_{|J|=2} \widetilde{H}^0(\mathcal{K}_J) \cong \mathbb{Z}^4.$$

Next we have the following computational result.

Proposition 3.3. *Let P be a simple 4-polytope with m facets, and let $\mathcal{K} = \mathcal{K}_P$ be the dual simplicial complex of P. Then, we have*

$$H^l(\mathcal{Z}_P) = \begin{cases} \widetilde{H}^{-1}(\mathcal{K}_\varnothing) \cong \mathbb{Z}, & l = 0, \\ \bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J), & 1 \leq l \leq 5, \\ \bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J) \\ \oplus \bigoplus_{|J|=l-3} \widetilde{H}^2(\mathcal{K}_J), & 6 \leq l \leq 7, \\ \bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J) \\ \oplus \bigoplus_{|J|=l-3} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-4} \widetilde{H}^3(\mathcal{K}_J), & 8 \leq l \leq m+3 \\ 0, & \text{otherwise}, \end{cases}$$

where $Z_P = Z_K = Z_{K_P}$.

Proof. By definition, we can show the following facts.

(1)

$$\begin{aligned} H^0(\mathcal{Z}_P) &= \bigoplus_{J \subseteq [m]} \widetilde{H}^{0-|J|-1}(\mathcal{K}_J) \\ &= \bigoplus_{J \subseteq [m]} \widetilde{H}^{-|J|-1}(\mathcal{K}_J) \\ &= \widetilde{H}^{-1}(\mathcal{K}_\varnothing) \cong \mathbb{Z}. \end{aligned}$$

(2)

$$\begin{aligned} H^1(\mathcal{Z}_P) &= \bigoplus_{J \subseteq [m]} \widetilde{H}^{1-|J|-1}(\mathcal{K}_J) \\ &= \bigoplus_{J \subseteq [m]} \widetilde{H}^{-|J|}(\mathcal{K}_J) \\ &= \widetilde{H}^0(\mathcal{K}_\varnothing) \oplus \bigoplus_{|J|=1-2} \widetilde{H}^1(\mathcal{K}_J) \ (= 0). \end{aligned}$$

$$
\begin{aligned}
H^2(\mathcal{Z}_P) &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{2-|J|-1}(\mathcal{K}_J) \\
&= \bigoplus_{J\subseteq[m]} \widetilde{H}^{1-|J|}(\mathcal{K}_J) \\
&= \bigoplus_{|J|=0} \widetilde{H}^1 \oplus \bigoplus_{|J|=1} \widetilde{H}^0(\mathcal{K}_J).
\end{aligned}
$$

$$
H^l(\mathcal{Z}_P) = \begin{cases}
\widetilde{H}^{-1}(\mathcal{K}_\varnothing) \cong \mathbb{Z}, & l=0, \\
\bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J), & 1 \le l \le 5, \\
\bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J) & \\
\oplus \bigoplus_{|J|=l-3} \widetilde{H}^2(\mathcal{K}_J), & 6 \le l \le 7, \\
\bigoplus_{|J|=l-1} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-2} \widetilde{H}^1(\mathcal{K}_J) & \\
\oplus \bigoplus_{|J|=l-3} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=l-4} \widetilde{H}^3(\mathcal{K}_J), & 8 \le l \le m+1, \\
\bigoplus_{|J|=m} \widetilde{H}^1(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-1} \widetilde{H}^2(\mathcal{K}_J) & \\
\oplus \bigoplus_{|J|=m-2} \widetilde{H}^3(\mathcal{K}_J), & l=m+2, \\
\bigoplus_{|J|=m} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-1} \widetilde{H}^3(\mathcal{K}_J), & l=m+3, \\
\mathbb{Z}, & l=m+4, \\
0, & \text{otherwise.}
\end{cases}
$$

$$
= \bigoplus_{|J|=2-1=1} \widetilde{H}^{1-1=0}(\mathcal{K}_J) \oplus \bigoplus_{|J|=0=2-2} \widetilde{H}^1(\mathcal{K}_J) \oplus \cdots .
$$

$$
\begin{aligned}
H^6(\mathcal{Z}_P) &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{6-|J|-1}(\mathcal{K}_J) = \bigoplus_{J\subseteq[m]} \widetilde{H}^{5-|J|}(\mathcal{K}_J) \\
&= \bigoplus_{|J|=3} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=4} \widetilde{H}^1(\mathcal{K}_J) \oplus \bigoplus_{|J|=5} \widetilde{H}^0(\mathcal{K}_J).
\end{aligned}
$$

$$
\begin{aligned}
H^7(\mathcal{Z}_P) &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{7-|J|-1}(\mathcal{K}_J) = \bigoplus_{J\subseteq[m]} \widetilde{H}^{6-|J|-1}(\mathcal{K}_J) \\
&= \bigoplus_{|J|=3} \widetilde{H}^3(\mathcal{K}/J) \oplus \bigoplus_{|J|=4} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=5} \widetilde{H}^1(\mathcal{K}_J) \\
&\oplus \bigoplus_{|J|=6} \widetilde{H}^0(\mathcal{K}_J). \\
H^8(\mathcal{Z}_P) &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{7-|J|}(\mathcal{K}_J) \\
&= \bigoplus_{|J|=4} \widetilde{H}^3(\mathcal{K}_J) \oplus \bigoplus_{|J|=5} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=6} \widetilde{H}^1(\mathcal{K}_J) \\
&\oplus \bigoplus_{|J|=7} \widetilde{H}^0(\mathcal{K}_J). \\
H^9(\mathcal{Z}_P) &= \bigoplus_{J\subseteq[m]} \widetilde{H}^{8-|J|}(\mathcal{K}_J) \\
&= \oplus \bigoplus_{|J|=5} \widetilde{H}^3(\mathcal{K}_J) \oplus \bigoplus_{|J|=6} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=7} \widetilde{H}^1(\mathcal{K}_J) \\
&\oplus \bigoplus_{|J|=8} \widetilde{H}^0(\mathcal{K}_J).
\end{aligned}
$$

For $q \leq l \leq m+3$, we can show the theorem in a similar way.

Finally, for the case of $l = m+4$ we have

$$H^{m+4}(\mathcal{Z}_P) \cong \mathbb{Z}.$$

This is because $\mathcal{Z}_P$ is smooth manifold for P polytope. Here, note that

$$H^{m+4}(\mathcal{Z}_P) = \bigoplus_{J\subseteq[m]} \widetilde{H}^{m-|J|+3}(\mathcal{K}_J).$$

This completes the proof of Proposition 3.3. □

Remark 3.4. As a consequence of the above computations, we can see that all non-trivial products in $H^*(\mathcal{Z}_P)$ are of the form

(1) $\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^0(\mathcal{K}_J) \longrightarrow \widetilde{H}^1(\mathcal{K}_{I\cup J})$ $I \cap J \neq \varnothing$,

(2) $\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_J) \longrightarrow \widetilde{H}^2(\mathcal{K}_{I\cup J})$, or
$\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^2(\mathcal{K}_{[m]-I}) \longrightarrow \widetilde{H}^3(\mathcal{K}_P) \cong \mathbb{Z}$, or
$\widetilde{H}^1(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_{[m]-I}) \longrightarrow \widetilde{H}^3(\mathcal{K}_P) \cong \mathbb{Z}$.

Note that $\widetilde{H}^3(\mathcal{K}_J) = 0$ if $\mathcal{K}_J$ is 3-dimensional and has a boundary.

Chapter 4

Main results: Proof of Theorem 1.1

The aim of this chapter is to give a proof of our main Theorem 1.1.

To do so, let P be a generalized flag 4-polytope, and let $\mathcal{K} = \mathcal{K}_P$ be the dual simplicial complex of dimension 3. Assume that the above two maps

(1) $\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^0(\mathcal{K}_J) \longrightarrow \widetilde{H}^1(\mathcal{K}_{I\cup J})$ $I \cap J \neq \varnothing$, and

(2) $\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_J) \longrightarrow \widetilde{H}^2(\mathcal{K}_{I\cup J})$, or
$\widetilde{H}^0(\mathcal{K}_I) \otimes \widetilde{H}^2(\mathcal{K}_{[m]-I}) \longrightarrow \widetilde{H}^3(\mathcal{K}_P) \cong \mathbb{Z}$, or
$\widetilde{H}^1(\mathcal{K}_I) \otimes \widetilde{H}^1(\mathcal{K}_{[m]-I}) \longrightarrow \widetilde{H}^3(\mathcal{K}_P) \cong \mathbb{Z}$.

are surjective. Then, we have the following lemma that is analogous to Lemma 4.5 of the paper [4].

Lemma 4.1. *Let P and $\mathcal{K}_P$ be as above. Then the ring*

$$H^*(\mathcal{Z}_P) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathcal{K}_J)$$

is multiplicatively generated by $\bigoplus_{J\subseteq[m]} \widetilde{H}^0(\mathcal{K}_J)$.

We also need the following lemma.

Lemma 4.2. *A simple 4-polytope $P \neq \triangle^4$ with m facets is generalized-flag if and only if any non-trivial cohomology class in $H^{m-3}(\mathcal{Z}_P)$ is decomposable. As a consequence, if $H^{m-3}(\mathcal{Z}_P) = 0$, then either P is generalized-flag or $P \simeq \triangle^4$.*

Proof. ($\Rightarrow$) Suppose that P is not a generalized flag polytope. Then, there is a generalized 4-belt $\{F_{j1}, F_{j2}, F_{j3}, F_{j4}\}$. Equivalently, there is a minimal missing 3-face $J = \{j_1, j_2, j_3, j_4\}$. This gives a non-zero cohomology class

$$\alpha \in H^{-1,2J}(\mathcal{Z}_P) \subset H^{-1+2|J|}(\mathcal{Z}_P) = H^7(\mathcal{Z}_P)$$

represented by, say, $u_{j1} \cdot v_{j1}v_{j2}v_{j3}v_{j4}$. Note that

$$d(u_{j1} \cdot v_{j1} \cdots v_{j4}) = v_{j1}^2 v_{j2} \cdots v_{j4} = 0.$$

Now, consider the Poincaŕe duality pairing

$$H^{m-3}(\mathcal{Z}_P) \otimes H^7(\mathcal{Z}_P) \longrightarrow H^{m+4}(\mathcal{Z}_P) \cong \mathbb{Z}$$

which restricts to

$$H^{-(m-5),2([m]-J)} \otimes H^{-1,2J}(\mathcal{Z}_P) \longrightarrow H^{-(m-4),2[m]}(\mathcal{Z}_P) \cong \mathbb{Z}.$$

Let $\beta \in H^{-(m-5),2([m]-J)}(\mathcal{Z}_P) \subset H^{m-3}(\mathcal{Z}_P)$ such that $\alpha \cdot \beta$ is a generator of $H^{-(m-4),2[m]}(\mathcal{Z}_P) \cong \mathbb{Z}$. Note that

$$H^{-(m-5),2([m]-J)}(\mathcal{Z}_P) \cong \widetilde{H}^0(\mathcal{K}_{[m]-J})$$

and that $\widetilde{H}^0(\mathcal{K}_{[m]-J})$ has no indecomposable element. Hence, we have found an in-decomposable element $\beta \in H^{m-3}(\mathcal{Z}_P)$.

Next, suppose that P is a generalized flag polytope. It follows from Proposition 3.3 that

$$H^{m-3}(\mathcal{Z}_P) \cong \bigoplus_{|J|=m-4} \widetilde{H}^0(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-5} \widetilde{H}^1(\mathcal{K}_J)$$

$$\oplus \bigoplus_{|J|=m-6} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-7} \widetilde{H}^3(\mathcal{K}_J).$$

By considering the Poincaŕe duality pairing

$$\widetilde{H}^0(\mathcal{K}_J) \otimes \widetilde{H}^2(\mathcal{K}_{[m]-J}) \longrightarrow \widetilde{H}^3(\mathcal{K}_{[m]}) \cong \mathbb{Z}$$

and $\mathcal{K}$ being generalized-flag, note that $\widetilde{H}^2(\mathcal{K}_{[m]-J}) = 0$ for $|J| = m-4$, since there is no missing faces with 4 vertices. Thus, we can obtain

$$\bigoplus_{|J|=m-4} \widetilde{H}^0(\mathcal{K}_J) = 0,$$

and so we have

$$H^{m-3}(\mathcal{Z}_P) = \bigoplus_{|J|=m-5} \widetilde{H}^1(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-6} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=m-7} \widetilde{H}^3(\mathcal{K}_J).$$

This implies that each non-zero element of $H^{m-3}(\mathcal{Z}_P)$ is decomposable. □

Theorem 4.3. *let P be a generalized flag 4-polytope. Assume that there is a ring isomorphism*

$$H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$$

for some other simple 4-polytope P'. Then, P' is also generalized-flag. Thus, the property of being a generalized flag 4-polytope is B-rigid.

Proof. Suppose that $P' \neq \triangle^4$ is not generalized flag. Then, there is a non-trivial indecomposable element in

$$H^{m-3}(\mathcal{Z}_P) \cong H^{m-3}(\mathcal{Z}_{P'}).$$

Thus, there is also an indecomposable element in $H^{m-3}(\mathcal{Z}_P)$. This implies that P cannot be a generalized flag polytope. This is a contradiction. Now, it remains to show that $P' \neq \triangle^4$. If $P' \simeq \triangle^4$, then P' has 5 facets. Since $H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$, $J - H$. P has also 5 facet. Hence $m = 5$, that is, $P \simeq \triangle^4$ up to the combinatorial equivalence. Note that $\triangle^4$ is not a generalized flag polytope. But it is again a contradiction. □

In order to deal with 6-belts, we need the following lemma.

Lemma 4.4. *Let P be a simple 4-polytope such that $\widetilde{H}^1(\mathcal{K}_J) = 0$ for $|J| = 7$. The product $H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P)$ is trivial if and only if P does not have any 6-belt.*

Proof. ($\Rightarrow$) Suppose that P has a 6-belt, say $\{F_{i1}, \cdots, F_{i6}\}$. Then, by the definition of a 6-belt there is a corresponding 2-cycle $\{i_1, i_2, \cdots, i_6\}$ in $\mathcal{K}_P$ that forms an octahedron. That is, we have the following Figure 4.1

Hence, $\{1, 2, 5\} \notin \mathcal{K} = \mathcal{K}_P$ and $\{3, 4, 6\} \notin \mathcal{K}$. Thus, we have a non-trivial product

$$\widetilde{H}^0(\mathcal{K}_{\{1,2,5\}}) \otimes \widetilde{H}^1(\mathcal{K}_{\{3,4,6\}}) \longrightarrow \widetilde{H}^3(\mathcal{K}_{\{1,2,\cdots,6\}}).$$

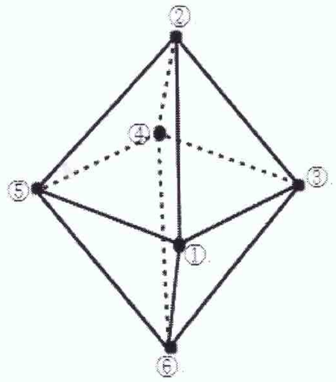

Figure 4.1

($\Leftarrow$) Suppose that there is a non-trivial product

$$H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P).$$

Note that

$$H^9(\mathcal{Z}_P) = \bigoplus_{|J|=5\subseteq[m]} \widetilde{H}^3(\mathcal{K}_J) \oplus \bigoplus_{|J|=6} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=7} \widetilde{H}^1(\mathcal{K}_J) \oplus \bigoplus_{|J|=8} \widetilde{H}^0(\mathcal{K}_J)$$

and that elements of $\widetilde{H}^0(\mathcal{K}_J)$ are indecomposable. By assumption, $\widetilde{H}^1(\mathcal{K}_J) = 0$ for $|J| = 7$. Thus, we have

$$H^9(\mathcal{Z}_P) = \bigoplus_{|J|=5} \widetilde{H}^3(\mathcal{K}_J) \oplus \bigoplus_{|J|=6} \widetilde{H}^2(\mathcal{K}_J) \oplus \bigoplus_{|J|=8} \widetilde{H}^0(\mathcal{K}_J).$$

Note that an element of $\widetilde{H}^3(\mathcal{K}_J)$ with $|J| = 5$ can be decomposed into a product if and only if there should be a product

$$\widetilde{H}^0(\mathcal{K}_{J_1}) \otimes \widetilde{H}^2(\mathcal{K}_{J_2}) \longrightarrow \widetilde{H}^3(\mathcal{K}_J)$$

with $|J_1| = 2$ (resp. 3) and $|J_2| = 3$ (resp. 2) such that $J = J_1 \bigcup J_2$ and $J_1 \bigcap J_2 = \varnothing$. We next need to consider the following two cases separately.

1) $|J_1| = 2$ and $|J_2| = 3$ case;

$$\widetilde{H}^0(\mathcal{K}_{J_1}) \subseteq H^3(\mathcal{Z}_P) \text{ and } \widetilde{H}^2(\mathcal{K}_{J_2}) \subseteq H^6(\mathcal{Z}_P)$$

This does not give a product of $H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P)$.

2) $|J_1| = 3$ and $|J_2| = 2$ case;

$$\widetilde{H}^0(\mathcal{K}_{J_1}) \subseteq H^4(\mathcal{Z}_P) \text{ and } \widetilde{H}^2(\mathcal{K}_{J_2}) \subseteq H^5(\mathcal{Z}_P).$$

But we have $\widetilde{H}^2(\mathcal{K}_{J_2}) = 0$, since K_{J_2} with $|J_2| = 2$ is a 1-dimensional simplicial complex. Thus, this case does not yield a non-trivial product of

$$H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P),$$

either.

As a consequence, this case does not yield a non-trivial product of

$$H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P).$$

Hence, it follows from $\widetilde{H}^2(\mathcal{K}_J)$ with $|J| = 6$ that we should have a 6-belt.

This completes the proof of Lemma 4.4. □

Remark 4.5. We remark that an argument similar to the proof of Lemma 4.4 does not rule out the case of $\widetilde{H}^1(\mathcal{K}_J)$ for $|J| = 7$. This is because it is possible for

$$\widetilde{H}^0(\mathcal{K}_{J_1}) \otimes \widetilde{H}^0(\mathcal{K}_{J_2}) \longrightarrow \widetilde{H}^1(\mathcal{K}_J)$$

with $J = J_1 \bigcup J_2$ and $J_1 \bigcap J_2 = \varnothing$ such that $|J_1| = 3$ and $|J_2| = 4$ to give a non-trivial product of

$$H^4(\mathcal{Z}_P) \otimes H^5(\mathcal{Z}_P) \longrightarrow H^9(\mathcal{Z}_P).$$

Theorem 4.6. *Let P be a simple 4-polytope without generalized 6-belts such that $\widetilde{H}^1(\mathcal{K}_J) = 0$ with $|J| = 7$. Assume there is a ring isomorphism*

$$H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$$

for some other simple 4-polytope P'. Then, P' also does not have any 6-belts.

Proof. The proof follows immediately form the above Lemma 4.4. □

Finally, we can conclude this thesis with the following theorem (Theorem 1.1).

Theorem 4.7. *Let P be a simple 4-polytope such that*

(1) *it does not have any generalized 4-belts,*

(2) *it does not have any generalized 6-belts, and*

(3) $\widetilde{H}^1(\mathcal{K}_J) = 0$ *for* $|J| = 7$.

Assume that there is a ring isomorphism

$$H^*(\mathcal{Z}_P) \cong H^*(\mathcal{Z}_{P'})$$

for some other simple 4-polytope P'. Then P' also satisfies the above three conditions (1), (2), *and* (3).

Proof. The proof follows immediately from Theorems 4.3 and 4.7. □

Finally, it remains to prove Lemma 4.1 given at the beginning of this chapter. To do so, we begin with recalling a definition. That is, an element in a graded ring is called *decomposable* if it can be written as a sum of non-trivial products of elements of non-zero degree.

Now, we want to give a proof of the following claim.

Lemma 4.8. *Let P be a 4-polytope, and let be its dual simplicial complex. Then, the ring $H^*(\mathcal{Z}_P) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}^*(\mathcal{K}_J)$ is multiplicatively generated by $\bigoplus_{J\subseteq[m]} \widetilde{H}^0(\mathcal{K}_J)$.*

To show this Lemma 4.8, we want to show that each non-trivial cohomology class in $\widetilde{H}^1(\mathcal{K}_I) \subseteq H^*(Z_P)$ is decomposable, that is, the product map

$$\bigoplus_{I=I_1 \dot\bigcup I_2} \widetilde{H}^0(\mathcal{K}_{I_1}) \otimes \widetilde{H}^0(\mathcal{K}_{I_2}) \longrightarrow \widetilde{H}^1(\mathcal{K}_I)$$

with $I_1 \cap I_2 \neq \varnothing$ is surjective.

To prove it, we first set up some notations, as follows. Assume that P has the facets $F_1, \cdots, F_m$, and we identify the set $\{F_1, \cdots, F_m\}$ of facets with $[m]$. For $I \subseteq [m]$, we set

$$P_I := \bigcup_{i\in I} F_i \subseteq \partial P$$

Recall that there are Poincaŕe duality isomorphisms

$$H_{3-i}(P_I, \partial P_I) \;\cong\; H^i(\mathcal{K}_I),$$

where

$$\partial P_I = \{x \in P_I \mid \text{there exists a } j \notin I \;;\; x \in F_j,\; i = 0,1,2,3\}.$$

As in the case of 3-polytope, P_I is just a disjoint union of several 3-handlebodies with some smaller 3-handlebodies deleted, and ∂P_I is a disjoint union of several 2-spheres. In terms of the cellular homology theory, $H_j(P_I, \partial P_I)$ has the following interpretation. That is, let

$$P_I \;:=\; P_{I'} \dot{\bigcup} \cdots \dot{\bigcup} P_{I^k}$$

that is the decomposition of P_I into its connected components. Then, the following statements are true:

(1) $H_3(P_I, \partial P_I)$ is a free abelian group with a basis of homology classes

$$[P_{I^j}] \;=\; \sum_{s \,\in\, I_J} [F_s], \;\; 1 \le j \le k.$$

(2) $H_2(P_I, \partial P_I) \;=\; \bigoplus_{j=1}^k H_2(P_{I^j}, \partial P_{I^j})$ and $H_2(P_{I^j}, \partial P_{I^j})$ is a free abelian group with a basis consisting of 2-dimensional surfaces whose boundaries lie on ∂P_{I^j}.

(3) $H_1(P_I, \partial P_I) \;=\; \bigoplus_{j=1}^k H_1(P_{I^j}, \partial P_{I^j})$ and each $H_1(P_{I^j}, \partial P_{I^j})$ is a free abelian group with rank one less than the number of boundary components of ∂P_{I^j}.

(4) $H_0(P_I, \partial P_I) \;=\; \mathbb{Z}$, if $I = [m]$, and 0, otherwise.

Let

$$\widetilde{H}_i(P_I, \partial P_I) \;:=\; \begin{cases} H_i(P_I, \partial P_I), & i = 0,1,2, \\ H_3(P_I, \partial P_I)/\langle\, \sum_{i \in I}[F_i] \,\rangle, & i = 3. \end{cases}$$

Then, we have

$$\widetilde{H}_{3-j}(P_I, \partial P_I) \cong \widetilde{H}^i(\mathcal{K}_I),\ i = 0, 1, 2, 3,$$

and so

$$\bigoplus_{I=I_1} \dot{\bigcup} I_2 \widetilde{H}^0(\mathcal{K}_{I_1}) \otimes \widetilde{H}^0(\mathcal{K}_{I_2}) \longrightarrow \widetilde{H}^1(\mathcal{K}_I)$$

can be re-written as

$$(4.1) \qquad \bigoplus_{I=I_1 \dot{\bigcup} I_2} \widetilde{H}_3(P_{I_1}, \partial P_{I_1}) \otimes \widetilde{H}_3(P_{I_2}, \partial P_{I_2}) \longrightarrow \widetilde{H}_2(P_I, \partial P_I),$$

where

$$[P_{I_1^p}] \otimes [P_{I_2^q}] \longmapsto [P_{I_1^p} \cap P_{I_2^q}].$$

Here,

$$I_1 = I_1^1 \dot{\bigcup} \cdots \dot{\bigcup} I_1^{s_1} \text{ and } I_2 = I_2^1 \dot{\bigcup} \cdots \dot{\bigcup} I_2^{s_2},$$

where $1 \le p \le s_1,\ 1 \le q \le s_2$ for some $s_1, s_2 \in \mathbb{N}$.

$$[P_{I_1^p} \cap P_{I_2^q}] = [f_1] + \cdots + [f_r],$$

where each f_i is represented by a 2-dimensional surface S_i in $P_{I_1^P} \cap P_{I_2^q}$ whose boundaries lie on $\partial P_{I_1^P} \cup \partial P_{I_2^q}$. Let W_1 be the handlebody in P_I bounded by S_i whose boundaries lie on ∂P_I, and let W_2 be the complement of W_1 in P_I. We assume that W_i, $i = 1, 2$ consists of facets of P. So that

$$W_i \subset P_{I_j},\ i = 1, 2 \text{ and } I_1 \bigcup I_2 = I.$$

Let

$$a_i = \sum_{s \in I_i} [f_s] \in \widetilde{H}_2(P_{I_i}, \partial P_{I_i}),\ i = 1, 2.$$

Then, $a_1 \cdot a_2 = [f_i]$. This implies that the map (4.1) should be surjective. This completes the proof of Lemma 4.1.

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