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Simplicial wedge complexes and projectivity

조선대학교 교육대학원

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이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

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CONTENTS

국 문 초 록

단체 쐐기 복합체와 사영성에 관한 연구

임 영 신
지도교수 : 김 진 홍
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단체 쐐기 작용은 n개의 꼭짓점을 가진 단순복합체 K에서 부터 n+1개의 꼭짓점을 가진 새로운 단체복합체를 얻기 위한 유용한 방법이다. 본 논문의 목적은 비사영적이고 비특이적이며 완비적인 팬으로 $\mathrm{Proj}_{v_0}\Sigma$과 $\mathrm{Proj}_{v_1}\Sigma$이 모두 사영적인 경우가 없음을 증명했다. 또한, 완비인 단순 팬이 강한 폴리토팔이면 그들의 사영적인 팬인 $\mathrm{Proj}_{v_0}\Sigma$과 $\mathrm{Proj}_{v_1}\Sigma$도 강한 폴리토팔이고 그 역도 성립함을 보였다.

Simplicial wedge complexes and projectivity

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Abstract

A simplicial wedge operation is a useful method to produce a new abstract simplicial complex $\text{wedge}_v(K)$ with $n+1$ vertices from a given abstract simplicial complex K with n vertices. The aim of this paper is to show that there is no complete non-singular non-projective fan Σ over the simplicial wedge complex $\text{wedge}_v(K)$ whose projected fans $\text{Proj}_{v_0}\Sigma$ and $\text{Proj}_{v_1}\Sigma$ over the same K are both projective. In other words, if a complete simplicial fan Σ over $\text{wedge}_v(K)$ is strongly polytopal, then their projected fans $\text{Proj}_{v_0}\Sigma$ and $\text{Proj}_{v_1}\Sigma$ over K should be also strongly polytopal, and the converse is also true.

Contents

Chapter 1

Introduction

Our main concern of this paper is one special subject in toric topology. Toric topology is a field of mathematics that is currently very active and can be regarded as a topological generalization of toric algebraic geometry. It is closely related to many other areas of mathematics such as algebraic topology, symplectic geometry, convex geometry, and combinatorics ([2]).

In toric algebraic geometry, there is a well-known one-to-one correspondence between toric varieties and complete fans or underlying simplicial complexes, up to certain equivalence relations (refer to [5]). This means that in order to study toric varieties it suffices to study their corresponding fans or underlying simplicial complexes which are more tractable. Recall that a *fan* in the vector space $\mathbb{R}^k$ is a collection of strongly convex rational cones such that every face of cones and every intersection of a finite number of cones are also in the fan. In addition, a fan is called *complete* if the union of all cones covers the whole vector space $\mathbb{R}^k$, while a fan is called *non-singular* if one-dimensional faces of each cone are unimodular in the lattice $\mathbb{Z}^k$ embedded in $\mathbb{R}^k$. On the other hand, a fan is called *simplicial* if one-dimensional faces of each cone are linearly independent in $\mathbb{R}^k$.

It is possible to think of a complete non-singular fan Σ as a pair (K_Σ, λ) of a underlying simplicial complex K_Σ and a characteristic map λ, where λ

is a map from the vertex set of K to the lattice $\mathbb{Z}^n$ obtained by assigning a primitive integral vector to each vertex of K. Also, K_Σ (or Σ) is called *polytopal* if there is an embedding of the geometric realization $|K_\Sigma|$ of K into $\mathbb{R}^k$ such that $|K_\Sigma|$ is given by the boundary of the simplicial dual polytope P^* of a simple convex polytope P. If, in addition, P^* contains the origin and Σ is given by the positive hulls of proper faces of P^*, then K_Σ (or Σ) is said to be *strongly polytopal*. It is well known that the toric variety associated to a strongly polytopal fan is projective. By abuse of terminology, in this case we will just say that the corresponding fan or underlying simplicial complex is projective. Recall also that a simplicial complex K is *fan-like* if there is a complete fan Σ whose underlying simplicial complex K_Σ is exactly K.

Let λ be a characteristic map on K, and let σ be a face of K such that the vectors $\lambda(i)$ for $i \in \sigma$ are unimodular. Then the *projected characteristic map* $\mathrm{Proj}_\sigma\lambda$ of λ with respect to σ is defined by the map

$$\mathrm{Proj}_\sigma\lambda : \mathrm{Lk}_K(\sigma) \to \mathbb{Z}^n/\langle \lambda(i) \,|\, i \in \sigma\rangle \cong \mathbb{Z}^{n-|\sigma|},$$

where $\mathrm{Lk}_\sigma(K)$ denotes the link of σ in K. Similarly, there is a notion of the projected fan $\mathrm{Proj}_\sigma\Sigma$ of a fan Σ with respect to a face σ of the underlying simplicial complex K_Σ ([4] for more details).

There is a well-known operation, called a *simplicial wedge operation*, from abstract simplicial complexes with n vertices to another abstract simplicial complexes with $n+1$ vertices. That is, for a simplicial complex K with n vertices and any sequence $J = (j_1, j_2, \ldots, j_n)$ of positive integers we can construct a new simplicial complex $K(J)$ with $d(J) = j_1 + j_2 + \cdots + j_n$ vertices, called a *simplicial wedge complexes*. Such a simplicial wedge complex $K(J)$ is obtained inductively by starting from K and applying the simplicial wedge operation to one of the vertices of K. To be more precise, as above let K be a fan-like simplicial complex in $\mathbb{R}^{2m-1}$ with vertices $w_1, w_2, \ldots, w_n$, and let $v = w_1$ be a vertex of K. Let v_0 and v_1 denote two newly created vertices in the simplicial complex $\mathrm{wedge}_v(K) := K(2, 1, 1, \ldots, 1)$ obtained by applying the simplicial wedge operation to K at v. It is natural to ask whether or not there is a complete non-singular non-projective fan Σ over $\mathrm{wedge}_v(K)$ whose projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ over K are both projective.

The aim of this paper is to give a negative answer to this question. To be precise, our main result is

Theorem 1.1. *Let K be a fan-like simplicial complex with a vertex v and let Σ be the corresponding fan. Let $\mathrm{wedge}_v(K)$ be the simplicial complex obtained from K by applying the simplicial wedge operation to v, and let v_0 and v_1 denote two newly created vertices in $\mathrm{wedge}_v(K)$. If the complete simplicial fan Σ over $\mathrm{wedge}_v(K)$ is strongly polytopal, then their projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ over K should be also strongly polytopal, and the converse is also true.*

We organize this paper as follows. In Chapters 2 and 3, we briefly review basic facts regarding simplicial wedge complexes and Shephard criterion which play crucial roles in the proof of Theorem 1.1. In Chapter 4, we give a proof of Theorem 1.1.

Chapter 2

Wedge operation of simplicial complexes

The aim of this section is to collect some basic material regarding simplicial wedge complexes and Shephard criterion necessary for the proof of Theorem 1.1 (see [1], [6], and [7] for more details).

2.1 Simplicial wedge operations

A (convex) polytope P is the convex hull of a finite set of point in $\mathbb{R}^n$. Let P be a convex polytope of dimension n.

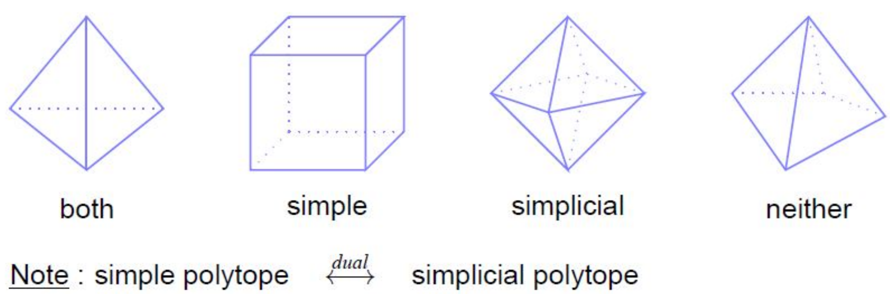

Figure 2.1: simple or simplicial

- P is simple if each vertex is the intersection of exactly n facet.
- P is simplicial if every facet is an $(n-1)$-simplex.

There are two equivalent ways to construct simplicial wedge complexes. One way is to use the notion of a minimal non-face of a simplicial complex and the fact that every simplicial complex is completely determined by all minimal non-faces (see [1] for more details).

A *simplicial complex* K on a finite set V is a collection of subsets of V satisfying

- if $v \in V$, then $\{v\} \in K$,
- if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$.

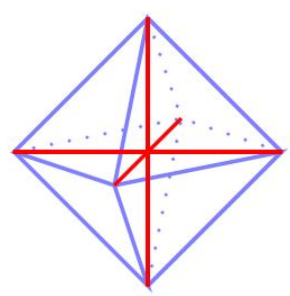

Figure 2.2: simplicial complex K

Each element $\sigma \in K$ is called a *face* of K. The dimension of σ is defined by $\dim \sigma = |\sigma| - 1$. Then *dimension* of K is defined by

$$\dim K = \max\{\dim \sigma \,|\, \sigma \in K\}.$$

There is a useful way to construct new simplicial complexes from a given simplicial complex introduced in [1]. We briefly present the construction here. Let K be a simplicial complex of dimension $n-1$ on vertices $V = [m] = \{1, 2, \ldots, m\}$. A subset $\tau \subset V$ is called a *non-face* of K if it is not a face of K. A non-face τ is *minimal* if any proper subset of τ is a face of K. Note that a simplicial complex is determined by its minimal non-faces.

In the setting above, let $J = (j_1, \ldots, j_m)$ be a vector of positive integers. Denote by $K(J)$ the simplicial complex on vertices

$$\{1_1, 1_2, \ldots, 1_{j_1}, 2_1, 2_2, \ldots, 2_{j_2}, \ldots, m_1, \ldots, m_{j_m}\}$$

with minimal non-faces

$$i_{11}, \cdots, i_{1j_1}, i_{21}, \cdots, i_{2j_2}, \cdots, i_{m1}, \cdots, i_{mj_m}$$

for each minimal non-faces $\{i_1, \ldots, i_k\}$ of K.

There is another way to construct $K(J)$ called the *simplicial wedge construction*. Recall that for a face σ of a simplicial complex K, the *link* of σ in K is the subcomplex

$$\mathrm{Lk}_K \sigma := \{\tau \in K \,|\, \sigma \cup \tau \in K, \sigma \cap \tau = \phi\},$$

and the *join* of two disjoint simplicial complexes K_1 and K_2 is defined by

$$K_1 \star K_2 = \{\sigma_1 \cup \sigma_2 \,|\, \sigma_1 \in K_1, \sigma_2 \in K_2\}.$$

Let K be a simplicial complex with vertex set $[m]$ and fix a vertex i in K. Consider a 1-simplex I whose vertices are i_1 and i_2 and denote by $\partial I = \{i_1, i_2\}$ the 0-skeleton of I. Now, let us define a new simplicial complex on $m+1$ vertices, called the (*simplicial*) *wedge* of K at i, denoted by $\mathrm{wedge}_i(K)$, by

$$\mathrm{wedge}_i(K) = (I \star \mathrm{Lk}_K\{i\}) \cup (\partial I \star (K \setminus \{i\})).$$

where $K \setminus \{i\}$ is the induced subcomplex with $m-1$ vertices except i. The operation itself is called the *simplicial wedge operation* or the (*simplicial*) *wedging*. See Figure 2.3.

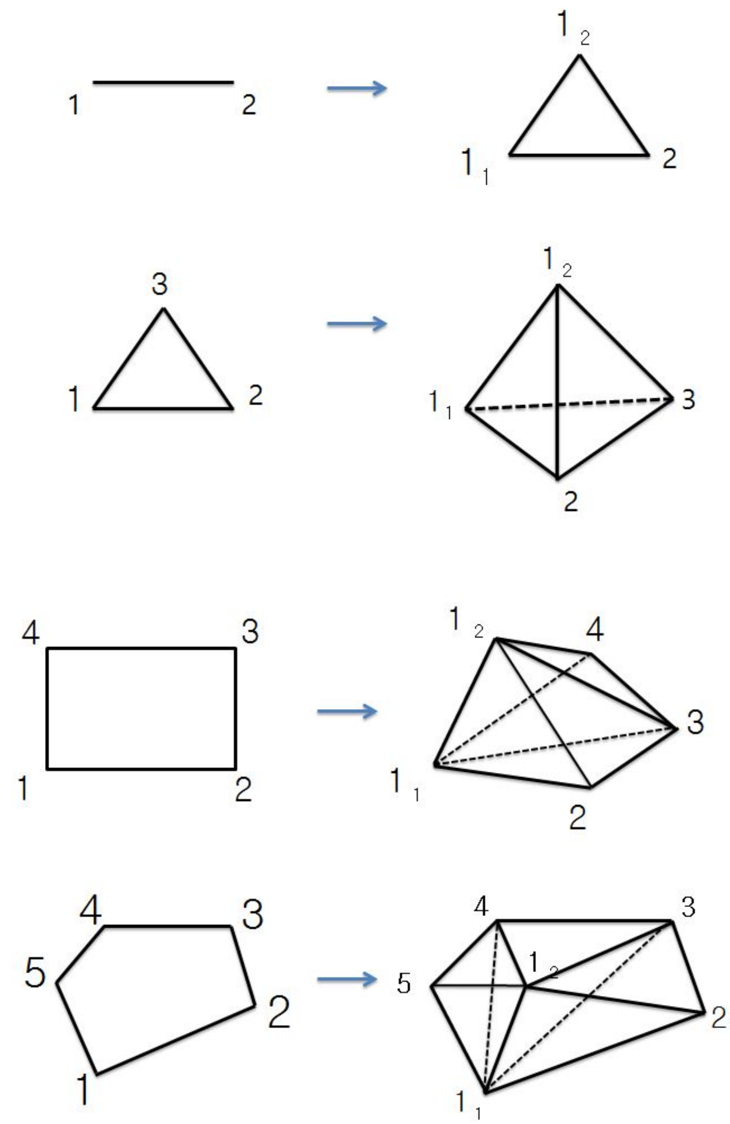

Figure 2.3: An illustration of a wedge of K

It is an easy observation to show that $\mathrm{wedge}_i(K) = K(J)$ where $J = (1,\ldots,1,2,1,\ldots,1)$ is the m-tuple with 2 as the i-th entry. By consecutive application of this construction starting from $J = (1,\ldots,1)$ we can produce $K(J)$ for any J. Although there is some ambiguity to proceed from $J = (j_1,\ldots,j_m)$ to $J' = (j_1,\ldots,j_{i-1},j_i+1,j_{i+1},\ldots,j_m)$ of $j_i \geq 2$, we have no problem since any choice of the vertex yields the same minimal non-face of the resulting of $K(J)$. In conclusion, one can obtain a simplicial complex $K(J)$ by successive simplicial wedge constructions starting from K, independent of order of wedgings.

Related to the simplicial wedging, we recall some hierarchy of simplicial complexes. Among simplicial complexes, simplicial spheres form a very important subclass.

Example 2.1. Let K be the boundary complex of a pentagon. Then the minimal non-faces of K are

$$\{1,3\},\ \{1,4\},\ \{2,4\},\ \{2,5\},\ \text{and}\ \ \{3,5\}.$$

Hence, the minimal non-face of $\mathrm{wedge}_1(K) := K(2,\ 1,\ 1,\ 1,\ 1)$ are

$$\{1_1,1_2,3\},\ \{1_1,1_2,4\},\ \{2,4\},\ \{2,5\},\ \text{and}\ \ \{3,5\}.$$

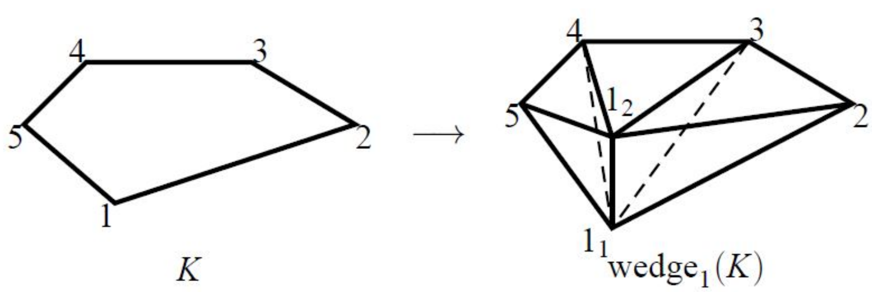

Figure 2.4: $\mathrm{wedge}_1(K)$

Definition 2.2. Let K be a simplicial complex of dimension $n-1$.

(1) K is called a *simplicial sphere* of dimension $n-1$ if its geometric realization $|K|$ is homoeomorphic to a sphere S^{n-1}.

(2) K is called *star-shaped in* p if there is an embedding of $|K|$ into $\mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ such that any ray from p intersects $|K|$ once and only once. The geometric realization $|K|$ itself is also called star-shaped.

(3) K is said to be *polytopal* if there is an embedding of $|K|$ in to $\mathbb{R}^n$ which is the boundary a simplicial n-polytopal P^*.

We have a chain of inclusions

simplicial complexes $\supset$ simplicial spheres $\supset$ star-shaped complexes $\supset$ polytopal complexes.

It is worthwhile to observe that each category of simplicial complexes above is closed under the wedge operation as follows.

Proposition 2.3. *Let K be a simplicial complex and v its vertex. The the followings hold:*

(1) *If K is a simplicial sphere, then so is* $\mathrm{wedge}_v(K)$.

(2) *K is star-shaped if and only if so is* $\mathrm{wedge}_v(K)$.

(3) *K is polytopal if and only if so is* $\mathrm{wedge}_v(K)$.

When K is polytopal, we often regard K as the boundary complex of a simple polytopal P. To be more precise, let K be the boundary of a simplicial polytope Q. Then the dual polytope to Q is a simple polytope P. Recall that an n-dimensional polytope P is called *simple* if exactly n facets (or codimension 1 faces) intersect at each vertex of P.

We next define the notion of the (polytopal) wedge. Let $P \subseteq \mathbb{R}^n$ be a polytope of dimension n and F a face of P. To do so, consider a polyhedron

$P\times[0,\infty)\subseteq\mathbb{R}^{n+1}$ and identify P with $P\times\{0\}$. Pick a hyperplane H in $\mathbb{R}^{n+1}$ so that $H\cap P=F$ and H intersects the interior of $P\times[0,\infty)$. Then H cuts $P\times[0,\infty)$ into two parts. The part which contains P is an $(n+1)$-polytope and combinatorially determined by P and F, and it is called the (*polytopal*) *wedge of* P *at* F and denoted by $\mathrm{wedge}_F(P)$. Note that $\mathrm{wedge}_F(P)$ is simple if P is simple and F is a facet of P. See Figure 2.5.

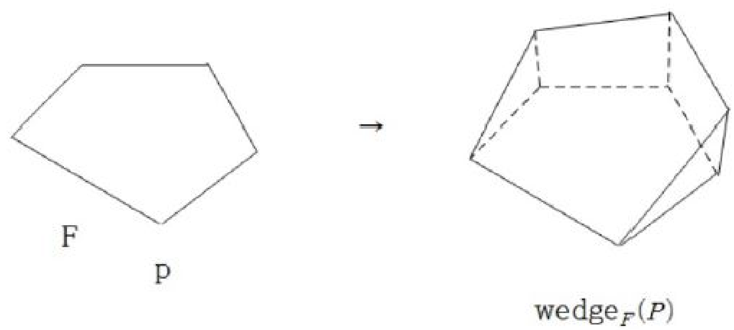

Figure 2.5: An illustration of a wedge of K

The next lemma is due to [4, Lemma 2.3].

Lemma 2.4. *Assume that P is a simple polytope and F is a facet of P. Then the boundary complex of* $\mathrm{wedge}_F(P)$ *is the same as the simplicial wedge of the boundary complex of P at F.*

Suppose P is an simple polytope and $\mathcal{F}=(F_1,\dots,F_m)$ is the set of facets of P. Let $J=(a_1,\dots,a_m)\in\mathbb{N}^m$ be a vector of positive integers. Then define $P(J)$ by the combinatorial polytope obtained by consecutive polytopal wedgings analogous to the construction of $K(J)$ with simplicial wedgings.

Lemma 2.4 guarantees that if the boundary complex of P is K, then the boundary complex of $P(J)$ is $K(J)$.

Remark 2.5. It is known that the converse of (1) in Proposition 2.3 does not hold, in general. This is due to the famous Double Suspension Theorem of Edwards and Cannon [3] which states that every double suspension of a homology n-sphere M is homeomorphic to an $(n+2)$-sphere.

2.2 Toric varieties and fans

Let us review the definition of a fan. For a subset $X \subset \mathbb{R}^n$, the *positive hull* of X, that is,

$$\operatorname{pos} X = \left\{ \sum_{i=1} a_i x_i \,|\, a_i \geq 0,\ a_i \in X \right\}.$$

By convention, we put pos $X = \{0\}$ if X is empty. A subset C of $\mathbb{R}^n$ is called a *polyhedral cone*, or simply a *cone*, if there is an finite set X of vectors, called the *set of generators* of the cone, such that $C = \operatorname{pos} X$. The elements of X is called *generators* of C. We also say that X positively spans the cone C. A subset D of C is called a *face* of C if there is a hyperplane H such that $C \cap H = D$ and C does not lie in both sides of H. A cone is by convention a face of itself and all other faces are called *proper*.

A cone is called *strongly convex* if it does not contain a nontrivial linear subspace. In this paper, every cone is assumed to be strongly convex. A polyhedral cone is called *simplicial* if its generators are linearly independent, and *rational* if every generator is in $\mathbb{Z}^n$. A rational cone is called *non-singular* if its generators are unimodular, i.e., they are a part of an integral basis of $\mathbb{Z}^n$.

A fan Σ of real dimension n is a set of cones in $\mathbb{R}^n$ such that

(1) if $C \in \Sigma$ and D is a face of C, then $D \in \Sigma$,

(2) and for $C_1, C_2 \in \Sigma$, $C_1 \cap C_2$ is a face of C_1 and C_2 respectively.

A fan Σ is said to be *rational* (resp. *simplicial* or *non-singular*) if every cone in Σ is rational (resp. simplicial or non-singular). Remark that the term "fan" is used for rational fans in most literature, especially among toric geometers. We will sometimes use the term "real fan" to emphasize that generators need not be integral vectors.

If a fan Σ is simplicial, then we can think of a simplicial complex K, called the *underlying simplicial complex* of Σ, whose vertices are generators of cones of Σ and whose faces are the sets of generators of cones in Σ (including the empty set). We also say that Σ is a fan over K. In this thesis, a fan is assumed to be simplicial unless otherwise mentioned.

A fan Σ is called *complete* if the union of cones in Σ covers all of $\mathbb{R}^n$. Observe that the underlying simplicial complex of a fan is a simplicial sphere if and only if the fan is complete. It is a well-known fact that a rational fan is complete (resp. non-singular) if and only if its corresponding toric variety is compact (resp.smooth). A compact smooth toric variety is called a *toric manifold* in this paper. We remark that a toric variety is an orbifold if and only if its corresponding fan is simplicial.

We close this section by giving definition of two notions relating a fan to a polytope. A fan is said to be *weakly polytopal* if its underlying simplicial complex is polytopal in the sense of Definition 2.2. A fan Σ is called *strongly polytopal* if there is a simplicial polytope P^*, called a *spanning polytope*, such that $0 \in \operatorname{int} P^*$ and

$$\Sigma = \{\operatorname{pos}\sigma \mid \sigma \text{ is a proper face of } P^*\}.$$

Observe that the underlying complex of Σ is ∂P^*. Therefore strong polytopalness implies weak polytopalness.

It is a well-known fact from convex geometry that a fan Σ is strongly polytopal if and only if Σ is the normal fan of a simple polytope P. For a given simple n-polytope $P \subset \mathbb{R}^n$, correspond to each facet F the outward normal vector $N(F)$. The *normal fan* of Σ of P is a collection of cones

$$\Sigma = \{\mathrm{pos}\,\{N(F)|F \supset f\}\,|\, f \text{ is a proper face of } P\}.$$

Chapter 3

Gale transforms, and Shephard's criterion

3.1 Projected characteristic map

A closed connected smooth orientable manifold M of dimensional $2n$ is called a *torus manifold* if is equipped with an effective T^n-action which has a nonempty fixed point set. Torus manifolds make a large class of manifolds properly containing topological toric manifolds. A torus manifold has its own combination object, called a multi-fan, which can be roughly understood as a collection of cones similar to a fan but the cones may overlap. Although we do not present the precise definition of multi-fans, we use the concept of overlapping cones to consider fan-givingness of a characteristic map.

Let (K, λ) be a characteristic map of dimensional n and $I \in K$ a face of K. One defines the cone over I be the positive hull $\operatorname{pos}\{\lambda(i)|i \in I\}$ and denote it by $\angle\lambda_I$. from now on, we assume that (K, λ) is complete. So we consider the simplicial complex $|K|$ which is an $(n-1)$-dimensional space. We set

$$\sigma_I := \left\{ \sum_{i=1} a_i e_i \,|\, \sum_{i=1} a_i = 1,\ a_i \geq 0, \right\} \subset \mathbb{R}^m \text{ for } I \in K,$$

where e_i is the i-th coordinate vector of $\mathbb{R}^m$. The geometric realization $|K|$ of K is given by

$$|K| = \bigcup_{I \in K} \sigma_I.$$

In this section, we study the relation of simplicial wedging and toric objects. First of all, we need the notion of "projected characteristic map".

Definition 3.1. Let (K, λ) be a characteristic map of dimension n and $\sigma \in K$ a face of K such that the set $\{\lambda(i) | i \in \sigma\}$ is unimodular. Let v be a vertex of $\mathrm{Lk}_K\sigma$. Then one maps v to $[\lambda(v)]$ which is an element of the quotient lattice of $\mathbb{Z}^m$ by the sublattice generated by $\lambda(i)$, $i \in \sigma$. This map, denoted by $\mathrm{Proj}_\sigma\lambda$, is called the *projected characteristic map*.

There is a similar notion called the *projected fans*. Note that projection characteristic maps generalize projected fans whenever it is applicable. we denoted by $\mathrm{Proj}_\sigma\Sigma$ the projected fan of Σ with respect to a face of $K(\Sigma)$.

Lemma 3.2. *Let K be a fan-like sphere. then for any proper face σ of K, $\mathrm{Lk_K}\sigma$ is a fan-like sphere. If is a complete non-singular characteristic map, then for any σ, Its projection $(\mathrm{Lk_K}\sigma, \mathrm{Proj}_\sigma\lambda)$ is also complete and non-singular. If λ is fan-giving, so is $\mathrm{Proj}_\sigma\lambda$.*

Suppose this is a topological toric version of projected fans and the proof is essentially the same. Since K is fan-like, there exist a complete real fan Σ over K. Its projected fan is complete and therefore $\mathrm{Lk}_K\sigma$ is a fan-like sphere.

Note that one can define projected topological fans in the same way. When σ is a vertex, the projection $\mathrm{Proj}_\sigma\lambda$ corresponds to a characteristic submanifold of $M(\lambda)$. We also remark that the above lemma shows that any multi-fan given by a complete characteristic map is complete. If (K, λ) is an oriented complete characteristic map, then a projected characteristic map $(\mathrm{Lk_K}\sigma, \mathrm{Proj}_\sigma\lambda)$ inherits an orientation so that

Let K be a fan-like sphere with $V(K) = [m] = \{1, \cdots, m\}$. A characteristic map $\lambda : V(K) \to \mathbb{Z}^n$ can be regarded as an $n \times m - matrix$, called the *characteristic matrix*, which is again denoted by λ. Each column is labeled by a vertex and the i-th column vector of the matrix λ corresponds to $\lambda(i)$.

Example 3.3. Let K be a simplicial complex and $\sigma \in K$ The link of σ is

$$\mathrm{Lk}_K \sigma := \{\tau \in K \,|\, \sigma \cup \tau \in K, \sigma \cap \tau = \phi\}.$$

Then $\mathrm{Lk}_{\mathrm{wedge}_1(K)} 1_2$ is equivalent to K.

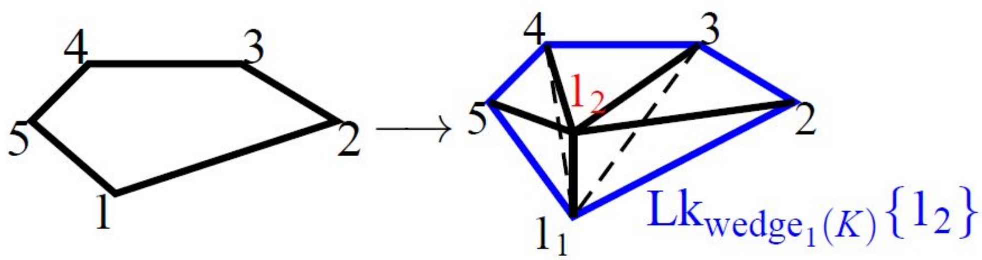

Figure 3.1: $\mathrm{Lk}_{\mathrm{wedge}_1(K)} 1_2 \cong K$

Let $\mathrm{wedge}_1(K)$ be the simplicial complex shown in Figure 3.1 and let λ be defined by the characteristic matrix

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose columns are labeled by the vertices $1_1, 1_2, 2, 3, 4, 5$, respectively. That is, we define

$$\lambda(1_1) = (0, 0, 1),$$

$$\lambda(1_2) = (1, 0, -1),$$

$$\lambda(2) = (0, 1, 0),$$

$$\lambda(3) = (-1, 1, 0),$$
$$\lambda(4) = (-1, 0, 0),$$
$$\lambda(5) = (0, -1, 0).$$

Since $\lambda(1_1)$ is a coordinate vector, the projection $\mathrm{Proj}_{1_1}\lambda$ is easily obtained by

$$\mathrm{Proj}_{1_1}\lambda = \begin{array}{c} \begin{matrix} 1_1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix} \end{array},$$

where the first row is for indicating column labeling. To complete $\mathrm{Proj}_3\lambda$, one should perform a row operation so that $\lambda(3)$ becomes a coordinate vector. Then add the second row of λ to the first one, and one obtains

$$\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathrm{Lk}_K\{3\}$ has vertices $1_1, 1_2, 2, 4$, its characteristic matrix looks like

$$\mathrm{Proj}_3\lambda = \begin{array}{c} \begin{matrix} 1_1 & 1_2 & 2 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \end{array}.$$

Example 3.4. Let us find $\mathrm{Proj}_5\lambda$. First of all, one should do the row operation by multiplying by -1 to the second row so that $\lambda(5)$ becomes a coordinate vector. That is, we have

$$\lambda = \begin{array}{c} \begin{matrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \sim \begin{array}{c} \begin{matrix} 1_1 & 1_2 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Since $\mathrm{Lk}_{\mathrm{wedge}_1(K)}\{5\}$ has vertices $1_1, 1_2, 4$, we have

$$\mathrm{Proj}_5\lambda = \begin{array}{c} \begin{matrix} 1_1 & 1_2 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}.$$

3.2 Gale transforms and Shephard's criterion

The aim of this chapter is to set up basic notations and definitions, and to collect some important facts necessary for the proof of Theorem 1.1. To do so, we first begin with reviewing linear transforms and Gale transforms. Refer to [7, Chapter II-Section 4] for more details.

Let $X = (x_1, x_2, \ldots, x_m)$ be a sequence of (not necessarily different) vector in an n-dimensional vector space U, and let $x_1, x_2, \ldots, x_m$ span U. We consider a m-dimensional vector space V and a basis $b_1, b_2, \ldots, b_m$ of V. Then there is a well-defined linear map

$$L : V \to U.$$

Let $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ be a finite sequence of vectors x_i in $\mathbb{R}^n$ which linearly spans $\mathbb{R}^n$. Then we consider the space of linear dependence (or linear relations) of X which is given by the $(m-n)$-dimensional space

$$\{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m | \sum_{i=1}^{m} \alpha_i x_i = 0\}.$$

By choosing a basis $\{\Theta^1, \ldots, \Theta^{m-n}\}$ of the space of linear dependencies as above, it is convenient to write it as a matrix of size $(m-n) \times m$, as follows.

$$\begin{aligned}(\Theta^1, \ldots, \Theta^{m-n})^T &= \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{(m-n)1} & \cdots & \alpha_{(m-n)m} \end{pmatrix}_{(m-n)\times m} \\ &= (\bar{x}_1, \ldots, \bar{x}_m) =: \bar{X}.\end{aligned}$$

The finite sequence $\bar{X}$ is called a *linear transform* (or *linear representation*) of X. Clearly, a linear transform is not unique and depends only on a choice of a basis. Note also that we have the following relationship between X and $\bar{X}$:

$$X\bar{X}^T = 0. \tag{3.1}$$

It is also easy to see that $\bar{X}X^T = 0$ by taking the transpose of the equation (3.1). Thus, if $\bar{X}$ is a linear transform of X, then X is also a linear transform of $\bar{X}$.

Example 3.5. Let $x_1, x_2, \ldots, x_6$ be the vertices of a prism in real affine 3-space H and consider H as a hyperplane in $\mathbb{R}^4$ such that 0 is not in H. Then $(-1, 1, 0, 1, -1, 0)$ and $(-1, 0, 1, 1, 0, -1)$ are linear dependencies and the columns of

$$A = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0-1 \end{pmatrix} = (\bar{x}_1, \ldots, \bar{x}_6)$$

are the elements of a linear transform of X

Lemma 3.6. *A linear transform $\bar{X}$ of X satisfies $\bar{x}_1 + \cdots + \bar{x}_m = 0$ if and only if the points x_i lie in a hyperplane H of $\mathbb{R}^n$ for which 0 $\notin H$.*

Note that one can assume that H is the hyperplane of points whose last coordinate is 1 since we can take $(1, \ldots, 1)$ for a linear dependency of $\bar{X}$. In general, for any strongly convex cone C, there is a hyperplane H which does not intersect the origin and $C \cap H = P$ is a convex polytope which has the same face poset with C.

Now we are ready to define the *Gale transform*. In order to define a Gale transform by using the notion of a linear transform, as before let $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ be a finite sequence of vectors $x_i \in \mathbb{R}^n$ which affinely spans $\mathbb{R}^n$. Then we identify $\mathbb{R}^n$ as an affine space with a hyperplane H in a linear space $\mathbb{R}^{n+1}$ by the natural embedding

$$j : \mathbb{R}^n \to \mathbb{R}^{n+1}, v \mapsto (v, 1).$$

Then $H = \{(v, 1) \in \mathbb{R}^{n+1} \,|\, v \in \mathbb{R}^n\}$ does not contain the origin of $\mathbb{R}^{n+1}$. Thus it follows from [7, Lemma 4.15] that a linear transform $\bar{\hat{X}} = (\bar{x}_1, \ldots, \bar{x}_m) \in$

$(\mathbb{R}^{m-n-1})^m$ of

$$j(X) = ((x_1, 1), \ldots, (x_m, 1)) =: (\hat{x}_1, \ldots, \hat{x}_m) =: \hat{X}$$

in $\mathbb{R}^{n+1}$ satisfies

$$\sum_{i=1}^{m} \bar{x}_i = 0,$$

and $\bar{\hat{X}}$ is called a *Gale transform* (or an *affine transform*) of $\hat{X}$.

Now, we are ready to characterize a complete fan that is strongly polytopal. To be more precise, we have the following criterion given by Shephard in the paper [8] (or [7, Theorem 4.8] and [6, Section 2]) for a complete fan to be strongly polytopal.

Theorem 3.7. *Let*

$$X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$$

be a finite sequence of lattice points $x_i \in \mathbb{Z}^n \subseteq \mathbb{R}^n$ that span the 1-dimensional cones of a complete fan Σ, and let $\bar{X}$ be a Gale transform of X for each proper face $\sigma = \operatorname{pos}\{x_{j_1}, \ldots, x_{j_k}\}$ of Σ, let $C(\sigma)$ denote the convex hull generated by

$$\bar{X} \setminus \{\bar{x}_{j_1}, \ldots, \bar{x}_{j_k}\}.$$

That is,

$$C(\sigma) = \operatorname{conv}(\bar{X} \setminus \{\bar{x}_{j_1}, \ldots, \bar{x}_{j_k}\}).$$

Then Σ is strongly polytopal if and only if we have

$$\bigcap_{\sigma \in \Sigma} \operatorname{relint} C(\sigma) \neq \emptyset$$

Here, $\operatorname{relint} C(\sigma)$ means the *relative interior* of $C(\sigma)$. Recall also that, when σ is a proper face of Σ generated by $\{x_{j_1}, \ldots, x_{j_k}\}$,

$$\bar{X} \setminus \{\bar{x}_{j_1}, \ldots, , \bar{x}_{j_k}\}$$

is called a *coface* of σ in X.

In fact, in order to use the Shepherd's criterion for a complete fan to be strongly polytopal, we shall start with a finite sequence X whose column sum is equal to zero. Then we obtain a linear transform $\hat{X}$ of X.

Theorem 3.8. *A complete fan Σ is strongly polytopal (or projective) if and only if*

$$\mathcal{S}(\Sigma, \hat{X}) := \bigcap_{\sigma \in \Sigma} \operatorname{relint} C(\sigma) \neq \emptyset.$$

Chapter 4

Proof of main results: Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1. To do so, it suffices to prove the following theorem.

Theorem 4.1. *Let Σ be a complete simplicial fan in $\mathbb{R}^m$ over* $\mathrm{wedge}_v(K)$ *on vertex set $\{v_0, v_1, \ldots, v_n\}$. Let x_i be a primitive integral generating vector for each 1-dimensional cone of Σ. Assume that*

$$X = (x_0, x_1, \ldots, x_n) \in (\mathbb{R}^m)^{n+1}, \ n > m$$

is an $(n+1)$-tuple of vectors in $\mathbb{R}^m$ which positively spans the origin in $\mathbb{R}^m$. Then the following statements hold:

(1) *If Σ is strongly polytopal (or projective), then so are both of the projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$.*

(2) *If either of the projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ is strongly polytopal (or projective), then so are both of them as well as Σ.*

In order to prove Theorem 4.1, by Theorem 3.8 again it suffices to prove the following theorem.

Theorem 4.2. *Let X be the same as in Theorem 4.1, and let*

$$\hat{X} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_n) \in (\mathbb{R}^{n-m+1})^{n+1}$$

be the Shephard transform for Σ. Then the following statements hold:

(1) *The subsequences*

$$\hat{X}\backslash\{\hat{x}_0\} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \text{ and } \hat{X}\backslash\{\hat{x}_1\} = (\hat{x}_0, \hat{x}_2, \dots, \hat{x}_n)$$

are Shephard transforms for the projected fans $\mathrm{Proj}_{v_0}\Sigma$ *and* $\mathrm{Proj}_{v_1}\Sigma$, *respectively.*

(2) $\mathcal{S}(\Sigma, \hat{X}) \neq \emptyset$ *if and only if* $\mathcal{S}(\mathrm{Proj}_{v_0}\Sigma, \hat{X}\backslash\{\hat{x}_0\}) \neq \emptyset$.

(3) $\mathcal{S}(\Sigma, \hat{X}) \neq \emptyset$ *if and only if* $\mathcal{S}(\mathrm{Proj}_{v_1}\Sigma, \hat{X}\backslash\{\hat{x}_1\}) \neq \emptyset$.

Proof. To begin with, we may assume without loss of generality that the first $m+1$ vectors $x_0, x_1, \dots, x_m$ is an affine basis of $\mathbb{R}^m$. Let

$$u_k = (x_1^k - x_0^k, x_2^k - x_0^k, \dots, x_n^k - x_0^k) \in \mathbb{R}^n, \quad 1 \leq k \leq m.$$

Then we fix a basis $\mathcal{B}$ of $\mathbb{R}^n$ whose first m basis vectors are u_k and whose last $n-m$ basis vectors are a part of the canonical basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. That is, we have

$$\mathcal{B} = \{u_1, u_2, \dots, u_m, e_{i_1}, \dots, e_{i_{(n-m)}}\}.$$

Let E (resp. F) be the m-dimensional (resp. $(n-m)$-dimensional) vector subspace of $\mathbb{R}^n$ generated by basis vectors u_k, $1 \leq k \leq m$, (resp. $e_{i_1}, e_{i_2} \dots, e_{i_{(n-m)}}$). So we have the direct sum decomposition such that $\mathbb{R}^n \cong E \oplus F$.

Now, let P be the square matrix of size $n \times n$ such that

$$P = (u_1, u_2, \dots, u_m, e_{i_1}, \dots, e_{i_{(n-m)}})_{n \times n},$$

and let J be a matrix of size $(n-m)\times n$ such that

$$J = (0_{(n-m)\times m} I_{(n-m)\times(n-m)})_{(n-m)\times n}.$$

Here $0_{(n-m)\times m}$ denotes the matrix of size $(n-m)\times m$ consisting of only zeros, while $I_{(n-m)\times(n-m)}$ denotes the identity matrix of size $(n-m)\times(n-m)$. Note that clearly P is invertible by its construction.

With these understood, let

$$\Pi = JP^{-1}$$

be the matrix of size $(n-m)\times n$, and let

$$L_\Pi : \mathbb{R}^n \to \mathbb{R}^{n-m}$$

be the linear map associated to the matrix Π. Then it follows from its construction that the linear map L_Π maps onto F and the kernel $\ker L_\Pi$ of L_Π is exactly equal to E.

As before, let e_i $(1 \le i \le n)$ denote the canonical basis vector of $\mathbb{R}^n$, and let

$$e_0 = -(e_1 + \cdots + e_n).$$

It will be important to note that by its construction of L_Π the set

$$Y := (L_\Pi(e_0), L_\Pi(e_1), \ldots, L_\Pi(e_n)) = (y_0, y_1, \ldots, y_n) \in (\mathbb{R}^{n-m})^{n+1}$$

positively spans $\mathbb{R}^{n-m}$. Since the kernel of the linear map L_Π is exactly E, this enables us to obtain a Shephard transform $\hat{Y}$ of Y such that

$$\hat{Y} = (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_n) \in (\mathbb{R}^{m+1})^{n+1},$$

where $\hat{y}_i = (x_i, 1) \in \mathbb{R}^{m+1}$ for $0 \le i \le n$. Since Y is a Gale transform of X by construction and $\hat{X}$ is a linear transform of X, it follows that $\hat{X}$ (resp. $\hat{Y}$) should be of the form

$$\begin{pmatrix} Y \\ 11\cdots 11 \end{pmatrix} \in (\mathbb{R}^{n-m+1})^{n+1} \left(\text{resp. } \begin{pmatrix} X \\ 11\cdots 11 \end{pmatrix} \in (\mathbb{R}^{m+1})^{n+1}\right).$$

Let

$$Y' = (y_1, y_2, \dots, y_n) \in (\mathbb{R}^{n-m})^n.$$

Then, since $\hat{X}$ is a Shephard transform of X, we see that

$$\hat{X}' = \begin{pmatrix} Y' \\ 11 \cdots 11 \end{pmatrix} \in (\mathbb{R}^{n-m+1})^n$$

is a Shephard transform of $X' = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^m)^n$ that can be also considered as a characteristic matrix of $\mathrm{Proj}_{v_0}\Sigma$. Similarly, let

$$Y'' = (y_0, y_2, \dots, y_n) \in (\mathbb{R}^{n-m})^n.$$

Then, we see that

$$\hat{X}'' = \begin{pmatrix} Y'' \\ 11 \cdots 11 \end{pmatrix} \in (\mathbb{R}^{n-m+1})^n$$

is a Shephard transform of $X'' = (x_0, x_2, \dots, x_n) \in (\mathbb{R}^m)^n$ that can be also considered as a characteristic matrix of $\mathrm{Proj}_{v_1}\Sigma$. This completes the proof of Theorem 3.7 (1).

For the proof of (2), we now suppose that $\mathrm{Proj}_{v_0}\Sigma$ is strongly polytopal (or projective). Then it follows from Theorem 3.8 that we have

$$\mathcal{S}(\mathrm{Proj}_{v_0}\Sigma, \hat{X}\backslash\{\hat{x}_0\}) \neq \emptyset.$$

That is, we have

$$\bigcap_{P' \in \Sigma'_{(m)}} C(P')^\circ \neq \emptyset, \tag{4.1}$$

where $P' = P\backslash\{0\} \subset \{1, 2, \dots, n\}$ for $P \in \Sigma_{(m)}$ and $\Sigma'_{(m)}$ denotes the collection of all such P''s. We then claim that we have

$$\mathcal{S}(\Sigma, \hat{X}) \neq \emptyset.$$

Indeed, note first that it follows from (4.1) that there exists $x \in C(P')^\circ$ for all $P' = \{i_1, \dots, i_{m-1}\} \in \Sigma'_{(m)}$. So we can write

$$x = \lambda_1 \hat{x}_{i_1} + \cdots + \lambda_{m-1}\hat{x}_{i_{m-1}}, \quad \sum_{j=1}^{m-1} \lambda_j = 1 \text{ with } \lambda_j \geq 0.$$

Then it is obvious to see

$$\frac{1}{2}\hat{x}_0 + \frac{1}{2}x = \frac{1}{2}\hat{x}_0 + \frac{1}{2}(\lambda_1 \hat{x}_{i_1} + \cdots + \lambda_{m-1}\hat{x}_{i_{m-1}}) \in C(P)^\circ$$

with $\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{m-1} \lambda_j = 1$ and $P = P' \cup \{0\}$. Since P' is arbitrary in $\Sigma'_{(m)}$, this implies that

$$\bigcap_{P \in \Sigma_{(m)}} C(P)^\circ \neq \emptyset,$$

where $P = P' \cup \{0\}$ for $P' \in \Sigma'_{(m)}$. Hence, it follows from Theorem 3.8 that Σ is strongly polytopal (or projective), as desired.

Conversely, suppose that Σ is strongly polytopal (or projective). Then it follows again from Theorem 3.8 that we have

$$\mathcal{S}(\Sigma, \hat{X}) \neq \emptyset.$$

Thus, by definition we have

$$\bigcap_{P \in \Sigma_{(m)}} C(P)^\circ \neq \emptyset.$$

For the sake of simplicity, we may assume that $\hat{x}_0 = 0$ in $\mathbb{R}^{n-m+1}$. Now, if $x \in C(P)^\circ$ for each $P = \{0, i_1, \dots, i_{m-1}\} \in \Sigma_{(m)}$, then there exists $\lambda > 0$ such that $\lambda x \in H'_0$, where H'_0 denotes a hyperplane of $\mathbb{R}^{n-m+1}$ which contains $\nu_j \hat{x}_{i_j}$ for $i_j \in P' = P \backslash \{0\} \subset \{0, 1, \dots, n\}$ and $\nu_j > 0$. So we can write

$$x = \sum_{j=1}^{m-1} \lambda_j \hat{x}_{i_j}, \quad \sum_{j=1}^{m-1} \lambda_j = 1 \text{ and } \lambda_j \geq 0. \tag{4.2}$$

It is also true that we can write

$$\lambda x = \sum_{j=1}^{m-1} \mu_j(\nu_j \hat{x}_{i_j}), \quad \sum_{j=1}^{m-1} \mu_j = 1, \tag{4.3}$$

since any $P' \in \Sigma'_{(m)}$ is an affine $\mathbb{R}$-basis of H'_0. Thus it follows from (4.2) and (4.3) that we have

$$\lambda\lambda_j = \mu_j\nu_j, \quad 1 \le j \le m-1,$$

which implies that $\mu_j \geq 0$ for all $1 \le j \le m-1$. Hence $\lambda x \in C(P')^\circ$. This immediately implies that $\mathrm{Proj}_{v_0}\Sigma$ is strongly polytopal (or projective), as desired.

It is obvious that the proof of Theorem 4.1 (3) can be dealt with in a similar way. This completes the proof of Theorem 4.1. □

Finally, we are ready to prove Theorem 1.1, as follows.

Proof of Theorem 1.1. For the proof, we continue to use the notations as in Theorem 4.1. Now suppose that the complete simplicial fan Σ over $\mathrm{wedge}_v(K)$ is strongly polytopal. Then, by Theorem 3.8 $\mathcal{S}(\Sigma, \hat{X})$ is non-empty. But then it follows from Theorem 4.2 (1) that their projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ over K should be also strongly polytopal. Furthermore, it is easy to see that by Theorem 4.2 (2) the converse is also true. This completes the proof of Theorem 1.1. □

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