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On winding numbers of ordinary polytopes and multi-polytopes

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폴리토프와 다중 폴리토프의 회전수에 관한 연구

문 경 연
지도교수 : 김 진 홍
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하토리와 마수다에 의해 발견된 다중 팬은 토러스 다양체와 기하학적으로 깊은 관련을 가지고 있다. 다중 팬은 보통의 팬과 여러 가지 다른 성질을 가지고 있는 반면, 또한 비슷한 중요한 성질도 함께 공유하고 있다. 본 논문에서는 하토리와 마수다의 결과를 확장하여, 다중 팬과 다중 폴리토프에 대응하는 회전수를 정의하고 다중 폴리토프가 보통 폴리토프가 될 필요충분조건을 회전수를 이용하여 찾았다.

좀 더 구체적으로, N를 n 차원 격자라 하고 V를 $N\otimes R$이라 할 때, V의 쌍대공간 V^*에 있는 아핀 초평면 F_i의 합집합 $\bigcup_{i=1}^{d} F_i$의 여집합 $V^* - \bigcup_{i=1}^{d} F_i$를 정의역으로 갖는 회전수 WN_P는 정수로 다음과 같이 정의된다.

$$\psi_*([\Delta]) = WN_p(u)[M_R - \{u\}]$$

본 논문에서 다중 폴리토프 P가 보통의 폴리토프이면 회전수 WN_P는 다중 폴리토프의 내부에서 1의 값을 갖고 외부에서는 0의 값을 가지며 그 역도 성립함을 월클로싱 공식(wall crossing formula)을 이용하여 증명하였다.

Chapter 1

Introduction

Our main aim of this thesis is to characterize the properties of a multi-fan and its associated multi-polytope in terms of the so-called winding numbers. A multi-fan is a generalization of an ordinary fan which has been introduced by Hattori and Masuda in the [4]. Similar but more restricted notions were previously introduced by Karshon and Tolman in [5] and also by Khovanskii and Pukhlikov in [6]. It was called a twisted polytope by Karshon and Tolman, and a virtual polytope by Khovanskii and Pukhlikov.

One importance of the notion of a multi-fan lies in the fact that we can associate a multi-polytope of dimension n to a torus manifold of dimension $2n$ with an effective action of the n-dimensional torus T^n. So this correspondence is at least surjective, but its injectivity is not yet obvious. However, if we restrict our attention to the class of ordinary polytopes, then it is known that this correspondence is bijective, up to certain equivalence (refer to [2, 3, 8]).

One crucial difference between a multi-fan and an ordinary fan is that in case of a multi-fan the maximal cones can overlap several times but also appears repeatedly. So we need to keep track of the occurrence of the maximal cones by using the notion of the weight function. Associated to a given multi-polytope $\mathcal{P}$, Hattori and Masuda have also defined a function $\mathrm{DH}_{\mathcal{P}}$, called the *Duistermaat-Heckman function*, which is completely analogous to the case of an ordinary polytope.

As expected, it turns out that the Duistermaat-Heckman function also reflects the important property of a multi-polytope that some maximal cones can appear repeatedly and also overlap several times. Moreover, it is easy to see that the Duistermaat-Heckman function of an ordinary polytope P has the value equal to one for each point in the interior of P and vanishes outside of it. This calculation indicates some strong possibility that one may detect an ordinary polytope in terms of the Duistermaat-Heckman function. This is a starting point of the thesis [1], which shows the validity of our expectation in detail.

In the paper [4], Hattori and Masuda have also introduced the function $\mathrm{WN}_{\mathcal{P}}$, called the *winding number*, of a multi-polytope $\mathcal{P}$. In some sense, the winding number $\mathrm{WN}_{\mathcal{P}}$ has more geometric meanings than the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$. Namely, one can figure out more easily how many times certain maximal cones overlap and appear repeatedly. Furthermore, it has been shown in [4] that the Duistermaat-Heckman function coincides with the winding number. Hence, it follows from a result of [1] that the winding number WN_P of an ordinary polytope P has the value equal to one for each point in the interior of P and vanishes outside of it. The aim of this thesis is to give a direct proof of this fact.

In order to describe our main result, we first need to set up some terminology and notation, in more detail. Indeed, let N be a lattice of rank n which is isomorphic to $\mathbb{Z}^n$, and let M be the dual lattice $\mathrm{Hom}(N, \mathbb{Z})$. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and let $M_{\mathbb{R}} = \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$. A multi-polytope $\mathcal{P}$ is a pair $(\Delta, \mathcal{F})$ of an n-dimensional multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ and an arrangement $\mathcal{F} = \{F_i\}$ of affine hyperplanes F_i in the dual space $M_{\mathbb{R}}$ with the same index set as the set of one-dimensional cones in Δ.

The primary aim of this thesis is to give some criteria for a multi-fan to be an ordinary fan in terms of the winding numbers. As said before, a cone in a multi-polytope can appear more than one time with different indices. If a maximal cone in $C(\Sigma)$ appears more than one time with different indices, then we shall assume that the weight function w has the same value for each repeated maximal cone. With these understood, our main result is

Theorem 1.1. *Let $\Delta = (\Sigma, C, \omega^{\pm})$ be a complete and simplicial multi-fan such that the weight function $w = w^{+} - w^{-}$ is not zero, and $\mathcal{P}$ be its associated multi-polytope. As a geometric realization, then Δ is an ordinary fan if and only if the winding number $\mathrm{WN}_{\mathcal{P}}$ defined over $V := M_{\mathbb{R}} \setminus \bigcup_{i=1}^{d} F_i$ satisfies*

$$\mathrm{WN}_{\mathcal{P}}(u) = \begin{cases} 1, & u \in \mathcal{P} \cap V, \\ 0, & \textit{otherwise} \end{cases}$$

We organize this thesis, as follows. In Chapter 2, we collect some basic material about ordinary fans and set up notations necessary for the rest of this thesis. Chapter 3 explains how to define a multi-fan as well as some easy exmaples, in detail. In Chapter 4, we compare two important notions of this thesis: an ordinary polytope and a multi-polytope. In the same chapter, we also provide many examples in order to help readers to understand certain similarities and differences between ordinary polytopes and multi-polytopes. In Chapter 5, we briefly explain how to define the Duistermaat-Heckman function of a multi-polytope. This chapter is necessary only for the proof of Theorem 1.1, but we do not pursue it in this thesis, in more detail. Finally, Chapter 6 is devoted to giving the proof of Theorem 1.1.

Chapter 2

Ordinary fans

The aim of this chapter is to collect basic material about ordinary fans and set up notations necessary for the rest of this thesis.

To do so, let N be a lattice of rank n which is isomorphic to $\mathbb{Z}^n$. We denote the real vector space $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* with apex at the origin if there exists a finite number of vectors $v_1, \ldots, v_m$ in N such that

$$\sigma = \{r_1 v_1 + \cdots + r_m v_m \mid r_i \in \mathbb{R} \text{ and } r_i \geq 0 \text{ for all } i\},$$

and

$$\sigma \cap (-\sigma) = \{0\}.$$

Here what we mean by rational is that it is generated by vectors in the lattice N, and a cone is said to be *strong* if it contains no line through the origin. As usual, we often call a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ simply a *cone* in N.

The dimension $\dim \sigma$ of a cone σ is defined to be the dimension of the linear spanned by vectors in σ. A subset τ of σ is called a face of σ if there is a linear function

$$l : N_{\mathbb{R}} \longrightarrow \mathbb{R}$$

such that l takes nonnegative values on σ on $\tau = l^{-1}(0) \cap \sigma$. A cone is regarded as a face of itself, while others are called *proper* faces.

Definition 2.1. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Δ is also a cone in Δ,

(2) The intersection of two cones in Δ is a face of each.

Definition 2.2. A fan Δ is said to be *complete* if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$.

A cone is called *simplicial* if it is generated by linearly independent vectors. If the generating vector can be taken as a part of a basis of N, then the cone is called *non-singular*.

Definition 2.3. A fan Δ is said to be *simplicial (resp. non-singular)* if every cone in Δ is *simplicial (resp. non-singular)*.

Let us denote by $\mathrm{Cone}\,(N)$ the set of all cones in N. An ordinary fan is a subset of $\mathrm{Cone}\,(N)$. The set $\mathrm{Cone}\,(N)$ has a partial ordering $\preceq$ defined by : $\tau \preceq \nu$ if τ is a face of ν, and $\tau \prec \nu$ if and only if τ is a proper face of ν. The cone $\{0\}$ consisting of the origin is the unique minimum element if $\mathrm{Cone}(N)$.

On the other hand, let Σ be a partial ordering finite set with a unique minimum element. We denote the strict partial ordering by $<$ and the minimum element by $*$. An example of Σ used later is an abstract simplicial set with an empty set added as a member, which we call an augmented simplicial set. In this case the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1)-simplex.

Suppose that there is a map

$$C : \Sigma \to \mathrm{Cone}\,(N)$$

such that

(1) $C(*) = \{0\}$,

(2) If $I < J$ for $I, J \in \Sigma$, then $C(I) < C(J)$,

(3) For any $J \in \Sigma$ the map C restricted on $\{I \in \Sigma \mid I \leq J\}$ is an isomorphism of ordered sets onto $\{K \in \mathrm{Cone}(N) \mid K \leq C(J)\}$.

For an integer m such that $0 \leq m \leq n$ we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \dim C(I) = m\}$$

One can check that $\Sigma^{(m)}$ does not depend on C. When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m, namely I is an $(m-1)$-simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notation $|I|$ for m when $I \in \Sigma^{(m)}$. The image $C(\Sigma)$ is a finite set of cones in N. We may think of a pair (Σ, C) as a set of cones in N labeled by the ordered set Σ. Cones in an ordinary fan intersect only at their faces, but cones in $C(\Sigma)$ may overlap, even the same cone may appear repeatedly with different labels. The pair (Σ, C) is almost what we call a multi-fan, but we incorporate a pair of weight functions on cones in $C(\Sigma)$ of the highest dimension $n = \mathrm{rank}\, N$. More precisely, we consider two functions

$$\omega^{\pm} : \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

We assume that $\omega^{+}(I) > 0$ or $\omega^{-}(I) > 0$ for every $I \in \Sigma^{(m)}$.

These two functions $\omega^{\pm}$ are called *weight functions*, and have its origin from geometry. In fact, if M is a torus manifold of dimension $2n$ and if $M_{i_1}, \cdots, M_{i_n}$ are characteristic submanifolds such that their intersection contains at least one T^n-fixed point, then the intersection

$$M_I = \bigcap_{i_j \in I} M_{i_j}, \ I = \{i_1, \ldots, i_n\}$$

consists of a finite number of T^n-fixed points. At each fixed point $p \in M_I$ the tangent space T_p has two orientations; one is endowed by the orientation of M and the other comes from the intersection of the oriented sub-manifolds M_{i_ν}. If we denote the ratio of the above two orientations by ϵ_p, then the number $\omega^+(I)$ can be thought of the number of points $p \in M_I$ with $\epsilon_p = +1$, and similarly for $\omega^-(I)$. In this thesis, only the difference

$$\omega = \omega^+ - \omega^-$$

plays an important role, as we can see in later chapters.

Chapter 3

Multi-fans

In this chapter, we briefly recall some basic terminology and definitions necessary for later discussion. Most of the material of this chapter is taken from the paper [4].

We begin with the definition of a multi-fan, as follows.

Definition 3.1. We call a triple $\Delta := (\Sigma, C, \omega^{\pm})$ a multi-fan in N. We define the dimension of Δ to be the rank of N (or the dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of Cone(N), one can view it as a multi-fan by taking $\Sigma = \Delta, C =$ the inclusion map, $\omega^{+} = 1$, and $\omega^{-} = 0$. In a similar way as in the case of ordinary fans, we say that a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is simplicial (resp. non-singular) if every cone in $C(\Sigma)$ is simplicial (resp. non-singular). The following lemma holds.

Lemma 3.2. *A multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

For the case of multi-fans, the definition of completeness is more involved than that of an ordinary fan. As in the case of ordinary fans, the completeness of a multi-fan might be defined to be the union of cones in $C(\Sigma)$ covers the entire space $N_{\mathbb{R}}$, but it turns out that it is not so useful.

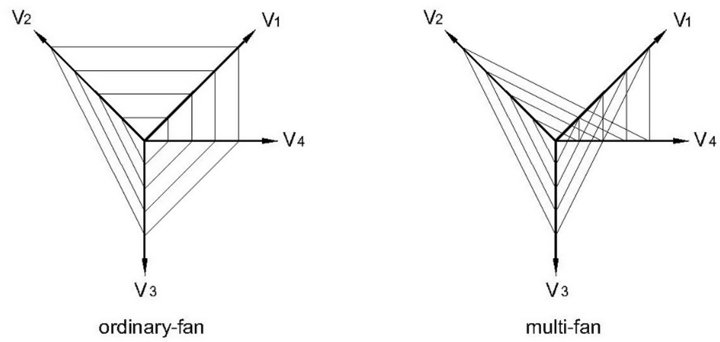

Figure 3.1: Examples of an ordinary fan and a multi-fan

In order to obtain more consistent and useful definition of the completeness of a multi-fan, we first need to introduce an intermediate notion of pre-completeness, as follows. A vector $v \in N_{\mathbb{R}}$ is called generic if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n. For a generic vector v, we set $d_v = \Sigma_{v \in C(I)} \omega(I)$, where the sum is understood to be zero if there is no such I.

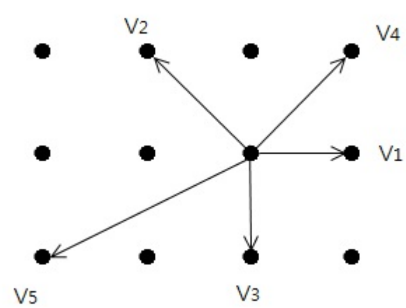

Figure 3.2: A complete non-singular multi-fan

Example 3.3. Here is an example of a complete non-singular multi-fan of degree two. Let $v_1, \ldots, v_5$ be integral vectors shown in Figure 3.2, where the dots denote lattice points. The vectors are rotating around the origin twice in counterclockwise. We take

$$\Sigma = \{\phi, \{1\}, \dots, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\},$$

define $C : \Sigma \to \mathrm{Cone}\,(N)$ by

$$C(\{i\}) = \text{the cone spanned by } v_i,$$
$$C(\{i, i+1\}) = \text{the cone spanned by } v_i \text{ and } v_{i+1},$$

where $i = 1, \dots, 5$ and 6 is understood to be 1, and take $\omega^{\pm}$ such that $\omega = 1$ on every two dimensional cone. Then $\Delta = (\Sigma, C, \omega^{\pm})$ is a complete non-singular two-dimensional multi-fan with $\deg\,(\Delta) = 2$.

With this understood, we are ready to give a definition of a complete multi-fan.

Definition 3.4. We call a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ of dimension n pre-complete if $\Sigma^{(n)} \neq 0$ and the integer d_v is independent of the choice of generic vectors v. We call this integer the degree of Δ and denote it by $\deg\,(\Delta)$.

Note that for an ordinary fan, pre-completeness is the same as completeness.

Assume $n = \dim\,\Delta > 1$. For each $\{i\} \in \Sigma^{(1)}$, the projected multi-fan $\Delta_{\{i\}} = (\Sigma_{\{i\}}, C_{\{i\}}, \omega^{\pm}_{\{i\}})$, which we abbreviate as $\Delta_i = (\Sigma_i, C_i, \omega^{\pm}_i)$, is defined on the quotient vector space $V \backslash V_i$ of V by the one-dimensional subspace V_i spanned by v_i.

To define the compeleteness for a multi-fan $\triangle$, we need to define a projected multi-fan with respect to an element in Σ. We do it as follows. For each $K \in \Sigma$, we set

$$\Sigma_k := \{J \in \Sigma \;\, K \leq J\}.$$

It inherits the partial ordering from Σ, and K is the unique minimum element in Σ_K. A map

$$C_K : \Sigma_k \to \mathrm{Cone}(N^{C(K)})$$

sending $J \in \Sigma_K$ to the cone $C(J)$ projected on $(N^{C(K)})_{\mathbb{R}}$ satisfies the three properties above required for C. We difine two functions

$$\omega_K{}^{\pm} : \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

to be the restrictons of $\omega^{\pm}$ to $\Sigma_K^{(n-|K|)}$. The triple $\triangle_K := (\Sigma_K, C_K, \omega_K{}^{\pm})$ is multi-fan in $N^{C(K)}$, and this is the desired *projected multi-fan* with respect to $K \in \Sigma$. When $\triangle$ is an ordinary fan, this definition agrees with the previous one.

Definition 3.5. A pre-compelete multi-fan $\triangle = (\Sigma, C, \omega^{\pm})$ is said to be *complete* if the projected multi-fan $\triangle_K$ is pre-complete for any $K \in \Sigma$.

Example 3.6. Here is an example of a multi-fan which is pre-complete but not complete. Let $v_1, \cdots, v_5$ be vectors shown in next Figure 3.3. We take

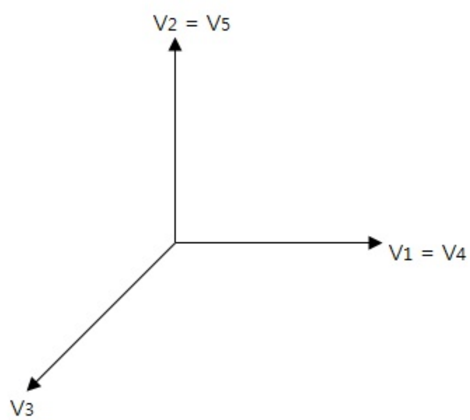

Figure 3.3: pre-complete

$$\Sigma = \{\phi, \{1\}, \ldots, \{5\}, \{1,2\}, \{2,3\}, \{3,1\}, \{4,5\}\},$$

define $C : \Sigma \to \mathrm{Cone}(N)$ as in Example 3.3 and take $\omega^{\pm}$ such that

$$\omega(\{1,2\}) = 2, \omega(\{2,3\}) = 1, \omega(\{3,1\}) = 1, \omega(\{4,5\}) = -1.$$

Then $\Delta = (\Sigma, C, \omega^{\pm})$ is a two-dimensional multi-fan which is pre-complete (in fact, $\deg(\Delta) = 1$) but not complete because the projected multi-fan $\Delta_{\{i\}}$ for $i \neq 3$ is not pre-complete.

Chapter 4

Polytopes and multi-polytopes

The aim of this chapter is to collect some fundamental facts regarding ordinary polytope and multi-polytopes. This chapter largely depends on the paper [4].

A convex polytope P in $V^* = \mathrm{Hom}\,(V, \mathbb{R})$ is the convex hull of a finite set of points in V^*. It is the intersection of a finite number of half space in V^* separated by affine hyperplanes, so that there are a finite number of nonzero vectors $v_1, \cdots, v_d$ in V and real numbers $c_1, \cdots, c_d$ such that

$$P = \{u \in V^* \mid \langle u, u_i \rangle \leq c_i \text{ for all } i\},$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between V^* and V.

A polytope gives rise to a multi-fan in this way, and note that a convex polytope gives rise to a complete fan. Conversely, we now start with a complete multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$. To do so, let $HP(V^*)$ be the set of all affine hyperplanes in V^*.

Definition 4.1. Let $\Delta = (\Sigma, C, \omega^{\pm})$ be a complete multi-fan and let $\mathcal{F} : \Sigma^{(1)} \to HP(V^*)$ be a map such that the affine hyperplane $\mathcal{F}(I)$ is perpendicular to the half line $C(I)$. That is, any element in $C(I)$ takes a constant on $\mathcal{F}(I)$. We call a pair $(\Delta, \mathcal{F})$ a multi-polytope, and denote it by $\mathcal{P}$. The dimension of a multi-polytope $\mathcal{P}$ is defined to be the dimension of the multi-fan Δ. We say that a multi-polytope $\mathcal{P}$ is *simple* if Δ is simplicial.

Example 4.2. A convex polytope determines a complete fan together with an arrangement of affine hyperplanes containing the facets of the polytope, so it uniquely determines a multi-polytope.

Example 4.3. Associated with the multi-fan in Example 3.3, one obtains the arrangement of lines with a suitable choice of the map $\mathcal{F}$. The pentagon produces the same arrangement of lines and can be viewed as a multi-polytope as explained in Example 4.2 above, but these two multi-polytopes are different because the underlying multi-fans are different; one is a multi-fan of degree two while the other is an ordinary fan. The reader will find a star-shaped figure in the former multi-polytope.

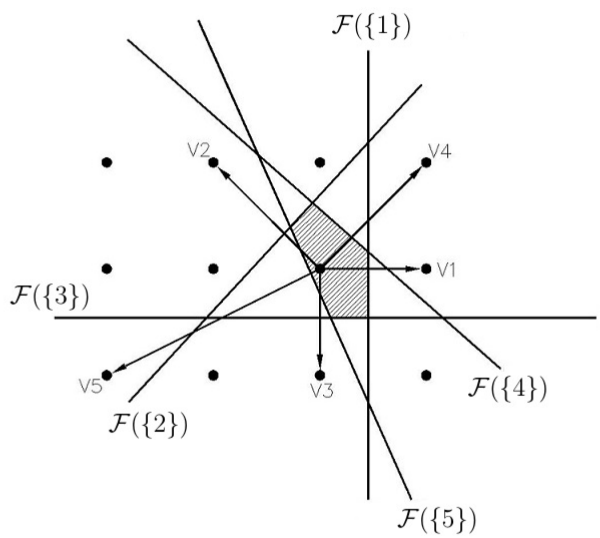

Figure 4.1: Multi-polytope for 3.3

Chapter 5

Duistermaat-Heckman functions

Before giving a proof of our main Theorem 1.1 in Chapter 6, in this chapter we review the definition of the Duistermaat-Heckman function associated to a complete and simplicial multi-polytope which is closely related to the notion of the winding number.

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ defines an arrangement of affine hyperplanes in V^*. Then we can associate with $\mathcal{P}$ a function on V^* minus the affine hyperplanes when $\mathcal{P}$ is simple. This function is locally constant and Guillemin-Lerman-Sternberg formula tells us that it agrees with the density function of a Duistermaat-Heckman measure when $\mathcal{P}$ arises from a moment map.

From now on, every multi-polytope $\mathcal{P}$ in this thesis will be assumed to be simple, so that the multi-fan $\Delta = (\Sigma, C, w^{\pm})$ is complete and simplicial unless otherwise stated. As before, we may assume that Σ consists of subsets of $\{1, \cdots, d\}$ and $\Sigma^{(1)} = \{\{1\}, \cdots, \{d\}\}$, and denote by v_i a nonzero vector in the one-dimensional cone $\mathrm{C}(\{i\})$. To simplify notation, we denote $\mathcal{F}(\{i\})$ by F_i and set

$$F_I := \bigcap_{i \in I} F_i \text{ for } I \in \Sigma.$$

Then F_I is an affine space of dimension $n - |I|$. In particular, if $|I| = n$ (i.e., $I \in \Sigma^{(n)}$), then F_I is a point, denoted by u_I.

Suppose now that $I \in \Sigma^{(n)}$. Then the set $\{v_i \mid i \in I\}$ forms a basis of V. Denote its dual basis of V^* by $\{u_i^I \mid i \in I\}$, i.e., $\langle u_i^I, v_j\rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Take a generic vector $v \in V$. Then $\langle u_i^I, v\rangle \neq 0$ for all $I \in \Sigma^{(n)}$ and $i \in I$. Set

$$(-1)^I := (-1)^{\#\{i\in I \mid \langle u_i^I, v_j\rangle > 0\}}, \quad \text{and} \quad (u_i^I)^+ := \begin{cases} u_i^I, & \text{if } \langle u_i^I, v_j\rangle > 0, \\ -u_i^I, & \text{if } \langle u_i^I, v_j\rangle < 0. \end{cases}$$

We denote by $C^*(I)^+$ the cone in V^* spanned by $(u_i^I)^+$'s $(i \in I)$ with apex at u_I, and by ϕ_I its characteristic function.

Definition 5.1. We define a function $\mathrm{DH}_{\mathcal{P}}$ on $V^* \backslash \bigcup_{i=1}^d F_i$ by

$$\mathrm{DH}_{\mathcal{P}} := \sum_{I \in \Sigma^{(n)}} (-1)^I \omega(I) \phi_I,$$

and call it the *Duistermaat-Heckman function* associated with $\mathcal{P}$.

Note that the function $\mathrm{DH}_{\mathcal{P}}$ depends on the choice of the generic vector $v \in V$. But it turns out that it is independent of v on $V^* \backslash \bigcup F_i$. This is the reason why we restricted the domain of the function to $V^* \backslash \bigcup F_i$.

In case of $\dim \mathcal{P} = 1$, it is more easier to see the independence of a generic vector v for the Duistermaat-Heckman function, as the following example shows.

Example 5.2. Suppose that $\dim \mathcal{P} = 1$, and identify V with $\mathbb{R}$. So V^* is also identified with $\mathbb{R}$. Let E be the subset of $\{1, \dots, d\}$ such that $i \in E$ if and only if $C(\{i\})$ is the half line consisting of nonnegative real numbers. Then the completeness of Δ means that

$$\sum_{i\in E} \omega(\{i\}) = \sum_{i \notin E} \omega(\{i\}) = \deg(\Delta). \tag{5.1}$$

Take a nonzero vector v. Since V^* is identified with $\mathbb{R}$, each affine hyperplane F_i is nothing but a real number. Suppose that v is toward the positive direction. Then

$$(-1)^{\{i\}} = \begin{cases} -1, & \text{if } i \in E, \\ 1, & \text{if } i \notin E, \end{cases} \tag{5.2}$$

and the support of the characteristic function $\phi_{\{i\}}$ is the half line given by

$$\{u \in \mathbb{R} \mid F_i \leq u\}.$$

Therefore

$$\mathrm{DH}_{\mathcal{P}}(u) = \sum_{i \in E \ \text{ s.t. } \ F_i < u} -w(\{i\}) + \sum_{i \notin E \ \text{ s.t. } \ F_i < u} w(\{i\}). \tag{5.3}$$

for $u \in \mathbb{R} \backslash \cup F_i$. If u is sufficiently small, then the sum above is empty, so it is zero. If u is sufficiently large, then the the sum is also zero by (5.1). Hence the support of the function $\mathrm{DH}_{\mathcal{P}}$ is bounded.

Now, suppose that v is toward the negative direction. Then $(-1)^{\{i\}}$ above is multiplied by -1 and the inequality $\leq$ above turns into $\geq$. Therefore, we have

$$\mathrm{DH}_{\mathcal{P}}(u) = \sum_{i \in E \ \text{ s.t. } \ u < F_i} w(\{i\}) + \sum_{i \notin E \ \text{ s.t. } \ u < F_i} (-w(\{i\})). \tag{5.4}$$

By subtracting the right hand side of (5.4) from that of (5.3), we can obtain

$$-\sum_{i \in E} w(\{i\}) + \sum_{i \notin E} w(\{i\}),$$

which is zero by (5.1). This shows that the function $\mathrm{DH}_{\mathcal{P}}$ is independent of v when $\dim \mathcal{P} = 1$.

Assume $n = \dim \Delta > 1$. For each $\{i\} \in \Sigma^{(1)}$, recall that the projected multi-fan $\Delta_{\{i\}} = (\Sigma_{\{i\}}, C_{\{i\}}, \omega^{\pm}_{\{i\}})$, denoted by $\Delta_i = (\Sigma_i, C_i, \omega^{\pm}_i)$, can be defined on the quotient vector space $V\backslash V_i$ of V by the one-dimensional subspace V_i spanned by v_i. Since Δ is complete and simplicial, so is Δ_i. We then identify the dual space $(V/V_i)^*$ with

$$(V^*)_i := \{u \in V^* | \langle u, v_i \rangle = 0\}$$

in a natural way. We choose an element $f_i \in F_i$ arbitrarily and translate F_i onto $(V^*)_i$ by $-f_i$. If $\{i, j\} \in \Sigma^{(2)}$, then F_j intersects F_i and their intersection will be translated into $(V^*)_i$ by $-f_i$. This observation leads us to consider the map

$$F_i : \Sigma_i \to \mathrm{HP}((V^*)_i)$$

sending $\{j\} \in \Sigma_i^{(1)}$ to $F_i \cap F_j$ translated by $-f_i$. The pair $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$ is a multi-polytope in $(V/V_i)^* \cong (V^*)_i$.

Let $I \in \Sigma^{(n)}$ such that $i \in I$. Since $\langle u_i^I, v_i \rangle = \delta_{ij}, u_i^I$ for $j \neq i$ is an element of $(V^*)_i$, which we also regard as an element of $(V/V_i)^*$ through the isomorphism $(V/V_i)^* \cong (V^*)_i$. We denote the projection image of the generic element $v \in V$ on V/V_i by $\bar{v}$. Then we have $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle$ for $j \neq i$, where u_j^I at the left-hand side is viewed as an element of $(V/V_i)^*$, while the one at the right-hand side is viewed as an element of $(V^*)_i$. Since $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle \neq 0$ for $j \neq i$, we use $\bar{v}$ to define $\mathrm{DH}_{\mathcal{P}_i}$.

The following lemma which is called a wall crossing formula plays an important role in the proof of Theorem 1.1 given in Chapter 6.

Theorem 5.3 (Wall crossing formula). *Let F be one of F_i's. Let u_α and u_β be elements in $V^* \backslash \bigcup_{i=1}^d F_i$ such that the segment from u_α to u_β intersects the wall F transversely at μ and does not intersect any other $F_j \neq F$. Then we have*

$$\mathrm{DH}_{\mathcal{P}_i}(u_\alpha) - \mathrm{DH}_{\mathcal{P}_i}(u_\beta) = \sum_{i : F_i = F} \mathrm{sign}\langle u_\beta - u_\alpha, v_i \rangle \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i).$$

Proof. For the sake of simplicity, we assume that there is only one i such thath $F_i = F$. We may assume that $\langle u_\beta - u_\alpha, v_i\rangle$ is positive without loss of generality. We can see one possible example in Figure 5.1.

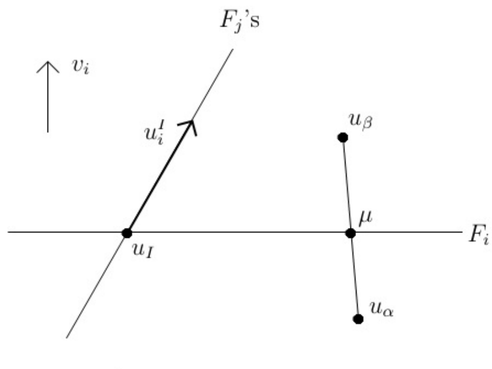

Figure 5.1: Wall crossing formula

It follows from the definition of $\mathrm{DH}_{\mathcal{P}}$ that the difference between $\mathrm{DH}(u_\alpha)$ and $\mathrm{DH}(u_\beta)$ arises from the cones $C^*(I)^+$'s for $I \in \Sigma^{(n)}$ such that $i \in I$ and $\langle u_I, v\rangle < \langle \mu, v\rangle$. In fact, one sees that

$$\mathrm{DH}(u_\alpha) + \sum_I \mathrm{sign}\langle u_i^I, v\rangle(-1)^I \omega(I)\phi_I(\mu) = \mathrm{DH}_{\mathcal{P}}(u_\beta),$$

where I runs over the elements as above. Since $\mathrm{sign}\langle u_i^I, v\rangle(-1)^I = -(-1)^{I\backslash\{i\}}$ and $\omega(I) = \omega_i(I\backslash\{i\})$, the equality above turns into

$$\mathrm{DH}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) \sum_I (-1)^{I\backslash\{i\}}\omega_i(I\backslash\{i\})\phi_I(\mu).$$

Here $\phi_I(\mu)$ may be viewed as the value at μ of the characteristic function of the cones in F_i with apex u_I spanned by $(u_j^I)^+$'s ($j \in I, j \neq i$). This show that the right-hand side at the equality above agrees with $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i)$, proving the theorem. □

Chapter 6

Proof of Theorem 1.1: winding numbers

The aim of this chapter is to give a proof of Theorem 1.1. To do so, as before we assume that the multi-polytope $\mathcal{P}$ with its associated multi-fan $\Delta = (\Sigma, \mathcal{F}, w^{\pm})$ is simple and that Σ is an augmented simplicial set consisting of subsets of $\{1, 2, \cdots, d\}$. As in the Duistermaat-Heckman functions, the winding numbers we deal with in this chapter is a locally constant function on $M_{\mathbb{R}} \setminus \bigcup_{i=1}^{d} F_i$.

Before starting the proof, we first set up some basic notations and definitions. Choose an orientation on $N_{\mathbb{R}}$, and fix it once and for all. Let $I = \{i_1, i_2, \cdots, i_n\} \in \Sigma^{(n)}$. Then I is said to have a *positive orientation* if the ordered basis $\{v_{i_1}, v_{i_2}, \cdots, v_{i_n}\}$ gives the chosen orientation of V, and is said to have a *negative orientation*, otherwise. We also define

$$\langle I \rangle := \begin{cases} \langle i_1, i_2, \cdots, i_n \rangle, & \text{if } \langle i_1, i_2, \cdots, i_n \rangle \text{ has a positive orientation,} \\ -\langle i_1, i_2, \cdots, i_n \rangle, & \text{if } \langle i_1, i_2, \cdots, i_n \rangle \text{ has a negative orientation.} \end{cases}$$

It can be shown that the completeness of Δ implies that

$$\sum_{I \in \Sigma^{(n)}} w(I) \langle I \rangle$$

is a cycle in the chain complex of the simplicial set Σ. In fact, the following lemma holds.

Lemma 6.1. *If a simplicial multi-fan Δ is complete, then $\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle$ is a cycle, and, moreover, the converse also holds.*

Proof. To prove it, we first need to show that

$$\partial\left(\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle\right) = 0.$$

To do so, write

$$\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle = \sum_{i=1}^{d}\sum_{i\in I\in\Sigma^{(n)}} w(I)\langle I\rangle.$$

For each $I\in\Sigma^{(n)}$ with $i\in I$, we also write $\langle I\rangle$, as follows.

$$\langle I\rangle = \varepsilon\langle i, j_1, j_2, \cdots, j_{n-1}\rangle,$$

where $\varepsilon = 1$ or -1. Then $\varepsilon\langle j_1, j_2, \cdots, j_{n-1}\rangle$ defines an oriented $(n-2)$-simplex $\langle I\backslash\{i\}\rangle$ in $\Sigma_i^{(n-1)}$. Thus we have

(6.1)

$$\begin{aligned}
\partial\left(\sum_{I\in\Sigma^{(n)}} w(I)\langle I\rangle\right) &= \sum_{i=1}^{d}\partial\left(\sum_{i\in I\in\Sigma^{(n)}} w(I)\langle I\rangle\right)\\
&= \sum_{i=1}^{d}\sum_{J\in\Sigma_i^{(n-1)}} w_i(J)\langle J\rangle, \quad w_i(J) = w(I) \text{ and } J = I\backslash\{i\},\\
&= \sum_{J\in\Sigma^{(n-1)}}\left(\sum_{I\geq J \text{ with } I\in\Sigma^{(n)}} w(I)\right)\langle J\rangle\\
&= \sum_{J\in\Sigma^{(n-1)}}\left(\sum_{I\geq J \text{ with } I\in\Sigma^{(n)}, v\in C(I)} w(I)\right.\\
&\quad \left. + \sum_{I\geq J \text{ with } I\in\Sigma^{(n)}, v\notin C(I)} w(I)\right)\langle J\rangle.
\end{aligned}$$

Here v is a suitable chosen generic vector. Now, note that the completeness of Δ implies that

$$\deg(\Delta) = \sum_{I \geq J \text{ with } I \in \Sigma^{(n)}, v \in C(I)} w(I) = - \sum_{I \geq J \text{ with } I \in \Sigma^{(n)}, v \notin C(I)} w(I),$$

since Δ_J is pre-complete for each $J \in \Sigma^{(n-1)}$. Thus it follows from (6.1) that we have

$$\partial \left(\sum_{I \in \Sigma^{(n)}} w(I) \langle I \rangle \right) = 0.$$

Conversely, since $\sum_{I \in \Sigma^{(n)}} w(I)\langle I \rangle$ is a cycle, the equation (6.1) implies that for each $J \in \Sigma^{(n-1)}$

$$\sum_{I \geq J \text{ with } I \in \Sigma^{(n)}, v \in C(I)} w(I) = - \sum_{I \geq J \text{ with } I \in \Sigma^{(n)}, v \notin C(I)} w(I).$$

But this implies that for each $J \in \Sigma^{(n-1)}$ the projected multi-fan $(\Delta_J, \Sigma_J, w_J^{\pm})$ is pre-complete. Therefore, Δ is also complete (refer to [4], Section 2).

This completes the proof of Lemma 6.1. $\square$

Definition 6.2. We shall denote by $[\Delta]$ the homology class that the cycle $\sum_{I \in \Sigma^{(n)}} w(I)\langle I \rangle$ defines in the reduced homology $\tilde{H}_{n-1}(\Sigma; \mathbb{Z})$.

Let S be the realization of the first barycentric subdivision of Σ. For each $i \in \{1, 2, \cdots, d\}$, we denote by S_i the union of simplices in S which contains the vertex $\{i\}$, and let $S_I = \cap_{i \in I} S_i$ for $I \in \Sigma$. Note that the boundary ∂S_i of S_i can be identified with the realization of the first barycentric subdivision of Σ_i, where Σ_i is the augmented simplicial set of the projected multi-fan $\Delta_i = (\Sigma_i, C_i, w_i^{\pm})$ in $M_{\mathbb{R}}/(M_{\mathbb{R}})_i$. Then, as before the cycle $[\Delta_i]$ defines an element in $\tilde{H}_{n-2}(\Sigma_i, \mathbb{Z}) = \tilde{H}_{n-2}(\partial S_i; \mathbb{Z})$ with respect to the compatible orientation.

The following lemma holds ([4], Lemma 6.1).

Lemma 6.3. *Under the compositions of the following maps*

$$\tilde{H}_{n-1}(\Sigma;\mathbb{Z}) \xrightarrow{i_*} H_{n-1}(S, S\backslash S_i^\circ;\mathbb{Z}) \cong H_{n-1}(S_i, \partial S_i;\mathbb{Z}) \xrightarrow{\partial} \tilde{H}_{n-2}(\partial S_i;\mathbb{Z}),$$

the $(n-1)$*-cycle* $[\Delta]$ *is mapped to the* $(n-2)$*-cycle* $[\Delta_i]$*.*

We also have the following lemma ([4], Lemma 6.2).

Lemma 6.4. *The following statements hold.*

(a) *The multi-polytope* $\mathcal{P}$ *gives rise to a continuous map*

$$\psi : S \to \cup_{i=1}^d F_i \subset M_{\mathbb{R}}$$

under which S_I *is mapped to* F_I *for each* $I \in \Sigma$*.*

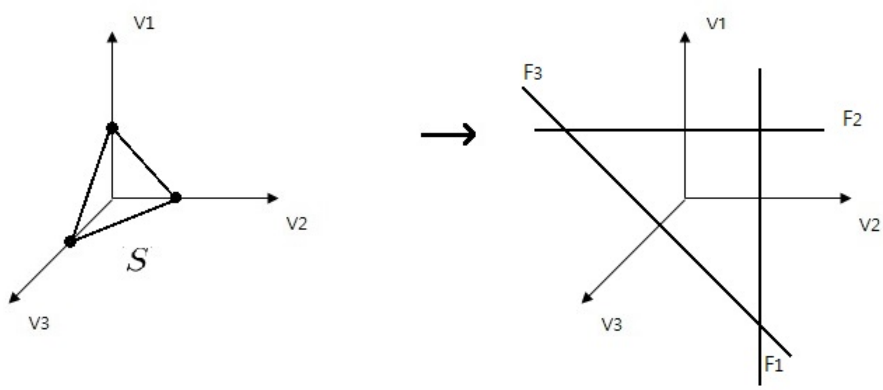

Figure 6.1

(b) *The map* ψ *induces a homomorphism*

$$\psi_* : \tilde{H}_{n-1}(S;\mathbb{Z}) \cong \tilde{H}_{n-1}(\Sigma;\mathbb{Z}) \to \tilde{H}_{n-1}(M_{\mathbb{R}} - \{u\};\mathbb{Z})$$

for each $u \in M_{\mathbb{R}} \backslash \cup_{i=1}^d F_i$*.*

We shall denote by $[M_{\mathbb{R}} - \{u\}]$ the fundamental class in $\tilde{H}_{n-1}(M_{\mathbb{R}} - \{u\}; \mathbb{Z})$ for each $u \in M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$.

Definition 6.5. For each $u \in M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$, we define an integer $\mathrm{WN}_{\mathcal{P}}(u)$ by

$$\psi_*([\Delta]) = \mathrm{WN}_{\mathcal{P}}(u)[M_{\mathbb{R}} - \{u\}],$$

and $\mathrm{WN}_{\mathcal{P}}(u)$ is called the *winding number* of the multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ around u.

Remark 6.6. (a) If u is an element in one of the unbounded regions of $M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$, then $\psi_*([\Delta])$ is homologous to zero. Thus the winding number $\mathrm{WN}_{\mathcal{P}}(u)$ is always equal to zero.

(b) $\mathrm{WN}_{\mathcal{P}}(u)$ is a locally constant function on $M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$, since $[M_{\mathbb{R}} - \{u_0\}]$ is homologous to $[M_{\mathbb{R}} - \{u_1\}]$, when u_0 and u_1 lie on the same component of $M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$.

(c) $\mathrm{WN}_{\mathcal{P}}(u)$ is independent of the choice of an orientation of V, since reversing the orientation of V changes the fundamental classes $[\Delta]$ and $[M_{\mathbb{R}} - \{u\}]$ simultaneously by $-[\Delta]$ and $-[M_{\mathbb{R}} - \{u\}]$.

We orient F_i so that the juxtaposition of a normal vector of F_i to F_i is oriented positively. Then ψ maps a pair $(S_i, \partial S_i)$ into a pair $(F_i, F_i - \{u\})$ for any $u \in F_i - \cup_{j \in \Sigma_i^{(1)}} F_j$. Moreover, when we identify F_i with $(M_{\mathbb{R}})_i$ through the translation by $-f_i$, up to homotopy the map $\psi|_{\partial S_i}$ agrees with the map constructed from the projected multi-polytope $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$. Hence we have

$$\psi_*([\Delta_i]) = \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i)[F_i - \{u\}],$$

where ν is an intersection point of the line segment γ staring from $u \in V$ with F_i, if it is not empty. Then the following lemma holds ([4], Lemma 6.4).

Lemma 6.7. *For each $u \in V := M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$, we have*

$$\mathrm{WN}_{\mathcal{P}}(u) = \sum_{i \text{ with } F_i \cap \gamma \neq \emptyset} \mathrm{sign}\langle \gamma, v_i \rangle \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i).$$

By using Lemma 6.7, it is immediate to obtain a wall crossing formula for the winding numbers, as in the case of the Duistermaat-Heckman functions.

Theorem 6.8. *Let F denote one of the hyperplanes F_i. Let u_α and u_β be two elements in $M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$ such that the segment γ from u_α to u_β intersects the wall F transversely at μ, and does not intersect any other $F_j \neq F$. Then, we have*

$$\mathrm{WN}_{\mathcal{P}}(u_\alpha) - \mathrm{WN}_{\mathcal{P}}(u_\beta) = \sum_{k \text{ with } F_k \cap \gamma \neq \emptyset} \mathrm{sign}\langle u_\beta - u_\alpha, v_k \rangle \mathrm{WN}_{\mathcal{P}_i}(\mu - f_k).$$

Proof. Let γ_α (resp. γ_β) be a line segment starting from u_α (resp. u_β) such that the ending point of γ_α coincides with that of γ_β. Then we may take $\gamma = \gamma_\alpha - \gamma_\beta$, and so it follows from Lemma 6.7 that

$$\begin{aligned} \mathrm{WN}_{\mathcal{P}}(u_\alpha) - \mathrm{WN}_{\mathcal{P}}(u_\beta) &= \sum_{i \text{ with } F_i \cap \gamma_\alpha \neq \emptyset} \mathrm{sign}\langle \gamma_\alpha, v_i \rangle \mathrm{WN}_{\mathcal{P}_i}(\mu_i - f_i) \\ &\quad - \sum_{j \text{ with } F_j \cap \gamma_\beta \neq \emptyset} \mathrm{sign}\langle \gamma_\beta, v_i \rangle \mathrm{WN}_{\mathcal{P}_i}(\mu_j - f_i) \\ &= \sum_{k \text{ with } F_k \cap \gamma \neq \emptyset} \mathrm{sign}\langle u_\beta - u_\alpha, v_k \rangle \mathrm{WN}_{\mathcal{P}_k}(\mu - f_i), \end{aligned}$$

as required. □

In fact, it turns out that the winding number $\mathrm{WN}_{\mathcal{P}}(u)$ coincides with the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}(u)$ for each $u \in M_{\mathbb{R}} \backslash \cup_{i=1}^{d} F_i$, as follows ([4], Theorem 6.6).

Theorem 6.9. *For any multi-polytope $\mathcal{P}$, we have $\mathrm{DH}_{\mathcal{P}} = \mathrm{WN}_{\mathcal{P}}$.*

The aim of this section is to give a direct proof of the following theorem without using Theorem 6.9.

Theorem 6.10. *Let Δ be a complete and simplicial multi-fan, and let $\mathcal{P}$ be its associated multi-polytope. Then $\mathcal{P}$ is an ordinary polytope if and only if the winding number* $\mathrm{WN}_{\mathcal{P}}$ *defined over* $V := M_{\mathbb{R}} \setminus \cup_{i=1}^{d} F_i$ *satisfies*

$$\mathrm{WN}_{\mathcal{P}}(u) = \begin{cases} 1, & u \in \mathcal{P} \cap V, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To prove it, assume first that the winding number $\mathrm{WN}_{\mathcal{P}}$ defined over $V = M_{\mathbb{R}} \setminus \cup_{i=1}^{d} F_i$ satisfies

$$\mathrm{WN}_{\mathcal{P}}(u) = \begin{cases} 1, & u \in \mathcal{P} \cap V, \\ 0, & \text{otherwise.} \end{cases}$$

Then we shall show that the multi-polytope $\mathcal{P}$ is a geometric realization of an ordinary polytope by the mathematical induction on $\dim \mathcal{P}$. This proof is similar to that of Theorem 1.1.

So assume that $\dim \mathcal{P} = 1$. As in the case of Duistermaat-Heckman functions, we identify $N_{\mathbb{R}}$ with $\mathbb{R}$, so that $M_{\mathbb{R}}$ is also identified with $\mathbb{R}$. Assume also that $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ have the standard orientations. Let E be a subset of $\{1, 2, \cdots, d\}$ such that v_i gives the orientation if and only if $i \in E$. Note now that

$$\begin{aligned} [\Delta] &= \sum_{I \in \Sigma^{(1)}} w(I)\langle I \rangle = \sum_{i \in E} w(\{i\})\langle i \rangle - \sum_{i \notin E} w(\{i\})\langle i \rangle \\ &= -\sum_{i=1}^{d} (-1)^{\{i\}} w(\{i\})\langle i \rangle. \end{aligned} \tag{6.2}$$

Recall that the completeness of Δ implies that

$$\sum_{i=1}^{d} (-1)^{\{i\}} w(\{i\}) = 0, \tag{6.3}$$

or, equivalently

$$\deg(\Delta) = \sum_{i \in E} w(\{i\}) = \sum_{i \notin E} w(\{i\}).$$

As before, let us assume that

$$E = \{j_1, j_2, \cdots, j_l\},$$
$$E^c = \{1, 2, \cdots, d\} - E = \{i_1, i_2, \cdots, i_k\}$$

such that

$$F_{j_1} < F_{j_2} < \cdots < F_{j_l}, \; F_{i_1} < F_{i_2} < \cdots < F_{i_k}, \text{ and } F_{i_k} < 0 < F_{j_1}.$$

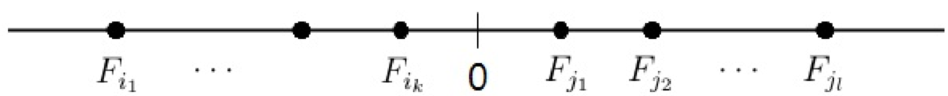

Figure 6.2

Then it is easy to see that $S_i = \{i\}$ and $\psi(\{i\}) = F_i$.

Next we claim that

$$\text{(6.4)} \qquad \mathrm{WN}_{\mathcal{P}}(u) = \sum_{F_i < u} (-1)^{\{i\}} w(\{i\}) = \sum_{F_i > u} (-1)^{\{i\}} w(\{i\}).$$

Indeed, for any $j \in \{1, 2, \cdots, d\}$, from the equations (6.2) and (6.3) we may write

$$[\Delta] = \sum_{i=1}^{d} (-1)^{\{i\}} w(\{i\})(\langle j \rangle - \langle i \rangle).$$

Thus we have

$$\begin{aligned} \psi_*([\Delta]) &= \sum_{i=1}^{d} (-1)^{\{i\}} w(\{i\})(\psi_*(\langle j \rangle) - \psi_*(\langle i \rangle)) \\ &= \sum_{i=1}^{d} (-1)^{\{i\}} w(\{i\})([F_j] - [F_i]) \qquad \text{(6.5)} \\ &= \mathrm{WN}_{\mathcal{P}}(u)[M_{\mathbb{R}} - \{u\}]. \end{aligned}$$

If u lies above F_{j_l} or lies below F_{i_1}, then $[F_j] = [F_i]$ in $M_{\mathbb{R}} - \{u\}$. Thus $\psi_*([\Delta]) = 0$, and so $\mathrm{WN}_{\mathcal{P}}(u) = 0$. On the other hand, if u lies between F_{i_1} and F_{i_2}, then we have

$$[F_{i_2}] = [F_{i_3}] = \cdots = [F_{i_k}] = [F_{j_1}] = \cdots = [F_{j_l}] \text{ in } M_{\mathbb{R}} - \{u\}.$$

Thus, by taking $j = j_l$ in the equation (6.5) we have

$$\psi_*([\Delta]) = (-1)^{i_1} w(\{i_1\})([F_{j_l}] - [F_{i_1}]) = \mathrm{WN}_{\mathcal{P}}(u)[M_{\mathbb{R}} - \{u\}].$$

That is, $\mathrm{WN}_{\mathcal{P}}(u) = (-1)^{i_1} w(\{i_1\})$. Similarly, if u lies between F_{i_1} and F_{j_l}, then we have

$$\mathrm{WN}_{\mathcal{P}}(u) = \sum_{F_i < u} (-1)^{\{i\}} w(\{i\}),$$

as claimed. The proof of other identity in (6.4) is similar, and will be left to the reader.

Next we are ready to finish the proof for the case of $\dim \mathcal{P} = 1$. So, assume that $l \geq 2$. Let u be an element between F_{j_1} and F_{j_2}. Then it follows from (6.4) that we have

$$1 = \mathrm{WN}_{\mathcal{P}}(u) = \sum_{s=1}^{k} w(\{i_s\}) - w(\{j_1\}),$$

where we use a generic vector v in the positive direction for the computations of the sign $(-1)^{\{i\}}$. On the other hand, if u lies between F_{i_k} and F_{j_1}, then we have

$$1 = \mathrm{WN}_{\mathcal{P}}(u) = \sum_{s=1}^{k} w(\{i_s\}).$$

Thus we should have $w(\{j_1\}) = 0$. Similarly, we can show that $w(\{j_t\}) = 0$ for all $1 \leq t \leq l$.

Using the second identity in (6.4) and a generic vector v in the negative direction, we can also show that $w(\{i_s\}) = 0$ for all $1 \leq s \leq k$. This implies that we would have $\mathrm{WN}_{\mathcal{P}}(u) = 0$ for all $u \in V$, provided that k or l is greater than or equal to 2. Therefore, $k = l = 1$. That is, $\mathcal{P}$ is actually a geometric realization of an ordinary simple convex polytope, as desired.

Next assume that $\mathcal{P} > 1$ and that the result holds for any multi-polytopes whose dimension is less than n. If there is an interior wall in the interior of the multi-polytope $\mathcal{P}$, then we let u_α and u_β be two interior points of $\mathcal{P}$ such that there is only one interior wall F_i which intersects the line segment between u_α and u_β. Then it follows from Theorem 6.8 that

$$\begin{aligned} 0 = 1 - 1 &= \mathrm{WN}_{\mathcal{P}}(u_\alpha) - \mathrm{WN}_{\mathcal{P}}(u_\beta) \\ &= \mathrm{sign}\langle u_\beta - u_\alpha, v_i \rangle \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i). \end{aligned}$$

Thus we have $\mathrm{WN}_{\mathcal{P}_i}(\mu - f_i) = 0$. On the other hand, by Lemma 6.3 and a diagram-chasing in the proof of [4], Lemma 6.4 we can see that

$$\begin{aligned} 0 &= \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i)[F_i - \{\mu\}] = \psi_*([\Delta_i]) \\ &= \psi_* \circ \partial \circ \mathcal{E} \circ i_*([\Delta]) = \partial \circ \psi_* \circ \mathcal{E} \circ i_*([\Delta]), \end{aligned}$$

where $\mathcal{E}$ denotes the excision between $H_{n-1}(S, S \backslash \cup_i S_i^\circ; \mathbb{Z})$ and $\oplus_i H_{n-1}(S_i, \partial S_i; \mathbb{Z})$. Since ∂ is an isomorphism, this implies that

$$\begin{aligned} 0 &= \psi_* \circ \mathcal{E} \circ i_*([\Delta]) \\ &= \mathrm{WN}_{\mathcal{P}}(u)[M_{\mathbb{R}} - \{u\}] = [M_{\mathbb{R}} - \{u\}]. \end{aligned}$$

This is a contradiction, which means that the multi-polytope $\mathcal{P}$ is actually a geometric realization of an ordinary polytope.

For the proof of the converse, we also use the mathematical induction on the dimension $\dim \mathcal{P}$. However, the case of $\dim \mathcal{P} = 1$ can be dealt with as above, so it will be left to the reader.

Next, assume that $\dim \mathcal{P} > 1$. Note that it can be shown as in the case of the Duistermaat-Heckman functions that the winding number $\mathrm{WN}_{\mathcal{P}}(u)$ vanishes for any u in an unbounded region of V. So it suffices to show that $\mathrm{WN}_{\mathcal{P}}(u)$ is equal to one for any u in the interior of $\mathcal{P}$. To do so, we want to use the wall crossing formula (Theorem 5.3). Indeed, let u_α (resp. u_β) be an element in the interior of $\mathcal{P}$ (resp. in an unbounded region of V) such that the segment from u_α to u_β intersects a wall F_i transversely at μ only once. Then it follows from Theorem 5.3 that

$$\begin{aligned}\mathrm{WN}_{\mathcal{P}}(u_\alpha) - \mathrm{WN}_{\mathcal{P}}(u_\beta) &= \mathrm{WN}_{\mathcal{P}}(u_\alpha)\\ &= \mathrm{sign}\langle u_\beta - u_\alpha, v_i\rangle \mathrm{WN}_{\mathcal{P}_i}(\mu - f_i)\\ &= 1,\end{aligned}$$

where we used $\mathrm{sign}\langle u_\beta - u_\alpha, v_i\rangle = 1$ and $\mathrm{WN}_{\mathcal{P}_i}(\mu - f_i) = 1$ by the mathematical induction.

This completes the proof of Theorem 6.10. □

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