2016년 8월
교육학석사(수학교육)학위논문

The Existence and the Completeness of Some Metrics on Lorentzian Warped Product Manifolds with Fiber Manifold of Class (B)

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엽다양체가 (B)형인 로렌쯔 휜곱다양체위의 거리의 존재성과 완비성

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이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

2016년 4월

조선대학교 교육대학원

수학교육전공

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김아용의 교육학 석사학위 논문을 인준함.

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CONTENTS

국 문 초 록

The Existence and the Completeness of Some Metrics on Lorentzian Warped Product Manifolds with Fiber Manifold of Class (B)

김 아 용
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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다. 연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

Kazdan and Warner [K.W.1, 2, 3]의 결과에 의하면 N 위의 함수 f가 N 위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있는 데 먼저

(A) N 위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 적당한 점에서 $f(x_0)<0$일 때이다. 특히 N 위에 negative constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(B) N 위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 항등적으로 $f\equiv 0$이거나 모든 점에서 $f(x)<0$인 경우이다. 특히 N 위에서 zero scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(C) N 위의 어떤 f라도 scalar curvature가 될 수 있는 적당한 Riemannian metric 이 존재하는 경우이다.

본 논문에서는 엽다양체 N이 (B)에 속하는 compact Riemannian manifold일 때, Riemannian warped product manifold인 $M=[a,\infty)\times_f N$ 위에 함수 $R(t,x)$가 적당한 조건을 만족하면 $R(t,x)$ 가 Riemannian warped product metric의 scalar curvature가 될 수 있는 warping function $f(x)$가 존재할 수 있음을 상해·하해 방법을 이용하여 증명하였다.

I. INTRODUCTION

In [L.1, 2], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function.

In this paper, we study the existence and the completeness of some metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner ([K.W.1, 2, 3]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three categories :

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [K.W.1, 2, 3], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold. In [L.1, 2], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [E.J.K.], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature. Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [E.J.K.], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [E.J.K.] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold of class (B), we discuss the method of using warped products to construct timelike or null future complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. And we prove the existence of warping functions on Lorentzian warped product manifolds and the completeness of the the resulting metrics with some prescribed scalar curvatures. These results of this paper are extensions of the results in [J.L.L.].

II. PRELIMINARIES

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. Let $\mathcal{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\Im(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

such that

(D1) $\nabla_V W$ is $\Im(M)$-linear in V,

(D2) $\nabla_V W$ is R-linear in W,

(D3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \Im(M)$,

(D4) $[V, W] = \nabla_V W - \nabla_W V$ and

(D5) $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$

for all $V, W, X \in \mathcal{X}(M)$.

If ∇ satisfies axioms (D1)$\sim$(D3), then $\nabla_V W$ is called the covariant derivative of W with respect to V for the connection ∇. If ∇ satisfies axioms (D1)$\sim$(D5), then ∇ is called the Levi-Civita connection of M, which is characterized by the $K\ddot{o}szul\ formula$ ([O.]).

A geodesic $c : (a, b) \to M$ is a smooth curve of M such that the tangent vector c' moves by parallel translation along c. In order words, c is a geodesic if

$$\nabla_{c'} c' = 0 \quad (\text{geodesic} \quad \text{equation}).$$

A pregeodesic is a smooth curve c which may be reparametrized to be a geodesic. Any parameter for which c is a geodesic is called an affine parameter. If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c'} c' = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U, (x^1, x^2, ..., x^n))$ be local coordinates on M and let $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ denote the natural basis with respect to these coordinates.

The connection coefficients Γ_{ij}^k of ∇ with respect to $(x^1, x^2, ..., x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j}\right) = \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad (\text{connection} \quad \text{coefficients}).$$

Using these coefficients, we may write equation as the system

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \quad \text{(geodesic equations in coordinates).}$$

Definition 2.2. The curvature tensor of the connection ∇ is a linear transformation valued tensor R in $\mathrm{Hom}(\mathcal{X}(M), \mathcal{X}(M))$ defined by :

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathcal{X}(M)$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well-known that $R(X, Y)Z$ at p depends only upon the values of X, Y and Z at p ([O.]).

If $\omega \in T_p^*(M)$ is a cotangent vector at p and $x, y, z \in T_p(M)$ are tangent vectors at p, then one defines

$$R(\omega, X, Y, Z) = (\omega, R(X, Y)Z) = \omega(R(X, Y)Z)$$

for X, Y and Z smooth vector fields extending x, y and z, respectively.

The curvature tensor R is a (1,3)-tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

Notice that $R(X,Y)Z = -R(Y,X)Z$, $R(\omega, X, Y, Z) = -R(\omega, Y, X, Z)$ and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \sum x^i \frac{\partial}{\partial x^i}$, $Y = \sum y^i \frac{\partial}{\partial y^i}$, $Z = \sum z^i \frac{\partial}{\partial z^i}$, and $\omega = \sum \omega_i dx^i$ then

$$R(X,Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} z^j x^k y^m \frac{\partial}{\partial x^i}$$

and

$$R(\omega, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} w_i z^j x^k y^m.$$

Consequently, one has $R(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}) = R^i_{jkm} w_i z^j x^k y^m$.

Definition 2.3. From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its components are $R_{ij} = \sum_{i=1}^{n} R_{ikj}^{k}$. The Ricci tensor is symmetric and its contraction $S = \sum_{ij=1}^{n} R_{ij} g^{ij}$ is called the scalar curvature ([A.], [B.E.], [B.E.E.]).

Definition 2.4. Suppose Ω is a smooth, bounded domain in R^n, and let $g = \Omega \times R \to R$ be a Caratheodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given.

Consider the equation

$$\Delta u = g(x, u) \quad \text{in} \quad \Omega$$

$$u = u_0 \quad \text{on} \quad \partial\Omega$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $\quad u \leq u_0 \quad \text{on} \quad \partial\Omega \quad$ and

$$\int_{\Omega} \nabla u \nabla \varphi dx + \int_{\Omega} g(x, u) \varphi dx \leq 0 \quad \text{for} \quad \text{all} \quad \varphi \in C_0^{\infty}(\Omega), \quad \varphi \geq 0.$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold. We briefly recall some results on warped product manifolds. Complete details may be found in [B.E.] or [O.]. On a semi-Riemannian product manifold $B \times F$. let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.5. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^* g_F$$

where g_B and g_F are metric tensors of B and F, respectively. In order words, if v is tangent to M at (p, q), then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber ([O.]).

We denote the metric g by $< \ \ , \ \ >$. In view of Remark 2.13 (1) and Lemma 2.14 we may also denote the metric g_B by $< \ \ , \ \ >$. The metric g_F will be denoted by $(\ \ , \ \ )$.

Remark 2.6. Some well known elementary properties of warped product manifold $M = B \times_f F$ are as follows :

(1) For each $q \in F$, the map $\pi \mid_{\sigma^{-1}(q) = B \times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma \mid_{\pi^{-1}(q) = p \times F}$ is a positive homothetic map onto F with homothetic factor $\frac{1}{f(p)}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and vertical fiber $\pi^{-1}(q) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = (f \circ \psi)$ then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification.

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p, q) by setting $\overline{X}(p, q) = (X_p, 0_q)$.

Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F.

Similarly, If Y is a vector field of F, $\overline{Y}$ is defined by $\overline{Y}(p, q) = (0_p, Y_q)$.

Lemma 2.7. If h is a smooth function an B, Then the gradient of the lift $(h \circ \pi)$ of h to M is the lift to M of gradient of h on B.

Proof. We must show that $grad(h\circ\pi)$ is horizontal and π-related to $grad(h)$ On B. If v is vertical tangent vector to M, then

$$< grad(h\circ\pi), v >= v(h\circ\pi) = d\pi(v)h = 0, \quad \text{since} \quad d\pi(v) = 0.$$

Thus $grad(h\circ\pi)$ is horizonal. If x is horizonal,

$$\begin{aligned} & < d\pi(grad(h\circ\pi)), d\pi(x) >=< grad(h\circ\pi), x > \\ = \quad & x(h\circ\pi) = d\pi(x)h < grad(h), d\pi(x) > . \end{aligned}$$

Hence at each point, $d\pi(grad(h\circ\pi)) = grad(h)$. □

In view of Lemma 2.14, we simplify the notations by writing h for $(h\circ\pi)$ and $grad(h)$ for $grad(h\circ\pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$ That is, if A is a (1,s)-tensor, and if $v_1, v_2, ..., v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, ..., v_s) = A(d\pi(v_1), ..., d\pi(v_s)) \in T_p(B)$.

Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $(f\circ\pi)$ generally only on horizontal vector. For detailed computations, see Lemma 5.1 in [B.E.P.].

Now we recall the formula for the Ricci curvature tensor Ric on the warped product maniford $M = B \times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.8. On a warped product maniford $M = B \times_f F$ with $n = dimF > 1$ let X, Y be horizontal and V, W vertical.

Then

(1) $Ric(X,Y) = Ric^B(X,Y) - \frac{n}{f}H^f(X,Y)$

(2) $Ric(X,Y) = 0$

(3) $Ric(V,W) = Ric^F(V,W) - < V,W > f^\sharp$,

Where $f^\sharp = \frac{\Delta f}{f} + (n-1)\frac{<grad(f),grad(f)>}{f^2}$ and $\Delta f = trace(H^f)$ is the Laplacian on B.

Proof. See Corollary 7.43 in [O.]. $\square$

On the given warped product manifold $M = B \times_f F$, we also write S^B For the pullback by π of the scalar curvature S_B of B and similarly for S^F From now on, we denote $grad(f)$ by Δf.

Lemma 2.9. If S is the scalar curvature of $M = B \times_f F$ with $n = dimF > 1$, then

$$(2.1) \qquad S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{<\nabla f, \nabla f>}{f^2},$$

where Δ is the Laplacian on B.

Proof. For each $(p,q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)}M$. we can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 =< \overline{d_j}, \overline{d_j} >= f(p)^2(d_j, d_j) = (f(p)d_j, f(p)d_j),$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF. By Lemma2.8 (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i})$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(p)g_F(d_j, d_j)(\frac{\Delta f}{f} + n(n-1)\frac{<\nabla f, \nabla f>}{f^2}).$$

Hence, for $\epsilon_i = g(\overline{e_i}, \overline{e_i})$ and $\epsilon_j = g(\overline{d_j}, \overline{d_j})$

$$
\begin{aligned}
S(p,q) &= \sum_{\alpha} \epsilon_\alpha R_{\alpha\alpha} \\
&= \sum_{i} \epsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_{j} \epsilon_j Ric(\overline{d_j}, \overline{d_j}) \\
&= S^B(p,q) + \frac{S^F(p,q)}{f^2} - 2n\frac{\nabla f}{f} - n(n-1)\frac{< \nabla f, \nabla f >}{f^2}
\end{aligned}
$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$. $\square$

III. MAIN RESULTS

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive constant. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g. \tag{3.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$ (For details, cf. [D.D.] or [G.L.]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [E.J.K.], we have the following proposition.

Proposition 3.1. If the scalar curvature of the fiber manifold N is an arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.

Proposition 3.1 implies that in Lorentzian warped product there is no obstruction of the existence of metric with positive scalar curvature. However, the results of [J.] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 3.2. Theorem 5.5 in [P.] implies that all timelike geodesics are future (resp. past) complete on $(-\infty, +\infty) \times_v N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) for some t_0 and Remark 2.58 in [B.E.] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) for some t_0 (cf. Theorem 4.1 and Remark 4.2 in [B.E.P.]. In this reference, the warped product metric is $g' = -dt^2 + v(t)g$).

If N is in class (B), then we assume that N admits a Riemannian metric of zero scalar curvature. In this case, equation (3.3) is changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) = 0. \tag{3.4}$$

If N admits a Riemannian metric of zero scalar curvature, then we let $u(t) = t^{\alpha}$ in equation(3.4), where $\alpha \in (0,1)$ is a constant, and we have

$$R(t,x) = -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > a.$$

There, from the above fact, Remark 3.2 implies the following:

Theorem 3.3. For $n \geq 3$, let $M = [a, \infty) \times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (B), then on M there is a future geodesically complete Lorentzian metric of negative scalar curvature outside a compact set.

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = t^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

If $R(t, x)$ is the function of only $t-$variable, then we have the following proposition, whose proof is similar to that of Lemma 1.8 in [L.2].

Proposition 3.4. If $R(g) = 0$, then there is no positive solution to equation (3.4) with

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. See Proposition 2.4 in [J.]. □

In particular, if $R(g) = 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

Proposition 3.5. Suppose that $R(g) = 0$ and $R(t, x) = R(t) \in C^\infty([a, \infty))$. Assume that for $t > t_0$, there exist an (weak) upper solution $u_+(t)$ and a (weak) lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (3.4) such that for $t > t_0$, $\quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. See Theorem 2.5 in [J.]. □

Theorem 3.6. Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^{\infty}([a,\infty))$ is a function such that

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq \frac{4n}{n+1}b\frac{1}{t^2} \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a$, $0 < c < 1$ and $0 < b < \frac{(n+1)(n+3)}{4}$ are constants. Then equation (3.4) has a positive solution on $[a,\infty)$ and on M the resulting Lorentzian warped product metric is a future geodesically complete one.

Proof. Since $R(g) = 0$, put $u_+(t) = t^{\alpha}$. Then $u_+''(t) = \alpha(\alpha-1)t^{\alpha-2}$ for $0 < \alpha < \frac{1}{2}$. Hence

$$\begin{aligned}
\frac{4n}{n+1}u_+''(t) - R(t)u_+(t) &\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2}u_+(t) \\
&= \frac{4n}{n+1}\alpha(\alpha-1)t^{\alpha-2} + \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2}t^{\alpha} \\
&= \frac{4n}{n+1}\frac{1}{4}t^{\alpha-2}[4\alpha(\alpha-1)+c] \\
&< 0,
\end{aligned}$$

If α is sufficiently close to $\frac{1}{2}$.

Therefore $u_+(t)$ is our (weak) upper solution.

And put $u_-(t) = t^{-\beta}$, where β is a positive constant with $0 < \beta < \frac{n+1}{2}$.

Then $u_-''(t) = \beta(\beta+1)t^{-\beta-2}$. Hence

$$\begin{aligned}
\frac{4n}{n+1}u_-''(t) - R(t)u_-(t) &\geq \frac{4n}{n+1}u_-''(t) - \frac{4n}{n+1}b\frac{1}{t^2}u_-(t) \\
&= \frac{4n}{n+1}\beta(\beta+1)t^{-\beta-2} - \frac{4n}{n+1}b\frac{1}{t^2}t^{-\beta} \\
&= \frac{4n}{n+1}t^{-\beta-2}[\beta(\beta+1) - b] \\
&> 0
\end{aligned}$$

if β is sufficiently close to $\frac{n+1}{2}$.

Thus $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u(t) < u_+(t)$ for large t.

And since β is sufficiently close to $\frac{n+1}{2,}$ $-\frac{2\beta}{n+1} + 1 > 0$. Therefore

$$\begin{aligned}
&\int_{t_0}^{+\infty} \left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \\
\geq\ &\int_{t_0}^{+\infty} \frac{u_-(t)^{\frac{2}{n+1}}}{\sqrt{1+u_-(t)^{\frac{4}{n+1}}}}dt = \int_{t_0}^{+\infty} \frac{t^{-\frac{2\beta}{n+1}}}{\sqrt{1+t^{-\frac{4\beta}{n+1}}}}dt \\
\geq\ &\frac{1}{\sqrt{2}}\int_{t_0}^{+\infty} t^{-\frac{2\beta}{n+1}}dt = +\infty
\end{aligned}$$

and

$$\int_{t_0}^{+\infty} f(t)dt = \int_{t_0}^{+\infty} u(t)^{\frac{2}{n+1}} dt$$
$$\geq \int_{t_0}^{+\infty} u_-(t)^{\frac{2}{n+1}} dt = \int_{t_0}^{+\infty} t^{-\frac{2\beta}{n+1}} dt = +\infty,$$

□

Example 3.7. If $R(g) = 0$ and $R(t) = \frac{4n}{n+1}\frac{n(n+2)}{4}\frac{1}{t^2}$, then there is a positive solution to equation (3.4)

$$t^2 u''(t) - \frac{n(n+2)}{4} u(t) = 0. \tag{3.5}$$

By Euler-Cauchy equation method, we put $u(t) = t^m$ then

$$m(m-1)t^{m-2}t^2 - \frac{n(n+2)}{4} t^m = 0,$$

$$(m^2 - m - \frac{n(n+2)}{4})t^m = 0$$

and

$$(m - \frac{n+2}{2})(m + \frac{n}{2}) = 0$$

so $m = \frac{n+2}{2}, -\frac{n}{2}$. Thus $u(t) = c_1 t^{\frac{n+2}{2}} + c_2 t^{-\frac{n}{2}}$ is solution of equation (3.5), where c_1 and c_2 are constants.

Therefore $u(t) = c_2 t^{-\frac{n}{2}}$ is our (weak) solution in the sense of Theorem 3.5 such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. And Remark 3.2 implies that the resulting Lorentzian warped product metric is a future geodesically complete one. $\square$

Theorem 3.8. Suppose that $R(g) = 0$. Assume that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a function such that

$$\frac{4n}{n+1} bt^{-2} < R(t) < \frac{4n}{n+1} dt^s,$$

where b, d, and s are positive constants.

If $b > \frac{(n+1)(n+3)}{4}$, then equation (3.4) has a positive solution on $[a, \infty)$ and on M the resulting Lorentzian warped product metric is not a future geodesically complete metric.

Proof. See Theorem 2.7 in [J.L.L.]. $\square$

Lemma 3.9. Prove

$$\int_{t_0}^{\infty} e^{-\alpha\sqrt{t}} dt < \infty$$

t_0 is a constant and α is a positive constant.

Proof. Since $\int_{t_0}^{\infty} e^{-\alpha\sqrt{t}}dt = \int_{t_0}^{\infty} \frac{-2\alpha\sqrt{t}}{-2\alpha\sqrt{t}} e^{-\alpha\sqrt{t}}dt$, integration by part implies that

$$\begin{aligned} & [\frac{-2\sqrt{t}}{\alpha}e^{-\alpha\sqrt{t}}]_{t_0}^{\infty} + \int_{t_0}^{\infty} \frac{1}{\alpha}\frac{1}{\sqrt{t}}e^{-\alpha\sqrt{t}}dt \\ = & \frac{2\sqrt{t_0}}{\alpha}e^{-\alpha\sqrt{t_0}} + [\frac{-2}{\alpha^2}e^{-\alpha\sqrt{t}}]_{t_0}^{\infty} \\ = & \frac{2\sqrt{t_0}}{\alpha}e^{-\alpha\sqrt{t_0}} + \frac{2}{\alpha^2}e^{-\alpha\sqrt{t_0}} < \infty. \end{aligned}$$

□

Theorem 3.10. Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^{\infty}([a,\infty))$ is a function such that

$$\frac{4n}{n+1}dt^s \leq R(t) < \frac{4n}{n+1}be^{ct},$$

where b, c, d, and s are positive constants. Then equation (3.4) has a positive solution on $[a,\infty)$ and on M the resulting Lorentzian warped product metric is not a future geodesically complete metric.

Proof. Since $R(g) = 0$, put $u_-(t) = e^{-e^{\beta t}}$, where β is a positive large constant. Then $u_-''(t) = -\beta^2 e^{\beta t} e^{-e^{\beta t}} + \beta^2 e^{2\beta t} e^{-e^{\beta t}}$.

Hence

$$
\begin{aligned}
& \frac{4n}{n+1}u_-''(t) - R(t)u_-(t) \\
= & \frac{4n}{n+1}[-\beta^2 e^{\beta t} e^{-e^{\beta t}} + \beta^2 e^{2\beta t} e^{-e^{\beta t}}] - R(t)e^{-e^{\beta t}} \\
\geq & \frac{4n}{n+1} e^{-e^{\beta t}}[-\beta^2 e^{\beta t} + \beta^2 e^{2\beta t} - be^{ct}] \\
> & 0
\end{aligned}
$$

for large β. Thus, for large β, $u_-(t)$ is a (weak) lower solution.

Since $R(g) = 0$ and $R(t) \geq \frac{4n}{n+1}dt^s$, we take the upper solution $u_+(t) = e^{-\sqrt{t}}$ and $u_+''(t) = \frac{1}{4}\frac{1}{t\sqrt{t}}e^{-\sqrt{t}} + \frac{1}{4t}e^{-\sqrt{t}}$.

Hence

$$
\begin{aligned}
\frac{4n}{n+1}u_+''(t) - R(t)u_+(t) &\leq \frac{4n}{n+1}u_+''(t) - \frac{4n}{n+1}dt^s u_+(t) \\
&= \frac{4n}{n+1}[\frac{1}{4}\frac{1}{t\sqrt{t}}e^{-\sqrt{t}} + \frac{1}{4t}e^{-\sqrt{t}} - dt^s e^{-\sqrt{t}}] \\
&= \frac{4n}{n+1}\frac{1}{4}e^{-\sqrt{t}}[\frac{1}{t\sqrt{t}} + \frac{1}{4t} - 4dt^s] \\
&\leq 0
\end{aligned}
$$

for large t, where $s > 0$ is a constant. Thus $u_+(t)$ is a (weak) upper solution and $0 < u_-(t) < u_+(t)$ for large t.

Hence Proposition 3.5 implies that equation (3.4) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$ for large t.

Therefore, by Lemma 3.9,

$$\begin{aligned}
&\int_{t_0}^{+\infty} \left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}} dt \\
\leq\ & \int_{t_0}^{+\infty} \frac{u_+(t)^{\frac{2}{n+1}}}{\sqrt{1+u_+(t)^{\frac{4}{n+1}}}} dt = \int_{t_0}^{+\infty} \frac{e^{-\sqrt{t}\frac{2}{n+1}}}{\sqrt{1+e^{-\sqrt{t}\frac{4}{n+1}}}} dt \\
\leq\ & \int_{t_0}^{+\infty} e^{-\sqrt{t}\frac{2}{n+1}} dt < +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{+\infty} f(t) dt = \int_{t_0}^{+\infty} u(t)^{\frac{2}{n+1}} dt \\
\leq\ & \int_{t_0}^{+\infty} u_+(t)^{\frac{2}{n+1}} dt = \int_{t_0}^{+\infty} e^{-\sqrt{t}\frac{2}{n+1}} dt < +\infty,
\end{aligned}$$

which by Remark 3.2 implies that the resulting warped product metric is not a future geodesically complete one. $\square$

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