[UCI]I804:24011-200000265621

2016년 8월
교육학석사(수학교육)학위논문

The Existence of Warping Functions on Riemannian Warped Product Manifolds with Fiber Manifolds of Class (A)

조선대학교 교육대학원

수학교육전공

윤 미 라

The Existence of Warping Functions on Riemannian Warped Product Manifolds with Fiber Manifolds of Class (A)

엽다양체가 (A)류인 경우의 리만 휜곱다양체 위의 휜함수의 존재성

2016년 8월

조선대학교 교육대학원

수학교육전공

윤 미 라

The Existence of Warping Functions on Riemannian Warped Product Manifolds with Fiber Manifolds of Class (A)

지도교수 정 윤 태

이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

2016년 4월

조선대학교 교육대학원

수학교육전공

윤 미 라

윤미라의 교육학 석사학위 논문을 인준함.

심사위원장	조선대학교 교수	김 남 권	인
심사위원	조선대학교 교수	김 진 홍	인
심사위원	조선대학교 교수	정 윤 태	인

2016년 6월

조선대학교 교육대학원

CONTENTS

국 문 초 록

The Existence of Warping Functions on Riemannian Warped Product Manifolds with Fiber Manifolds of Class (A)

-엽다양체가 (A)류인 경우의 리만 휜곱다양체 위의 휜함수의 존재성-

윤 미 라
지도교수 : 정 윤 태
조선대학교 교육대학원 수학교육전공

미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다.

연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

Kazdan and Warner ([K.W.1,2,3])의 결과에 의하면 N 위의 함수 f가 N 위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있는 데 먼저

(A) N 위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 적당한 점에서 $f(x_0) < 0$ 일 때이다. 특히, N 위에 nagative constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(B) N 위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 항등적으로 $f \equiv 0$ 이거나 모든 점에서 $f(x) < 0$ 인 경우이다. 즉, N 위에서 zero scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(C) N 위의 어떤 f 라도 scalar curvature가 되는 적당한 Riemannian metric이 존재하는 경우이다.

본 논문에서는 다양체 N이 (A)에 속하는 compact Riemannian manifold 일 때, Riemannian warped product manifold인 $M = [a, \infty) \times_f N$ 위에 함수 $R(t,x)$가 적당한 조건을 만족하면 적당한 metric의 scalar curvature가 될 수 있는 지를 상해·하해 방법을 이용하여 보였다.

I. INTRODUCTION

One of the basic problems in the differential geometry is studying the set of curvature functions which a given manifold possesses.

The well-known problem in differential geometry is that of whether there exists a warping function of warped metric with some prescribed scalar curvature function. One of the main methods of studying differential geometry is by the existence and the nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. In order to study these kinds of problems, we need some analytic methods in differential geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [B.K.], [B.O.], [D.D.], [G.L.], [K.K.P.], [L.M.], [M.M.]), and also in studying L_2-cohomology (cf. [Z.]).

In a study [L. 1, 2], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and the nonexistence of Riemannian warped metric with some prescribed scalar curvature function.

In this paper, we also study the existence and the nonexistence of Riemannian warped product metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. So, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f N$ in terms of its warping function f and the scalar curvatures of B and N.

By the results of Kazdan and Warner (cf. [K.W. 1, 2, 3]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three categories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [K.W. 1, 2, 3], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on open manifold. Results of Gromov and Lawson (cf. [G.L.]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded (cf. [G.L.], [L.M., p.322]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature (cf. [B.K.]). It follows from the results of Aviles and McOwen (cf.

[A.M.]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct Riemannian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. It is shown that if the fiber manifold N belongs to class (A), then M admits a Riemannian metric with some prescribed scalar curvature outside a compact set.

Although we will assume throughout this paper that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the given data is Hölder continuous.

II. PRELIMINARIES

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. Let $\mathcal{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\Im(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

such that

(D1) $\nabla_V W$ is $\Im(M)$-linear in V,

(D2) $\nabla_V W$ is R-linear in W,

(D3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \Im(M)$.

(D4) $[V, W] = \nabla_V W - \nabla_W V$, and

(D5) $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$

for all $X, V, W \in \mathcal{X}(M)$.

If ∇ satisfies axioms (D1)$\sim$(D3), then $\nabla_V W$ is called the covariant derivative of W with respect to V for the connection ∇. If ∇ satisfies axioms (D1)$\sim$(D5), then ∇ is called the Levi - Civita connection of M, which is characterized by the $K\ddot{o}szul formula$ ([O.]).

A geodesic $c : (a, b) \to M$ is a smooth curve of M such that the tangent vector c' moves by parallel translation along c. In order words, c is a geodesic if

$$\nabla_{c'} c' = 0. \quad \text{(geodesic} \quad \text{equation)}$$

A pregeodesic is a smooth curve c which may be reparametrized to be a geodesic. Any parameter for which c is a geodesic is called an affine parameter. If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c'} c' = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U, (x^1, x^2, ..., x^n))$ be local coordinates on M and let $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ denote the natural basis with respect to these coordinates. The connection coefficients Γ^k_{ij} of ∇ with respect to $(x^1, x^2, ..., x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) = \sum_{i=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \text{(connection coefficients)}.$$

Using these coefficients, we may write equation as the system

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \quad \text{(geodesic equations in coordinates)}.$$

Definition 2.2. The curvature tensor of the connection ∇ is a linear transformation valued tensor R in Hom$(\mathcal{X}(M), \mathcal{X}(M))$ defined by :

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathcal{X}(M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well-known that $R(X,Y)Z$ at p depends only upon the values of X, Y and Z at p ([O.]).

If $\omega \in T_p^*(M)$ is a cotangent vector at p and $x, y, z \in T_p(M)$ are tangent vectors at p, then one defines

$$R(\omega, X, Y, Z) = (\omega, R(X,Y)Z) = \omega(R(X,Y)Z)$$

for X, Y and Z smooth vector fields extending x, y and z, respectively.

The curvature tensor R is a (1,3) tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

.

Notice that $R(X,Y)Z = -R(X,Y)Z$, $R(\omega,X,Y,Z) = -R(\omega,Y,X,Z)$ and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \sum x^i \frac{\partial}{\partial x^i}$, $Y = \sum y^i \frac{\partial}{\partial y^i}$, $Z = \sum z^i \frac{\partial}{\partial z^i}$, and $\omega = \sum \omega_i dx^i$ then

$$R(X,Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} z^j x^k y^m \frac{\partial}{\partial x^i}$$

and

$$R(\omega,X,Y,Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \omega_i z^j x^k y^m.$$

Consequently, one has $R(dx^i, \frac{\partial}{\partial x^K}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}) = R^i_{jkm}$.

Definition 2.3. From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its components are $R_{ij} = \sum_{i=1}^n R^k_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{ij=1}^n R_{ij} g^{ij}$ is called the scalar curvature ([A.],[B.E.],[B.E.E.]).

Definition 2.4. Suppose Ω is a smooth, bounded domain in R^n, and let $g = \Omega \times R \to R$ be a Caratheodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation

$$\Delta u = g(x,u) \quad \text{in} \quad \Omega$$

$$u = u_0 \quad \text{on} \quad \partial\Omega$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $\quad u \leq u_0 \quad$ on $\quad \partial\Omega \quad$ and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega g(x,u)\varphi dx \leq 0 \quad \text{for} \quad \text{all} \quad \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0.$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold. We briefly recall some results on warped product manifolds. Complete details may be found in [B.E.] or [O.]. On a semi-Riemannian product manifold $B \times F$. let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.5. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^* g_F,$$

where g_B and g_F are metric tensors of B and F, respectively. In order words, if v is tangent to M at (p,q), then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber ([O.]).

We denote the metric g by $< \ , \ >$. In view of Remark2.13 (1) and Lemma2.14 we may also denote the metric g_B by $< \ , \ >$. The metric g_F will be denoted by $(\ , \ )$.

Remark 2.6. Some well known elementary properties of warped product manifold $M = B \times_f F$ are as follows :

(1) For each $q \in F$, the map $\pi \mid_{\sigma^{-1}(q)=B\times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma \mid_{\pi^{-1}(q)=p\times F}$ is a positive homothetic map onto F with homothetic factor $\frac{1}{f(p)}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) the horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodetic submanifold of M and Vertical fiber $\pi^{-1}(q) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = (f \circ \psi)$ then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p,q) by setting $\overline{X}(p,q) = (X_p, 0_q)$.

Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F.

Similarly, If Y is a vector field of F, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

Lemma 2.7. If h is a smooth function an B, then the gradient of the lift $(h \circ \pi)$ of h to M is the lift to M of gradient of h on B.

Proof. We must show that $grad(h \circ \pi)$ is horizontal and π-related to $grad(h)$ On B. If v is vertical tangent vector to M, then

$$< grad(h \circ \pi), v >= v(h \circ \pi) = d\pi(v)h = 0, \quad \text{since} \quad d\pi(v) = 0.$$

Thus $grad(h \circ \pi)$ is horizonal. If x is horizonal,

$$< d\pi(grad(h \circ \pi)), d\pi(x) >=< grad(h \circ \pi), x >= x(h \circ \pi) = d\pi(x)h$$

$$< grad(h), d\pi(x) > .$$

Hence at each point, $d\pi(grad(h \circ \pi)) = grad(h)$.

In view of Lemma 2.14, we simplify the notations by writing h for $(h \circ \pi)$ and $grad(h)$ for $grad(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$ That is, if A is a (1,s)-tensor, and if $v_1, v_2, ..., v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, ..., v_s) = A(d\pi(v_1), ..., d\pi(v_s)) \in T_p(B)$.

Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. this agrees with the Hessian of the lift $(f \circ \pi)$ generally only on horizontal vector. For detailed computations, see Lemma 5.1 in [B.E.P.].

Now we recall the formula for the Ricci curvature tensor Ric on the warped product maniford $M = B \times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.8. On a warped product maniford $M = B \times_f F$ with $n = dimF > 1$ let X, Y be horizontal and V, W vertical.

Then

(1) $Ric(X,Y) = Ric^B(X,Y) - \frac{n}{f}H^f(X,Y)$,

(2) $Ric(X,Y) = 0$,

(3) $Ric(V,W) = Ric^F(V,W) - < V, W > f^\sharp$,

Where $f^\sharp = \frac{\Delta f}{f} + (n-1)\frac{<grad(f), grad(f)>}{f^2}$ and $\Delta f = trace(H^f)$ is the Laplacian on B.

Proof. See Corollary 7.43 in [O.].

On the given warped product manifold $M = B \times_f F$, we also write S^B For the pullback by π of the scalar curvature S_B of B and similarly for S^F From now on, we denote $grad(f)$ by Δf. $\square$

Lemma 2.9. If S is the scalar curvature of $M = B \times_f F$ with $n = dimF > 1$, then

$$(2.1) \qquad S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{<\nabla f, \nabla f>}{f^2},$$

where Δ is the Laplacian on B.

Proof. For each $(p,q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)}M$. we can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 =< \overline{d_j}, \overline{d_j} >= f(p)^2(d_j, d_j) = (f(p)d_j, f(p)d_j),$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF. By Lemma 2.8 (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i})$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(p)g_F(d_j, d_j)(\frac{\Delta f}{f} + n(n-1)\frac{<\nabla f, \nabla f>}{f^2}).$$

Hence, for $\epsilon_i = g(\overline{e_i}, \overline{e_i})$ and $\epsilon_j = g(\overline{d_j}, \overline{d_j})$

$$\begin{aligned} S(p,q) &= \sum_{\alpha} \epsilon_\alpha R_{\alpha\alpha} \\ &= \sum_i \epsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_j \epsilon_j Ric(\overline{d_j}, \overline{d_j}) \\ &= S^B(p,q) + \frac{S^F(p,q)}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{< \nabla f, \nabla f >}{f^2}, \end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$. $\square$

III. MAIN RESULTS

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = dt^2 + f^2(t)g. \tag{3.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t, x) = \frac{1}{f^2(t)}\{R(g)(x) - 2nf(t)f''(t) - n(n-1)|f'(t)|^2\} \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$. (For details, cf. [D.D.] or [G.L.]). Here we also know that if $R(g)(x)$ is constant, then $R(t, x)$ is a function of only $t-$ variable.

Now we consider the following problem:

Problem I: Given a fiber N with constant scalar curvature c, can we find a warping function $f > 0$ on $B = [a, \infty)$ such that for any smooth function $R(t, x) = R(t)$, the warped metric g admits $R(t)$ as the scalar curvature on $M = [a, \infty) \times_f N$?

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u''(t) + R(t)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

If N belongs to (A), then a negative constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g) = -\dfrac{4n}{n+1}k$, where k is a positive constant. Then equation (3.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}ku(t)^{1-\frac{4}{n+1}} + R(t)u(t) = 0. \tag{3.4}$$

In order to prove the nonexistence of some Riemannian warped product metric with fiber manifolds of class (A), we have the following theorem whose proof is similar to that of Lemma 3.3 in [J.].

Theorem 3.1. Let $u(t)$ be a positive smooth function on $[a, \infty)$. If $u(t)$ satisfies

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2}$$

for some constant $C > 0$, then there exists $t_0 > a$ such that for all $t > t_0$

$$u(t) \leq c_0 t^{\epsilon}$$

for some positive constants c_0 and $\epsilon > 1$.

Proof. Since $C > 0$, we can choose $\epsilon > 1$ such that $\epsilon(\epsilon - 1) = C$. Then from the hypothesis, we have

$$t^{\epsilon} u''(t) \leq \epsilon(\epsilon - 1) t^{\epsilon - 2} u(t).$$

Upon integration from $t_1 (\geq a)$ to $t (> t_1 \geq a)$, and using integration by parts, we obtain

$$
\begin{aligned}
t^{\epsilon}u'(t) \quad &- \quad \epsilon t^{\epsilon-1}u(t) - t_1^{\epsilon}u'(t_1) + \epsilon t_1^{\epsilon-1}u(t_1) + \epsilon(\epsilon-1)\int_{t_1}^{t} s^{\epsilon-2}u(s)ds \\
&\leq C\int_{t_1}^{t} s^{\epsilon-2}u(s)ds.
\end{aligned}
$$

Therefore we have

$$
t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq t_1^{\epsilon}u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1). \tag{3.5}
$$

We consider two following cases:

[Case 1] There exists $t_1 \geq a$ such that $u'(t_1) \leq 0$.

If there is a number $t_1 \geq a$ such that $u'(t_1) \leq 0$, then we have

$$
t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq 0.
$$

This gives

$$
(\ln u(t))' \leq \epsilon(\ln t)'.
$$

Hence

$$u(t) \le c_1 t^{\epsilon}$$

for all $t > t_1$, where c_1 is a positive constant.

[Case 2] There does not exist $t_1 \ge a$ such that $u'(t_1) \le 0$.

In other words, if $u'(t) > 0$ for all $t \ge a$, then $u(t) \ge c'$ for some positive constant c'. Let c_2 be a positive constant such that

$$t_1^{\epsilon} u'(t_1) - \epsilon t_1^{\epsilon-1} u(t_1) \le c_2,$$

then equation (3.5) gives

$$t^{\epsilon} u'(t) - \epsilon t^{\epsilon-1} u(t) \le c_2$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \le \frac{\epsilon}{t} + \frac{c_2}{u(t)t^{\epsilon}} \le \frac{\epsilon}{t} + \frac{c_2}{c' t^{\epsilon}}.$$

Integrating from t_1 to t we have

$$\ln \frac{u(t)}{u(t_1)} \le \epsilon \ln \left(\frac{t}{t_1} \right) + \frac{c_2}{(\epsilon - 1) c' t_1^{\epsilon-1}} \le \epsilon \ln \left(\frac{c_3 t}{t_1} \right),$$

as $\epsilon > 1$. Here c_3 is a positive constant such that $\ln c_3 \geq \frac{c_2}{\epsilon(\epsilon-1)c' t_1^{\epsilon-1}}$. Hence we again obtain the inequality

$$u(t) \leq bt^{\epsilon}$$

for some positive constant b and for all $t \geq t_1$.

Thus from two cases we always find $t_0 > a$ and a constant $c_0 > 0$ such that

$$u(t) \leq c_0 t^{\epsilon}$$

for all $t \geq t_0$.

Using the above theorem, we can prove the following theorem about the nonexistence of warping function, whose proof is similar to that of Lemma 3.3 in [L.2].

Theorem 3.2 Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n(\geq 3)$. We may assume that $R(g) = -\dfrac{4n}{n+1}k$, where k is a positive constant. On $M = [a, \infty) \times_f N$, there does not exist a Riemannian warped product metric

$$g' = dt^2 + f^2(t)g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

Proof. Assume that we can find a warped product metric on $M = [a, \infty) \times_f N$ with

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$. In equation (3.4), we have

$$\text{(3.6)} \qquad \frac{4n}{n+1}\left[\frac{u''(t)}{u(t)} + \frac{k}{u(t)^{\frac{4}{n+1}}}\right] = -R(t) \leq \frac{n(n-1)}{t^2}$$

and

$$\text{(3.7)} \qquad \frac{u''(t)}{u(t)} \leq \frac{\frac{(n-1)(n+1)}{4}}{t^2}.$$

In equation (3.7), we can apply Theorem 3.1 and take $\epsilon = \frac{n+1}{2}$. Hence we have $t_0 > a$ such that

$$u(t) \leq c_0 t^{\frac{n+1}{2}}$$

for some positive constants c_0 and all $t > t_0$.

Then

$$\frac{k}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{'}}{t^2}$$

where $0 < c^{'} \leq \frac{k}{c_0^{\frac{4}{n+1}}}$ is a positive constant. Hence equation (3.6) gives

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n+1)(n-1)-\delta}{4t^2},$$

where $4c^{'} \geq \delta \geq 0$ is a constant. We can choose $\delta^{'} > 0$ such that

$$\frac{(n+1)(n-1)-\delta}{4} = \left(\frac{n+1}{2} - \delta^{'}\right)\left(\frac{n-1}{2} - \delta^{'}\right)$$

for small positive δ. Applying Theorem 3.1 again, we have $t_1 > a$ such that

$$u(t) \leq c_1 t^{\frac{n+1}{2} - \delta^{'}}$$

for some $c_1 > 0$ and all $t > t_1$. And

$$\frac{k}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{''}}{t^{2-\epsilon}}, \tag{3.8}$$

where $\epsilon = \dfrac{4}{n+1}\delta^{'}$ and $0 < c^{''} \leq \dfrac{k}{c_1^{\frac{4}{n+1}}}$. Thus equation (3.7) and (3.8) give

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n-1)(n+1)}{4t^2} - \frac{c^{''}}{t^{2-\epsilon}},$$

which implies that

$$u^{''}(t) \leq 0$$

for t large. Hence $u(t) \leq c_2 t$ for some constant $c_2 > 0$ and large t. From equation (3.5) we have

$$\frac{u^{''}(t)}{u(t)} \leq \frac{-c_3}{t^{\frac{4}{n+1}}} + \frac{(n+1)(n-1)}{4t^2} \leq -\frac{c_3}{t}$$

for t large enough, as $n \geq 3$. Here c_3 is a positive constant. Multiplying $u(t)$ and integrating from $t^{'}$ to t, we have

$$u^{'}(t) - u^{'}(t^{'}) \leq -c_3 \int_{t^{'}}^{t} \frac{u(s)}{s} ds, \quad t > t^{'}.$$

We consider two following cases :

[Case 1] There exists $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$. If $u^{'}(t^{'}) \leq 0$ for some $t^{'}$, then $u^{'}(t) \leq -c_4$ for some positive constant c_4. Hence $u(t) \leq 0$ for t large enough, contradicting the fact that u is positive.

[Case 2] There does not exist $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$. In order words, if $u^{'}(t) > 0$ for all t large, then $u(t)$ is increasing, hence

$$\int_{t'}^{t} \frac{u(s)}{s} ds \geq u(t^{'}) \int_{t'}^{t} \frac{1}{s} ds \to \infty.$$

Thus $u^{'}(t)$ has to be negative for some t large, which is a contradiction to the hypothesis. Therefore there does not exist such warped product metric.

$\square$

In particular, Theorem 3.2 implies that if $R(g) = -\frac{4n}{n+1}k$, then using Lorentzian warped product it is impossible to obtain a Riemannian metric of positive or zero scalar curvature outside a compact subset.

Proposition 3.3 Suppose that $R(g) = -\dfrac{4n}{n+1}k$ and $R(t) \in C^{\infty}([a, \infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (3.4) such that for $t > t_0, 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty)$, $\tilde{u}_-(t) \leq \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a, \infty))$, there exists a positive constant b such that $|R(t)| \leq \frac{4n}{n+1}b^2$ for $t \in [a, t_0]$. Since

$$
\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + R(t)u_+(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} \\
&\leq \frac{4n}{n+1}(u_+''(t) + b^2u_+(t) + ku_+(t)^{1-\frac{4}{n+1}}),
\end{aligned}
$$

if we divide the given interval $[a, t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_{i+}(t)$ by parallel transporting $\cos Bt$ such that $0 < \frac{1}{\sqrt{2}} \leq u_{i+}(t) \leq 1$. That is to say, for each interval I_i,

$$
\begin{aligned}
&\frac{4n}{n+1}u_{i+}''(t) + R(t)u_{i+}(t) + \frac{4n}{n+1}ku_{i+}(t)^{1-\frac{4}{n+1}} \\
&\leq \frac{4n}{n+1}(u_{i+}''(t) + b^2u_{i+}(t) + ku_{i+}(t)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1}(-B^2\cos Bt + b^2\cos Bt + k(\cos Bt)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1}\cos Bt(-B^2 + b^2 + k(\cos Bt)^{-\frac{4}{n+1}}) \\
&\leq \frac{4n}{n+1}\cos Bt(-B^2 + b^2 + k2^{\frac{2}{n+1}}) \\
&\leq 0
\end{aligned}
$$

for large B, which means that $u_{i+}(t)$ is an (weak) upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_{i+}(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $\frac{1}{\sqrt{2}} \le \tilde{u}_+(t) \le 1$ for all $t \in [a, t_0]$. Put $\tilde{u}_-(t) = \frac{1}{\sqrt{2}} e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$,

$$
\begin{aligned}
&\frac{4n}{n+1} u''_{i-}(t) + R(t) u_{i-}(t) + \frac{4n}{n+1} k u_{i-}(t)^{1-\frac{4}{n+1}} \\
&\ge \frac{4n}{n+1} (u''_{i-}(t) - b^2 u_{i-}(t)) \\
&= \frac{4n}{n+1} \frac{1}{\sqrt{2}} e^{-\alpha t} (\alpha^2 - b^2) \\
&\ge 0
\end{aligned}
$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$, $0 < \tilde{u}_-(t) \le \tilde{u}_+(t)$. □

However, in this paper, when N is a compact Riemannian manifold of class (A), we consider the existence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures. If $R(t, x)$ is also the function of only t-variable, then we have the following theorems.

Theorem 3.4 Suppose that $R(g) = -\frac{4n}{n+1}k$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a negative function such that

$$-\frac{4n}{n+1}be^{t^s} \le R(t) \le -\frac{4n}{n+1}\frac{C}{t^\alpha}, \quad for \quad t \ge t_0,$$

where $t_0 > a, 0 < \alpha < 2, C$ and $b, s > 1$ are positive constants. Then equation (3.4) has a positive solution on $[a,\infty)$.

Proof. We let $u_+(t) = t^m$, where m is some positive number. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\le \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m[\frac{m(m-1)}{t^2} + \frac{k}{t^{\frac{4}{n+1}m}} - \frac{C}{t^\alpha}] \\
&\le 0, \quad t \ge t_0,
\end{aligned}$$

for some large t_0, which is possible for large fixed m since $0 < \alpha < 2$. Hence, $u_+(t)$ is an upper solution. Now put $u_-(t) = e^{-t^\beta}$, where β is a positive constant, which will be determined later. Then

$$
\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} + R(t)u_-(t)\\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}be^{t^s}u_-(t)\\
&= \frac{4n}{n+1}e^{-t^\beta}[\beta^2t^{2\beta-2} - \beta(\beta-1)t^{\beta-2} + ke^{t^\beta\frac{4}{n+1}} - be^{t^s}]\\
&\geq 0, \quad t \geq t_0
\end{aligned}
$$

for some large t_0 and large β such that $\beta > s$, which means that $u_-(t)$ is a lower solution. And we can take β so large that $0 < u_-(t) < u_+(t)$. So by Proposition 3.3, we obtain a positive solution. □

The above theorem implies that if $R(t)$ is not rapidly decreasing and less than some negative function, then equation (3.4) has a positive solution.

Theorem 3.5 Suppose that $R(g) = -\frac{4n}{n+1}k$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a negative function such that

$$
-\frac{4n}{n+1}be^{t^s} \leq R(t) \leq -\frac{C}{t^2}, \quad for \quad t \geq t_0
$$

where $t_0 > a, b$ and $C, s > 1$ are positive constants. If $C > n(n-1)$, then equation (3.3) has a positive solution on $[a, \infty)$.

Proof. In case that $C > n(n-1)$, we may take $u_+(t) = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$
\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq C_+\frac{4n}{n+1}t^{\frac{n-3}{2}}[\frac{n^2-1}{4} + kC_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C] \\
&\leq 0,
\end{aligned}
$$

which is possible if we take C_+ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C < 0$. Thus $u_+(t)$ is an upper solution. And we take $u_-(t)$ as in Theorem 3.4. In this case, we also obtain a positive solution. $\square$

Remark 3.6 The results in Theorem 3.4, and Theorem 3.5 are almost sharp as we can get as close to $-\frac{n(n-1)}{t^2}$ as possible. For example, let $R(g) = -\frac{4n}{n+1}k$ and $f(t) = t\ln t$ for $t > a$. Then we have

$$
R = -\frac{1}{t^2}[\frac{4n}{n+1}\frac{k}{(\ln t)^2} + \frac{2n}{\ln t} + n(n-1)(1+\frac{1}{\ln t})^2],
$$

which converges to $-\frac{n(n-1)}{t^2}$ as t goes to ∞.

REFERENCES

[A.] T. Aubin "Nonlinear analysis on manifold" , Monge-Ampere equations, Springer-verlag New York Heidelberg Berlin, 1982

[A.M.] P. Aviles and R.C. McOwen, Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds, Diff. Geom. 27(1998), 225-239.

[B.E.] J.K. Beem and P.E.Ehrlich, "Global Lorentzian Geometry", Pure and Applied Mathematics, Vol 67, Dekker, New York, 1981.

[B.E.E.] J.K. Beem, P.E.Ehrlich and K.L.Easley, "Global Lorentzian Geometry", Pure and Applied Mathematics, Vol. 202, Dekker, New York, 1996.

[B.E.P.] J.K. Beem, P.E.Ehrlich and Th.G.Powell, Warped product manifolds in relativity, Selected Studies (Th.M.Rassias, eds.), North-Holland, 1982, 41-56.

[B.K.] J. Bland and M. Kalka, Negative scalar curvature metrics on noncompact manifolds, Trans. Amer. Math. Soc. 316(1989), 433-446.

[B.O.] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145(1969), 1-49.

[D.D.] F. Dobarro and E. Lami Dozo, Positive scalar curvature and the Dirac operater on complete Riemannian manifolds, Publ. Math.I.H.E.S. 58(1983), 295-408.

[G.L.] M. Gromov and H.B. Lawson, positive scalar curvature and the Dirac operater on complete Riemannian manifolds, Math. I.H.E.S. 58(1983), 295-408.

[J.] Yoon-Tae Jung, Partial differential equations on semi-Riemmannian manifolds, J. Math. Anal. Appl. 241(2000), 238-253.

[K.K.P.] H. Kitabara, H. Kawakami and J.S. Pak, On a construction of completely simply connected Riemmannian manifolds with negative curvature, Nagoya Math. J.113(1980), 7-13.

[K.W.1] J.L. Kazdan and F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Diff. Geo 10(1975), 113-134.

[K.W.2] J.L. Kazdan and F.W. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, Ann. of Math. 101(1975), 317-331.

[K.W.3] J.L. Kazdan and F.W. Warner, Curvature functions for compact 2-manifolds, Ann. of Math. 99(1974), 14-74.

[L.1] M.C. Leung, Conformal scalar curvature equations on complete manifolds, Commum. Partial Diff. Equation 20(1995), 367-417.

[L.2] M.C. Leung, Conformal deformation of warped products and scalar curvature functions on open manifolds, Bulletin des Science Math-ematiques. 122(1998), 369-398.

[L.M.] H.B. Lawson and M. Michelsohn, Spin geometry, Princeton University Press, Princeton, (1989).

[M.M.] X. Ma and R.C. McOwen, The Laplacian on complete manifolds with warped cylindrical ends, Commum. Partial Diff. Equation 16(1991),1583-1614.

[O.] B. O'Neill. "Semi-Riemannian geometry with applocations to relativity", Academic Press, New York, 1983.

[Z.] S. Zucker, L_2 cohomology of warped products and arithmetic groups, Invent. Math. 70(1982), 169-218.