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지도교수 김 진 홍

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조선대학교 교육대학원

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지도교수 : 김 진 홍
조선대학교 교육대학원 수학교육전공

히가시타니-마수다는 폴리곤에 관한 잘 알려진 피크 정리를 일반화하여 다중 폴리곤에 관한 일반화된 피크 정리를 증명하였다. 한편, 이화는 그의 석사 논문에서 하토리-마수다에 의해 소개된 다중 폴리토프에 관한 일반화된 피크 정리를 도출해 냈다. 일반적으로 다중 폴리토프의 개념과 성질은 다중 폴리곤의 개념과 성질을 포함하고 있으며 더 일반적이다.

본 논문에서는 이러한 두 가지 형태의 일반화된 피크 정리가 2차원 다중 폴리곤의 경우에는 서로 일치함을 보였다. 좀 더 구체적으로, 히가시타니-마수다는 $\widetilde{P}$가 격자 다중 폴리곤일 때 $A(\widetilde{P}), B(\widetilde{P}), C(\widetilde{P})$라는 불변량 정의하고

$$\widetilde{P} = A(\widetilde{P}) + \frac{1}{2}B(\widetilde{P}) + C(\widetilde{P})$$

이 성립함을 보였다. 또한, 이화는 석사논문에서 P가 2차원의 단순 격자 다중 폴리토프일 때,

$$vol(P) = \#(P) + \frac{1}{2}\#(\partial P) - \deg(\triangle)$$

이 성립함을 보였고 본 논문에서는

$$A(\widetilde{P}) = vol(P), B(\widetilde{P}) = \#(\partial P), C(\widetilde{P}) = \deg(\triangle)$$

을 보여 위의 두 방정식이 서로 동등함을 증명하였다.

Chapter 1

Introduction

In the paper [6], Higashitani and Masuda introduced the notion of a multi-polygon, and studies its general properties. Among other things, in particular, they proved certain generalized Pick's theorem in the same paper [6]. There is also a similar, but more general, notion of a multi-polytope which has been introduced by Hattori and Masuda in the paper [5]. Recently, in [9] Lee gave some similar generalized Pick's theorem for multi-polytopes by using the notion of the Duistermaat-Heckman function. Those two generalized Pick's theorems have seemingly different expressions, but it has been expected that they are actually equivalent in case of multi-polygons which are also multi-polytopes of dimension 2. The aim of this thesis is to give an affirmative answer to the expectation. In other words, we show that two versions of a generalized Pick's theorem proved in [6] and [9] are equivalent for such multi-polygons.

In order to explain our results precisely, we first need to set up some terminology and notations (see Chapters 2 and 3 for more details). Indeed, let $\tilde{\mathcal{P}}$ be a convex lattice polygon whose only interior lattice point is the origin and let $v_1, \ldots, v_{d+1}$ be the vertices of $\mathcal{P}$ arranged counterclockwise such that $v_{d+1} = v_1$. Then every v_i is primitive, and the triangle with the vertices $0, v_i$ and v_{i+1} has no lattice point in the interior for each i. We set $|v_i v_{i+1}|$ be the number of lattice points on the side v_iv_{i+1} minus 1. With these understood, we define

$$B(\tilde{\mathcal{P}}) = \sum_{i=1}^{d} \det(v_i, v_{i+1})|v_iv_{i+1}|.$$

Now, let $\tilde{\mathcal{P}}$ be a multi-polygon with a sign assignment ϵ in such a way that $\tilde{\mathcal{P}}$ is an oriented piecewise linear loop with signs attached to sides. For $i = 1\dots, d$, let n_i denote a normal vector to each side v_iv_{i+1} such that the 90 degree counterclockwise rotation of $\epsilon(v_iv_{i+1})n_i$ has the same direction as v_iv_{i+1}. The winding number $d_{\tilde{\mathcal{P}}}(v)$ of $\tilde{\mathcal{P}}$ around a point $v \in \mathbb{R}^2 \setminus \tilde{\mathcal{P}}$ is a locally constant function on $\mathbb{R}^2\setminus\tilde{\mathcal{P}}$, where $\mathbb{R}^2\setminus\tilde{\mathcal{P}}$ means the set of element in $\mathbb{R}^2$ which does not belong to any side of $\tilde{\mathcal{P}}$. We can also define

$$\begin{aligned} A(\tilde{\mathcal{P}}) &:= \int_{v\in\mathbb{R}^2\setminus\tilde{\mathcal{P}}} d_{\tilde{\mathcal{P}}}(v)\, dv, \\ C(\tilde{\mathcal{P}}) &:= \text{the rotation number of the sequence of } n_1, \dots, n_d. \end{aligned}$$

Let $\tilde{\mathcal{P}}_+$ be an oriented loop obtained from $\tilde{\mathcal{P}}$ by pushing each side v_iv_{i+1} slightly in the direction of n_i. Under suitable conditions on $\tilde{\mathcal{P}}$, $\tilde{\mathcal{P}}_+$ misses all lattice points, so the winding number $d_{\tilde{\mathcal{P}}_+}(u)$ can be defined for any lattice point u using $\tilde{\mathcal{P}}_+$. Using this winding number, we now define

$$\sharp\tilde{\mathcal{P}} := \sum_{u\in\mathbb{Z}^2} d_{\tilde{\mathcal{P}}_+}(u).$$

In the paper [5], Higashitani and Masuda proved that the following generalized Pick's formula for lattice multi-polygons holds.

Theorem 1.1. *Let $\tilde{\mathcal{P}}$ be a lattice multi-polygon. Then the following identity holds:*

$$\sharp\tilde{\mathcal{P}} = A(\tilde{\mathcal{P}}) + \frac{1}{2}B(\tilde{\mathcal{P}}) + C(\tilde{\mathcal{P}}).$$

Next, we want to explain what a multi-polytope is. To do so, let N be a lattice of rank n which is isomorphic to $\mathbb{Z}^n$, and let M be the dual lattice

$\mathrm{Hom}(N, \mathbb{Z})$. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and let $M_{\mathbb{R}} = \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$. A multi-polytope $\mathcal{P}$ is a pair $(\Delta, \mathcal{F})$ of an n-dimensional multi-fan Δ and an arrangement $\mathcal{F} = \{F_i\}$ of affine hyperplanes F_i in the dual space $M_{\mathbb{R}}$ with the same index set as the set of one-dimensional cones in Δ (refer to Chapter 2 for a more precise definition). A *lattice polytope* P means that each vertex of P lies in the lattice M of $M_{\mathbb{R}}$. For a convex lattice polytope P of dimension n in $M_{\mathbb{R}}$ and a positive integer ν, let νP be

$$\nu P = \{\nu u \mid u \in P\}.$$

Then νP is again a lattice multi-polytope in $M_{\mathbb{R}}$. Let us denote by $\#(\nu P)$ (resp. $\#(\nu P^{\circ})$) the number of lattice points in νP (resp. in the interior νP° of νP). Let us also denote by $\#(\partial(\nu P))$ be the number of lattice points on the boundary $\partial(\nu P)$ of νP. Then clearly we have

$$\#(\partial(\nu P)) = \#(\nu P) - \#(\nu P^{\circ}).$$

In this thesis, as in [5] we will normalize a volume element on $M_{\mathbb{R}}$ so that the volume of the unit cube determined by a basis of M is equal to one. Then $\#(\nu P)$ and $\#(\nu P^{\circ})$ are polynomials in ν of degree n, which are called the *Ehrhart polynomials* of P and P°, respectively. Recall that the number of lattice points of a simple, regular convex polytope is equal to its corresponding Riemann-Roch number (refer to, e.g., [3], [7], [4], and [10] for more details).

In the Master's thesis [9], Kim and Lee has shown the following generalized Pick's formula.

Theorem 1.2. *Let $\mathcal{P}$ be a complete and simple lattice multi-polytope of dimension 2. Then the following identity holds.*

$$\mathrm{vol}(\mathcal{P}) = \#(\mathcal{P}^{\circ}) + \frac{1}{2}\#(\partial\mathcal{P}) - \deg(\Delta),$$

or, equivalently

$$\#(\mathcal{P}) = \mathrm{vol}(\mathcal{P}) + \frac{1}{2}\#(\partial\mathcal{P}) + \deg(\Delta).$$

As an immediate consequence, we can easily obtain the well-known Pick's formula for simple convex polytopes.

Corollary 1.3. *Let P be a simple convex lattice polytope of dimension 2. Then the following identity holds.*

$$\mathrm{vol}(P) = \#(P^\circ) + \frac{1}{2}\#(\partial P) - 1.$$

Our main result is to show that two Theorems 1.1 and 1.2 are actually equivalent for multi-polygons which are also multi-polytopes, as follows.

Theorem 1.4. *Let $\tilde{\mathcal{P}}$ be an integral oriented multi-polygon of dimension 2. Then there is an associated complete, simple, and integral multi-polytope $\mathcal{P}$ of dimension 2 such that*

$$A(\tilde{\mathcal{P}}) = \mathrm{vol}(\mathcal{P}),\ B(\tilde{\mathcal{P}}) = \#(\partial\mathcal{P}),\ C(\tilde{\mathcal{P}}) = \deg(\Delta).$$

Note that by definition multi-polytopes are allowed to have three consecutive points on a same line, while essentially we may assume that all of multi-polygons do not have three consecutive points on a same line. As a consequence, we can see that the generalized Pick's theorem given in [9] is more general than that given in [6].

Finally, we are in a position to explain the structure of the thesis which goes as follows.

In Chapter 2, we briefly review the notion of a lattice multi-polygon and then state the generalized Pick's formula for lattice multi-polygons. A lattice multi-polygons is a piecewise linear loop with vertices in $\mathbb{Z}^2$ together with a sign function which assigns either $+$ or $-$ to each side and satisfies some mild conditions. The piecewise linear loop may have a self-intersection and we think of it as a sequence of points in $\mathbb{Z}^2$. A lattice polygon can naturally be regarded as a lattice multi-polygon.

In Chapter 3, we give a definition of multi-fan and introduce certain related notions. Also we introduce the notion of a multi-polytope in the same chapter. As mentioned above, a multi-polytope is a pair $\mathcal{P} = (\Delta, \mathcal{F})$ of an n-dimensional complete multi-fan Δ and an arrangement of hyperplanes $\mathcal{F} = \{F_i\}$ in $H^2(BT; \mathbb{R})$ with the same index set as the set of 1-dimensional cones in Δ. Recall that it is called *simple* if the multi-fan Δ is simplicial.

In Chapter 4, we carefully compare two generalized Pick's theorems and, finally, we show that they are equivalent.

Chapter 2

Multi-polygons

In this chapter, we recollect some definitions and properties of multi-polygons, and explain how to obtain the generalized Pick's formula as in [6], relatively in detail. This chapter is largely taken from the paper [6].

We say that a sequence of vectors $v_1, \ldots, v_{d+1}$ in $\mathbb{Z}^2$ ($d \geq 2$) with $v_{d+1} = v_1$ is *unimodular* if each triangle with vertices $\mathbf{0}, v_i$ and v_{i+1} contains no lattice point except the vertices, where $\mathbf{0} = (0, 0)$. The vectors in the sequence are not necessarily counterclockwise or clockwise. Very often they may go back and forth, whenever we need to do so.

Set

$$\epsilon_i = \det(v_i, v_{i+1}), \text{ for } i = 1, \ldots, d. \tag{2.1}$$

This implies that $\epsilon_i = 1$ if the rotation from v_i to v_{i+1} (with angle less than π) satisfies the so-called right-hand rule and $\{v_i, v_{i+1}\}$ is a basis of $\mathbb{Z}^2$ for $i = 1, \ldots, d$ and $\epsilon_i = -1$, otherwise. It is straightforward to see that

$$(v_i, v_{i+1}) = (v_{i-1}, v_i) \begin{pmatrix} 0 & -\epsilon_{i-1}\epsilon_i \\ 1 & -\epsilon_i a_i \end{pmatrix}$$

with a unique integer a_i for each i. It is easy to show that this is equivalent to

$$\epsilon_{i-1} v_{i-1} + \epsilon_i v_{i+1} + a_i v_i = 0. \tag{2.2}$$

Note also that $|a_i|$ is same as twice the area of the triangle with vertices $\mathbf{0}, v_{i-1}$ and v_{i+1}.

The following theorem has been proved in the paper [5, Theorem 1.2].

Theorem 2.1. *The rotation number of a unimodular sequence* $v_1, \ldots, v_{d+1}$ $(d \geq 2)$ *around the origin such that* $v_{d+1} = v_1$ *is given by*

$$\frac{1}{12}\left(\sum_{i=1}^{d} a_i + 3\sum_{i=1}^{d} \epsilon_i\right),$$

where ϵ_i *and* a_i *are the integers defined as in* (2.1) *and* (2.2)*.*

A *lattice polygon* is a polygon where the vertices are elements of $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of integers. If $\mathcal{P}$ is a convex lattice polygon whose only interior lattice point is the origin and $v_1, \ldots, v_{d+1}$ are the vertices of $\mathcal{P}$ arranged counterclockwise, then every v_i is primitive and the triangle with the vertices $0, v_i$ and v_{i+1} has no lattice point in the interior for each i, where $v_{d+1} = v_1$ as usual. This observation motivates the following definition.

Definition 2.2. A sequence of vectors $\mathcal{P} = (v_1, \ldots, v_{d+1})$, where $v_1, \ldots, v_{d+1}$ are in $\mathbb{Z}^2$ and $d \geq 2$, is called a *legal loop* if every v_i is primitive and whenever $v_i \neq v_{i+1}, v_i$ and v_{i+1} are linearly independent, i.e., $v_i \neq -v_{i+1}$ and the triangle with the vertices $0, v_i$ and v_{i+1} has no lattice point in the interior. We say that a legal loop is *reduced* if $v_1 \neq v_{i+1}$ for any $i (i \neq d+1)$.

A (non-reduced) legal loop $\mathcal{P}$ naturally determines a reduced legal loop, denoted $\mathcal{P}_{red}$, by dropping all the redundant point. We define the *winding number* of a legal loop $\mathcal{P} = (v_1, \ldots, v_{d+1})$ to be the rotation number of the vectors $v_1, \ldots, v_{d+1}$ around the origin.

By joining successive points in a legal loop $\mathcal{P} = (v_1, \ldots, v_{d+1})$ by straight lines, we can form a lattice polygon which may have a self-intersection. A unimodular sequence $v_1, \ldots, v_{d+1}$ determines a reduced legal loop. Conversely, a reduced legal loop $\mathcal{P} = (v_1, \ldots, v_{d+1})$ determines a unimodular sequence by adding all the lattice points on the line segment v_iv_{i+1}, called a side of $\mathcal{P}$, connecting v_i and v_{i+1} for every i. To each side v_iv_{i+1} with $v_i \neq v_{i+1}$, we assign the sign of $\det(v_iv_{i+1})$, denoted $\mathrm{sign}(v_iv_{i+1})$.

Definition 2.3. Let $|v_iv_{i+1}|$ be the number of lattice points on the side v_iv_{i+1} minus 1. So $|v_iv_{i+1}| = 0$ when $v_i = v_{i+1}$. Then we define

$$B(\mathcal{P}) = \sum_{i=1}^{d} \mathrm{sign}(v_i, v_{i+1})|v_iv_{i+1}|.$$

It is immediate from its definition to obtain $B(\mathcal{P}) = B(\mathcal{P}_{red})$.

Definition 2.4. A lattice multi-polygon $\mathcal{P}$ can be equipped with the assignment ϵ of signs, $\epsilon = \pm$, satisfying the following two conditions: when there are consecutive three points v_{i-1}, v_i, v_{i+1} in $\mathcal{P}$ lying on a line,

(1) $\epsilon(v_{i-1}v_i) = \epsilon(v_iv_{i+1})$ if v_i is in between v_{i-1} and v_{i+1}.

(2) $\epsilon(v_{i-1}v_i) \neq \epsilon(v_iv_{i+1})$ if v_{i-1} lies on v_iv_{i+1} or v_{i+1} lies on $v_{i-1}v_i$.

Remark 2.5. In the paper [10], Masuda discussed lattice multi-polygons such that three consecutive points are not on a same line. But, if we require the condition (1) and (2) of Definition 2.4, then the argument developed in the paper [10] works for any lattice multi-polygons. A *shaven polygon* is the case of a lattice multi-polygon with $\epsilon = +$, so that v_i is allowed to lie on the line segment $v_{i-1}v_{i+1}$ but v_{i-1} (resp. v_{i+1}) is not allowed to lie on v_iv_{i+1}(resp. $v_{i-1}v_i$) by (2) of Definition 2.4.

In view of this discussion, it will suffice to consider lattice multi-polygons such that three consecutive points are not on a same line, or we may assume without loss of generality that all lattice multi-polygons do not have three consecutive points on a same line.

Let $\mathcal{P}$ be a multi-polygon with a sign assignment ϵ. We think of $\mathcal{P}$ as an oriented piecewise linear loop with signs attached to sides. For $i = 1, \ldots, d$, let n_i denote a normal vector to each side v_iv_{i+1} such that the 90 degree counterclockwise rotation of $\epsilon(v_iv_{i+1})n_i$ has the same direction as v_iv_{i+1}. The winding number of $\mathcal{P}$ around a point $v \in \mathbb{R}^2 \backslash \mathcal{P}$, denoted $d_{\mathcal{P}}(v)$, is a locally constant function on $\mathbb{R}^2 \backslash \mathcal{P}$, where $\mathbb{R}^2$ means the set of element in $\mathbb{R}^2$ which does not belong to any side of P.

As in [6], we then define

$$\begin{aligned}
A(\mathcal{P}) &:= \int_{v \in \mathbb{R}^2 \backslash \mathcal{P}} d_{\mathcal{P}}(v)\, dv, \\
B(\mathcal{P}) &:= \sum_{i=1}^{d} \epsilon(v_iv_{i+1})|v_iv_{i+1}|, \\
C(\mathcal{P}) &:= \text{the rotation number of the sequence of } n_1, \ldots, n_d.
\end{aligned}$$

In case of lattice polygons, recall that $A(P), B(P), \sharp P^\circ$ of lattice polygons P (not necessarily convex) are defined by

$$\begin{aligned}
A(P) &:= \text{the area of } P, \\
B(P) &:= |\partial P \cap \mathbb{Z}^2|, \\
\sharp P^\circ &:= |P^\circ \cap \mathbb{Z}^2|.
\end{aligned}$$

If a multi-polygon $\mathcal{P}$ happens to be a lattice polygon P, that is, P is a sequence of the vertices of P arranged in counterclockwise order and $\epsilon = +$, then it is clear that $A(\mathcal{P}) = A(P)$, $B(\mathcal{P}) = B(P)$, $C(\mathcal{P}) = 1$.

Next we want to define $\sharp\mathcal{P}$ in such a way that if $\mathcal{P}$ happens to arise from a lattice polygon, then $\sharp\mathcal{P} = \sharp P$, as follows. Indeed, let $\mathcal{P}_+$ be an oriented loop obtained from $\mathcal{P}$ by pushing each side v_iv_{i+1} slightly in the direction of n_i. Since $\mathcal{P}$ satisfies the conditions (1) and (2) of Definition 2.4, $\mathcal{P}_+$ misses all lattice points. So the winding number $d_{\mathcal{P}_+}(u)$ can be defined for any lattice point u using $\mathcal{P}_+$. We then define

$$\sharp\mathcal{P} := \sum_{u \in \mathbb{Z}^2} d_{\mathcal{P}_+}(u).$$

As mentioned before, lattice multi-polygons treated in [10] are required that consecutive tree points v_{i-1}, v_i, v_{i+1} do not lie on a same line. But if the sign assignment ϵ satisfies the condition (1) and (2) of Definition 2.4, then it can be shown as in [6, Theorem 3.1] or [10, Theorem 3.1] that the following generalized Pick's formula for lattice multi-polygons holds.

Theorem 2.6. *Let $\mathcal{P}$ be a lattice multi-polygon of dimension* 2*. Then we have the following identity:*

$$\sharp\mathcal{P} = A(\mathcal{P}) + \frac{1}{2}B(\mathcal{P}) + C(\mathcal{P}).$$

Proof. For the sake of reader's convenience, we give a sketch of proof. To do so, let $\mathcal{P} = (v_1, \ldots, v_d)$ be a lattice multi-polygon, and assume that $\mathcal{P}$ contains consecutive three points lying on a line, say, v_1, v_2 and v_3. We consider a new sequence $(v_1, v_3, \ldots, v_d)$ and assign the sign for each of its sides by removing the second point of consecutive three points lying on a line and assign the signs. So we obtain a lattice multi-polygon containing no consecutive three points lying on a line, denoted by $\tilde{\mathcal{P}}$. Since the sign assignment ϵ continues to satisfy (1) and (2) of Definition 2.4, it can shown that all of

$$\sharp\tilde{\mathcal{P}}, \sharp\tilde{\mathcal{P}}^\circ, A(\tilde{\mathcal{P}}), B(\tilde{\mathcal{P}}), C(\tilde{\mathcal{P}})$$

coincide with those of the original lattice multi-polygon $\mathcal{P}$, respectively. This completes the proof of Theorem 2.1. $\square$

Next we give an example to illustrate Theorem 2.6.

Example 2.7. Take a unimodular sequence

$$\mathcal{P} = (v_1, v_2, v_3, v_4, v_1) = ((1,0), (0,1), (-1,0), (0,-1), (1,0)),$$

see Figure 2.1. Then obtain

$$\epsilon_1 = \det(v_1, v_2) = 1,\ \ \epsilon_2 = \det(v_2, v_3) = 1 = \epsilon_3 = \epsilon_4$$

$$a_1 = a_2 = a_3 = a_4 = 0$$

and the rotation number of $\mathcal{P}$ is 1. Note that

$$\sharp\mathcal{P} = 5 \text{ and } A(\mathcal{P}) = 2,$$

and it follows from Theorem 2.1 and Definition 2.3 that

$$B(\mathcal{P}) = 4 \text{ and } C(\mathcal{P}) = \frac{1}{12}(0 + 3 \cdot 4) = 1.$$

Therefore we have

$$5 = \sharp\mathcal{P} = A(\tilde{\mathcal{P}}) + \frac{1}{2}B(\tilde{\mathcal{P}}) + C(\tilde{\mathcal{P}}) = 2 + \frac{1}{2}4 + 1,$$

as expected.

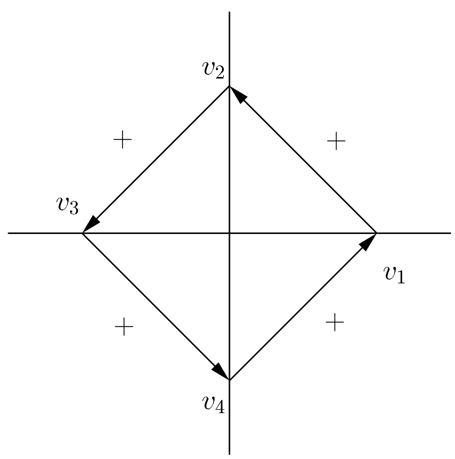


Figure 2.1: Figure

For the proof of our main Theorem 1.1 given in Chapter 4, we need to recall the Ehrhart polynomials of a multi-polygon $\mathcal{P}$. Indeed, for a given a positive integer m we can dilate $\mathcal{P}$ by m times, denoted $m\mathcal{P}$. In other words, if $\mathcal{P}$ is $(v_1, \ldots, v_d)$ with a sign assignment ϵ, then $m\mathcal{P}$ is $(mv_1, \ldots, mv_d)$ with $\epsilon(v_i v_{i+1})$ as the sign of the side $mv_i mv_{i+1}$ of $m\mathcal{P}$ for each i. Then we have

$$\sharp(m\mathcal{P}) = A(\mathcal{P})m^2 + \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}). \tag{2.3}$$

That is, $\sharp(m\mathcal{P})$ is a polynomial in m of degree at most 2 whose coefficients are as above.

Chapter 3

Multi-fans and multi-polytopes

The aim of this chapter is to review and set up some necessary notations and definitions for multi-fans and multi-polytopes and to explain how to derive the Pick's formula for multi-polytopes. This chapter is heavily dependent on the paper [5].

3.1 Ordinary fans

To do so, let N be a lattice of rank n which is isomorphic to $\mathbb{Z}^n$. We denote the real vector space $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral* with apex at the origin if there exists a finite number of $v_1, \ldots, v_m$ in N such that

$$\sigma = \{r_1 v_1 + \cdots + r_m v_m | r_i \in \mathbb{R} \text{ and } r_i \geq 0 \text{ for all } i\},$$

and $\sigma \bigcap (-\sigma) = \{0\}$. Here "rational" means that it is generated by vectors in the lattice N, and "strong" convexity that it contains no line through the origin. We often call a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ simply a cone in N. The dimension dim σ of a cone σ is the dimension of the linear space spanned by vectors in σ. A subset τ of σ is called a face of σ if there is a linear function $l : N_{\mathbb{R}} \to \mathbb{R}$ such that l takes nonnegative values on σ and

$\tau = l^{-1}(0) \cap \sigma$. A cone is regarded as a face of itself, while others are called proper faces.

Definition 3.1. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) each face of a cone in Δ is also a cone in Δ,

(2) the intersection of two cones in Δ is a face of each.

Definition 3.2. A fan Δ is said to be complete if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$. A cone is called *simplicial* if it is generated by linearly independent vectors. If the generating vectors can be taken as a part of a basis of N, then the cone is called *non-singular*.

Definition 3.3. A fan Δ is said to be *simplicial* (resp. *non-singular*) if every cone in Δ is *simplicial* (resp. *non-singular*).

For each $\tau \in \Delta$, we define N^τ to be the quotient lattice of N by the sublattice generated (as a group) by $\tau \cap N$; so the rank of N^τ is $n - \dim \tau$. We consider cones in Δ that contain τ as a face, and project them on $N^\tau_{\mathbb{R}}$. These projected cones form a fan in N^τ, which we denote by Δ_τ and call the projected fan with respect to τ. The dimensions of the projected cones decrease by dim τ. The completeness, simpliciality and non-singularity of Δ are inherited to Δ_τ for any τ.

3.2 Multi-fans

Let N be as lattice of rank n. Denote by $\mathrm{Cone}(N)$ the set of all cones in N. An ordinary fan is a subset of $\mathrm{Cone}(N)$. The set $\mathrm{Cone}(N)$ has a (strict) partial ordering $\prec$ defined by : $\tau \prec \sigma$ if and only if τ is a proper face of σ. The cone $\{0\}$ consisting of the origin is the unique minimum element in $\mathrm{Cone}(N)$.

On the other hand, let Σ be a partially ordered finite set with a unique minimum element. We denote the (strict) partial ordering by $<$ and the minimum element by $*$. An example of Σ used later is an abstract simplicial set with an empty set added as a member, which we call an augmented simplicial set. In this case the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1)-simplex.

Suppose that there is a map

$$C : \Sigma \ \rightarrow \ \mathrm{Cone}(N)$$

such that

(1) $C(*) = \{0\}$,

(2) If $I < J$ for $I, J \in \Sigma$, then $C(I) \prec C(J)$,

(3) For any $J \in \Sigma$ the map C restricted on $\{I \in \Sigma \mid I \leq J\}$ is an isomorphism of ordered sets onto $\{\kappa \in \mathrm{Cone}(N) \mid \kappa \preceq C(J)\}$.

For an integer m such that $0 \leq m \leq n$, we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \ \dim C(I) = m\}.$$

It can be shown that $\Sigma^{(m)}$ does not depend on C. When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m, namely I is an $(m-1)$-simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notion $|I|$ for m, when $I \in \Sigma^{(m)}$.

The image $C(\Sigma)$ is a finite set of cones in N. We may think of a pair (Σ, C) as a set of cones in N labeled by the ordered set Σ. Cones in an ordinary fan intersect only at their faces, but one special feature of a multi-fan is that cones in $C(\Sigma)$ may overlap, even the same cone may appear repeatedly with different labels. The pair (Σ, C) is almost what we call a multi-fan, but we incorporate a pair of weight functions on cones in $C(\Sigma)$ of the highest dimension $n = \text{rank } N$. More precisely, we consider two functions

$$\omega^{\pm} : \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

such that $\omega^{+} > 0$ or $\omega^{-} > 0$ for every $I \in \Sigma^{(n)}$. These two functions have its origin from geometry. In fact, if M is a torus manifold of dimension $2n$ and if $M_{i_1}, ..., M_{i_n}$ are characteristic submanifolds such that their intersection contains at least one $T-$fixed point, then the intersection $M_I = \bigcap_{\nu \in I} M_{i_\nu}$, consists of a finite number of $T-$fixed points. At each fixed point $p \in M_I$ the tangent space τ_p has two orientations; one is endowed by the orientation of M and the other comes from the intersection of the oriented submanifolds M_{i_ν}. Denoting the ratio of the above two orientations by ϵ_p we define the number $\omega^{+}(I)$ to be the number of points $p \in M_I$ with $\epsilon_p = +1$ and similarly for $\omega^{-}(I)$.

Definition 3.4. We call a triple $\Delta := (\Sigma, C, \omega^{\pm})$ a multi-fan in N. We define the *dimension* of Δ to be the rank of N (or the dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of Cone(N), one can view it as a multi-fan by taking $\Sigma = \Delta$, $C =$ the inclusion map, $\omega^{+} = 1$, and $\omega^{-} = 0$. In a similar way as in the case of ordinary fans, we say that a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is *simplicial* (resp. *non- singular*) if every cone in $C(\Sigma)$ is simplicial (resp. *non-singular*).

The following lemma holds.

Lemma 3.5. *A multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

The definition of completeness of a multi-fan Δ is rather involved. A naive definition of the completeness would be that the union of cones in

$C(\Sigma)$ covers the entire space $N_{\mathbb{R}}$. Although the two weighted functions $\omega^{\pm}$ are incorporated in the definition of a multi-fan, only the difference

$$\omega^{\pm} := \omega^{+} - \omega^{-}$$

is important in this thesis.

3.3 Completeness of a multi-fan

In order to define the completeness of a multi-fan, we first need to introduce the following intermediate notion of pre-completeness. A vector $v \in N_{\mathbb{R}}$ will be called *generic* if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n. For a generic vector v we set

$$d_v = \sum\nolimits_{v \in C(I),\ I \in \Sigma^{(n)}} \omega(I),$$

where the sum is understood to be zero if there is no such I.

Definition 3.6. We call a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ of dimension n is *pre-complete* if $\Sigma^{(n)} \neq \emptyset$ and the integer d_v is independent of the choice of generic vectors v. We call this integer the *degree* of Δ and denote it by $\deg(\Delta)$.

We remark that for an ordinary fan, pre-completeness is the same as completeness. Now we define the completeness for a multi-fan Δ. To do so, we need to define a projected multi-fan with respect to an element in Σ. We do it as follows. For each $K \in \Sigma$, we set

$$\Sigma_K := \{J \in \Sigma \mid K \leq J\}.$$

It inherits the partial ordering from Σ, and K is the unique minimum element in Σ_K. A map

$$C_K : \Sigma_K \rightarrow \mathrm{Cone}(N^{C(K)})$$

sending $J \in \Sigma_K$ to the Cone $C(J)$ projected on $(N^{C(K)})_{\mathbb{R}}$ satisfies the three properties above required for C. We define two functions

$$\omega_K{}^{\pm} : \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

to be the restrictions of $\omega^{\pm}$ to $\Sigma_K^{(n-|K|)}$. the triple $\Delta_K := (\Sigma, C, \omega^{\pm})$ is a multi-fan in $N^{C(K)}$, and this is the desired $projected\ multi-fan$ with respect to $K \in \Sigma$. When Δ is an ordinary fan, this definition agrees with the previous one.

Definition 3.7. A pre-complete multi-fan $\Delta = (\Sigma, C, \omega^{(\pm)})$ is said to be *complete* if the projected multi-fan Δ_K is pre-complete for any $K \in \Sigma$.

Lemma 3.8. *A multi-fan Δ is complete if and only if the projected multi-fan Δ_J is pre-complete for any $J \in \Sigma^{(n-1)}$.*

Proof. The argument for the proof goes as follows. Indeed, the pre-completeness of Δ_J for $J \in \Sigma^{(n-1)}$ implies that $d_v = \sum_{v \in C(I)} \omega(I)$ remains unchanged when v gets across the codimension one Cone $C(J)$, which means the pre-completeness of Δ. Since $\Sigma_K^{(n-|K|-1)}$ is contained in $\Sigma^{(n-1)}$ for any $K \in \Sigma$, the pre-completeness of Δ_J for any $J \in \Sigma^{(n-1)}$ also implies the pre-completeness of Δ_K for any $K \in \Sigma$, as desired. □

Example 3.9. Let v_1, v_2, v_3 be vectors shown in Figure 3.1, and Σ be an ordinary polytope given by the folowing simplicial complex

$$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}.$$

Then define a function $C : \Sigma \to \mathrm{Cone}(N)$ by

$$\begin{aligned} C(\{i\}) &= \text{the cone spanned by } v_i \\ C(\{i, i+1\}) &= \text{the cone spanned by } \ v_i \text{ and } \ v_{i+1}, \end{aligned}$$

Here we assume that $v_4 = v_1$. Let us also take weight functions $w^{\pm}$ such that $w = 1$ on every two dimensional cone in

$$\Sigma^{(2)} = \{\{1,2\},\{2,3\},\{3,1\}\}.$$

Then,

$$\Delta = (\Sigma, C, w),\ \ I\ \in\ \Sigma^{(2)}$$

is a complete non-singular two-dimensional multi-fan (actually, fan) with $\deg(\Delta) = 1$.

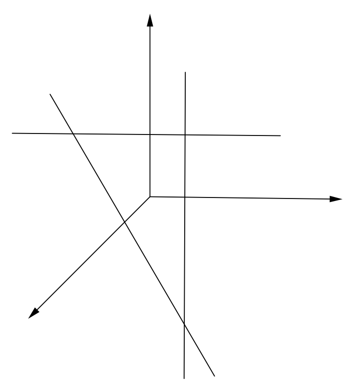


Figure 3.1

3.4 Multi-polytopes

From now on, let a convex polytope P in $V^* = \mathrm{Hom}(V, \mathbb{R})$ is the convex hull of a finite set of points in V^*, where $V = N_{\mathbb{R}}$. It is the intersection of a finite number of half space in V^* separated by affine hyperplanes, so there are a finite number of nonzero vectors $v_1, \ldots, v_d \in V$ and real numbers $c_1, \ldots, c_d$ such that

$$P = \{u \in V^* \mid \langle u, v_i \rangle \leq c_i \text{ for all } i\},$$

where $\langle \ , \ \rangle$ denotes the natural pairing between V^* and V. A polytope gives rise to a multi-fan in this way. Note that a convex polytope gives rise to a complete fan.

Now, for a complete multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ let us denote by $\mathrm{HP}(V^*)$ the set of all affine hyperplanes in V^*.

Definition 3.10. Let $\Delta = (\Sigma, C, \omega^{\pm})$ be a compete multi-fan and let

$$\mathcal{F} : \Sigma^{(1)} \ \rightarrow \ \mathrm{HP}(V^*)$$

be a map such that the affine hyperplane $\mathcal{F}(I)$ is perpendicular to the half line $C(I)$ for each $I \in \Sigma^{(1)}$, i.e., an element in $C(I)$ takes a constant on $\mathcal{F}(I)$. We call a pair $(\Delta, \mathcal{F})$ a *multi-polytope* and denote it by $\mathcal{P}$. The dimension of a multi-polytope $\mathcal{P}$ is *simple* if Δ is simplicial.

Chapter 4

Main results: comparison of two generalized Pick's Theorems

The goal of this chapter is to show that two version of generalized Pick's theorems are actually equivalent.

To be more precise, let $(\Sigma, C, \omega^{\pm})$ be a complete multi-polytope in the sense of the paper [5] of Hattori-Masuda, and let $\mathrm{HP}(V^*)$ be the set of all affine hyperplanes in $V^* = \mathrm{Hom}(V, \mathbb{R})$, where $V = N_{\mathbb{R}} = N \otimes \mathbb{R}$. Let $\mathcal{F} : \Sigma^{(1)} \to \mathrm{HP}(V^*)$ be a map such that the affine hyperplane $\mathcal{F}(I)$, $I \in \Sigma^{(1)}$, is perpendicular to the half line $C(I)$.

From now on, we assume that a multi-polytope $\mathcal{P}$ is simple, so that its corresponding multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is complete and simplicial, unless stated otherwise.

We also assume that Σ consists of subsets of $[d] := \{1, 2, \cdots, d\}$, and that $\Sigma^{(1)} = \{\{1\}, \cdots, \{d\}\}$. Let us denote by v_i a non-zero vector in the one-dimensional cone $C(\{i\})$, and let $F_i := \mathcal{F}(\{i\})$. Then, for $I \in \Sigma^{(n)}$ such that $|I| = n$, $F_I := \bigcap_{i \in I} F_i$ is just a point. This point will be denoted by u_I.

Throughout this chapter, we shall exclusively deal with the cone of $n = 2$ only. This our maximal dimensional cone is 2-dimensional, so that $u_I \in \mathbb{R}^2 \cong$

V^*, when $n = 2$. We assume also that $u_I \in \mathbb{Z}^2$, i.e., u_I is an integral lattice point in $\mathbb{R}^2$. In this case, a multi-polytope is called *integral*.

On the other hand, there is a notion of an integral oriented polygon in $\mathbb{R}^2$, which we want to recall here. To do so, recall that multi-polygon means a piecewise linear closed curve and that an integral vertex means that vertex lies in the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. With theses understood, let $\tilde{\mathcal{P}}$ be an integral oriented multi-polygon in $\mathbb{R}^2$ with sign attached to each side. Note that $\tilde{\mathcal{P}}$ may have self-intersections but do not have three consecutive vertices lying on a line.

From now on, let us denote the oriented sides of $\tilde{\mathcal{P}}$ by s_i $(i = 1, 2, \ldots, d)$, where we number s_i's so that the next side of s_i in $\tilde{\mathcal{P}}$ is s_{i+1}. Let $\mathrm{sign}(s_i)$ denote the assigned sign of s_i, and let v_i be a normal vector of s_i such that the 90° counter clockwise rotation of $\mathrm{sign}(s_i)v_i$ has the same direction on s_i. For examples, see Figures 4.1 and 4.2.

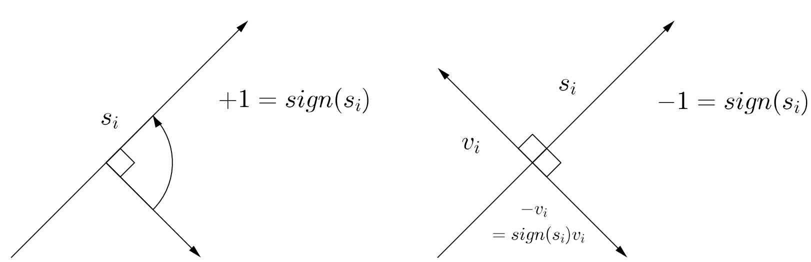
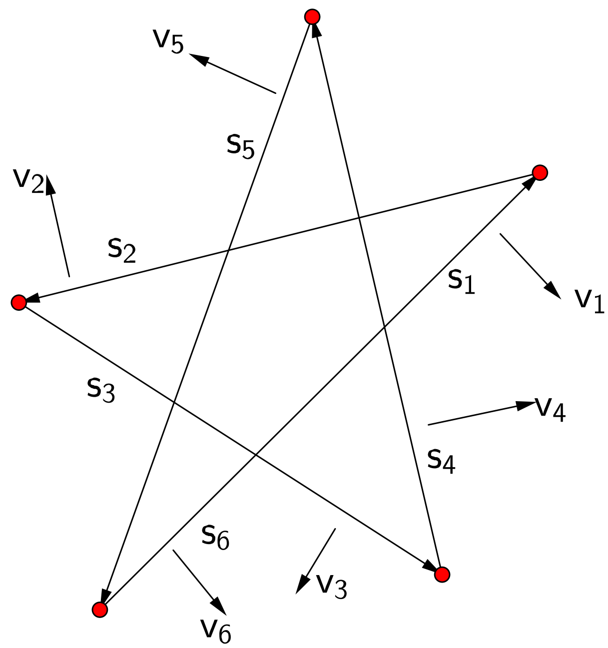


Figure 4.1: Figure

Next, we need to briefly recall the notion of a projected multi-fan with respect to an element in Σ. To be precise, let $K \in \Sigma$ and set

$$\Sigma_K := \{J \in \Sigma \mid K \leq J\}.$$



$sign(s_i) = +1$
$i = 1, 2, ..., 6$

Figure 4.2: Figure

Then K is the unique minimal element in Σ_K with respect to the partial ordering induced from Σ. Let $C_K : \Sigma_K \to \text{Cone}(N^{C(K)})$ be a map given by mapping $J \in \Sigma_K$ to the cone $C(J)$ projected on $(N^{C(K)})_{\mathbb{R}}$.

We also define two functions

$$\omega_K^{\pm} : \Sigma_K^{(n-|K|)} \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

by the restrictions of $\omega^{\pm}$ to $\Sigma_K^{(n-|K|)}$. Then the triple $\Delta_K := (\Sigma_K, C_K, \omega_K^{\pm})$ is a multi-fan in $N^{C(K)}$, called the projected multi-fan with respect to $K \in \Sigma$.

Recall that a vector $v \in V = \mathbb{N}_{\mathbb{R}}$ is called generic if v does not lie on any linear subspace spanned by a Cone in $C(\Sigma)$ of dimension $< n$. For a generic vector v,

$$d_v = \sum_{\substack{v \in C(I) \\ I \in \Sigma^{(n)}}} \omega(I).$$

Note that if for each $\{i\} \in \Sigma^{(1)}$, there are exactly two j and k ($j \neq k$) such that $\{i,j\}, \{i,k\} \in \Sigma^{(2)}$, then we have

$$d_v = \sum_{\substack{v \in C_{\{i\}}(I) \\ I \in \Sigma^{(1)}_{\{i\}}}} \omega_{\{i\}}(I) = \pm 1.$$

Then following proposition plays an important role in the proof of our main Theorem 1.1.

Proposition 4.1. *Let $\mathcal{P} = (\Delta, \mathcal{F})$ be a complete, simple and integral multi-polytope of dimension 2 such that for each $\{i\} \in \Sigma^{(1)}$, there are exactly two j and k ($j \neq k$) such that $\{i,j\}$ and $\{i,k\}$ are all elements of $\Sigma^{(2)}$. Then there is an integral oriented multi-polygon $\tilde{\mathcal{P}}$ associated to $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be an integral and simple multi-polytope as the proposition. Then it follows from its definition that there is a complete and simplicial multi-fan $\Delta = (\Sigma, \Delta, \omega^{\pm})$ of dimension 2 and a map

$$\mathcal{F} : \Sigma^{(1)} \ \rightarrow \ \mathrm{HP(V^*)}$$

such that $\mathcal{F}(\{i\})$ is perpendicular to the half-line $C(\{i\})$ spanned by v_i. Here

$$\Sigma^{(1)} = \{1, 2, \ldots, d\}, \ \ V^* \cong \mathbb{R}^2.$$

By assumption, for each $\{i\} \in \Sigma^{(1)}$, there are exactly two 2-dimensional cones, say $\{i,j\}$ and $\{i,k\}$. Hence, for each hyperplane $\mathcal{F}(\{i\}) = F_i$ which is a line in $\mathbb{R}^2$, there are exactly two vertices $u_{\{i,j\}}$ and $u_{\{i,k\}}$.

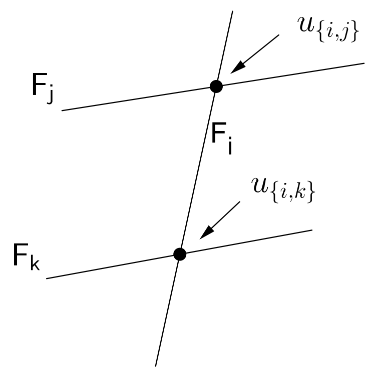


Figure 4.3

Now, in order to obtain our desired oriented multi-polygon, we form a "closed" piecewise linear curve by using the hyperplane F_i's and vertices $u_{\{i,j\}}$ and $u_{\{i,k\}}$. Note also that all vertices on the closed piecewise linear curve are integral by assumption. Next, we denote each side from F_i by s_i, and assign each normal vector v_i of F_i to the side s_i. See Figures 4.3 and 4.4. Finally, we can also assign the sign(s_i) to each s_i by requiring 90 degree counterclockwise rotation sign$(s_i)v_i$ to have the same direction on s_i. Hence we are done $\square$

Now recall that a multi-polytope is defined by the affine hyperplanes perpendicular to the half-line $C(\{i\})$ for $\{i\} \in \Sigma^{(1)}$.

Theorem 4.2. *Let $\tilde{\mathcal{P}}$ be an integral oriented multi-polygon of dimension 2. Then there is an associated complete, simple, and integral multi-polytope $\mathcal{P}$ of dimension 2 such that for each $\{i\} \in \Sigma^{(1)}$, there are exactly two j and k $(j \neq k)$ such that $\{i,j\}, \{i,k\} \in \Sigma^{(2)}$. Moreover,we have the following*

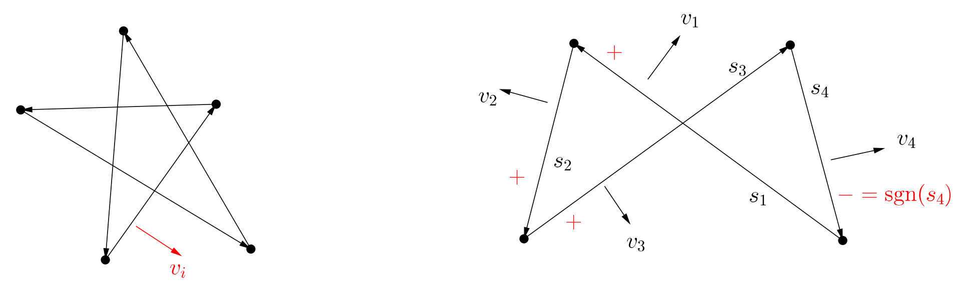


Figure 4.4

identity:

$$A(\tilde{\mathcal{P}}) = \mathrm{vol}(\mathcal{P}),\ B(\tilde{\mathcal{P}}) = \#(\partial\mathcal{P}),\ C(\tilde{\mathcal{P}}) = \deg(\Delta).$$

Proof. For the proof of the first statement, suppose that we have an integral oriented multi-polygon $\tilde{\mathcal{P}}$ with sign assigned to each side. Then we can easily recover the corresponding multi-polytope. To be more precise, for each side labelled s_i, assign an affine hyperplane F_i with normal vector v_i.

Using this information, we then form a map

$$C : \sum \ \to\ \mathrm{Cone}(V^*),\ V^*\ \cong\ \mathbb{R}^2$$

such that $C(\{i\})=$ one-dimensional half-line spanned by v_i. We also set $\omega(I) = +1$ for $I \in \Sigma$. Hence we can obtain a multi-fan $(\Sigma, C, \omega^{\pm})$, $\Sigma = \Delta$ and its corresponding multi-polytope $\mathcal{P} = (\Delta, F)$. Note that $\mathcal{P}$ is integral by the assumption of $\tilde{\mathcal{P}}$.

Since each side has only two vertices, there should be exactly two hyperplanes F_j and F_k intersecting F_i. But this implies that, for each $\{i\} \in \Sigma^{(1)}$ there are exactly two 2-dimensional cones $\{i, j\}$ and $\{i, k\}$. So $\mathcal{P}$ satisfies the assumption of the first item of the theorem.

Moreover, it follows from definition of $\tilde{\mathcal{P}}$ that $\mathcal{P}$ is always simple. So it remains to show that $\mathcal{P}$ is complete. Since $\omega(I) = +1$ for $I \in \Sigma$ and for each

$\{i\} \in \Sigma^{(1)}$, there are exactly two j and k ($j \neq k$) such that $\{i,j\}, \{i,k\} \in \Sigma^{(2)}$, it is easy to see

$$d_v = \sum_{\substack{v \in C_{\{i\}}(I) \\ I \in \Sigma^{(1)}_{\{i\}}}} \omega(I) = +1,$$

which is constant. Therefore, we can conclude that $(\Sigma, C, \omega^{\pm})$ is complete. This completes the proof of the first statement of Theorem 4.2.

Next, for the proof of the second statement we first recall the generalized Pick's theorem for multi-polytopes and multi-polygons. To do so, we begin with the case of multi-polygons $\tilde{\mathcal{P}}$, and we need to recall three invariants of $\tilde{\mathcal{P}}$. For this, we identify $\mathbb{R}^2$ (resp. $\mathbb{Z}^2$) with $H^2(BT : \mathbb{R})$ (resp. $H^2(BT : \mathbb{Z})$, and think of $\tilde{\mathcal{P}}$ on a polygon in $H^2(BT : \mathbb{R})$. We may also assume that each consecutive pair v_{i-1} and v_i is a basis of $H_2(BT : \mathbb{Z})$. Note also that by Masuda [10] there is a unitary toric manifold M of real dimension 4 whose multi-fan is $\tilde{\mathcal{P}}$. We may assume that the T-action on M is effective and $H^{\mathrm{odd}}(M; \mathbb{Z}) = 0$, where $T = S^1 \times S^1$ in this case.

For the associated multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$, there are real numbers c_i's given by

$$\mathcal{F}(\{i\}) = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_i \rangle = c_i\},$$

and these numbers c_i's determine an element

$$c_1^T(\mathcal{P}) = \sum_{i=1}^{d} c_i x_i \in H_T^2(\Delta; \mathbb{R})$$

called the *equivariant first Chern class* of $\mathcal{P}$. In particular, when Δ is non-singular, $\mathcal{P}$ is a lattice multi-polytope if and only if all c_i's are integers ([5, p. 26]).

With this set-up, let L be a complex T-line bundle over M such that

$$c_1^T(L) = \sum c_i \xi_i \in H_T^2(M; \mathbb{Z}),$$

where ξ_i's are dual elements of v_i's in $H^2(BT;\mathbb{Z})$ and c_i's are integers defined in (4.1). Note that the existence of L follows from [10, Lemma 3.2]. We now apply the Riemann-Roch formula for multi-polygons (see, e.g., [10, Theorem 7.2]) to the multi-polygon $\tilde{\mathcal{P}}$, so that we have

$$\#(\tilde{\mathcal{P}}) = \left\langle e^{c_1(L)} Td(M), [M] \right\rangle, \tag{4.1}$$

where $Td(M)$ denotes the Todd class of M. On the other hand, by applying the Riemann-Roch formula for multi-polytopes given in [5, Theorem 8.5] to our multi-polytope $\mathcal{P}$ we can also obtain

$$\#(\mathcal{P}) = \int_\Delta e^{c_1(\mathcal{P})} \mathcal{T}(\Delta), \tag{4.2}$$

where $c_1(\mathcal{P})$ and $\mathcal{T}(\Delta)$ denote the first Chern class and Todd class of $\mathcal{P}$, respectively (see [5, p. 28]). We then claim that

$$\#(\tilde{\mathcal{P}}) = \#(\mathcal{P}).$$

To see it, it suffices to note that by their constructions $c_1(L)$ and $Td(M)$ play the same roles as $c_1(\mathcal{P})$ and $\mathcal{T}(\Delta)$, respectively, in the above two equations (4.1) and (4.2).

Now, by applying the same arguments to $m\mathcal{P}$ and $m\tilde{\mathcal{P}}$ for $m \in \mathbb{Z}$, it is also easy to obtain

$$\begin{aligned} \#(m\tilde{\mathcal{P}}) &= A(\tilde{\mathcal{P}})m^2 + \frac{1}{2}B(\tilde{\mathcal{P}})m + C(\tilde{\mathcal{P}}), \\ = \#(m\mathcal{P}) &= A(\mathcal{P})m^2 + \frac{1}{2}B(\mathcal{P})m + C(\mathcal{P}), \end{aligned}$$

where

$$A(\mathcal{P}) = \mathrm{vol}(\mathcal{P}),\ B(\mathcal{P}) = \#(\partial\mathcal{P}),\ C(\mathcal{P}) = \deg(\Delta).$$

As a consequence, we can conclude that the following identities hold:

$$A(\tilde{\mathcal{P}}) = A(\mathcal{P}),\ B(\tilde{\mathcal{P}}) = B(\mathcal{P}),\ C(\tilde{\mathcal{P}}) = C(\mathcal{P}).$$

This completes the proof of Theorem 4.2. $\square$

Finally we close this chapter with two examples to show the validity of Theorem 4.2.

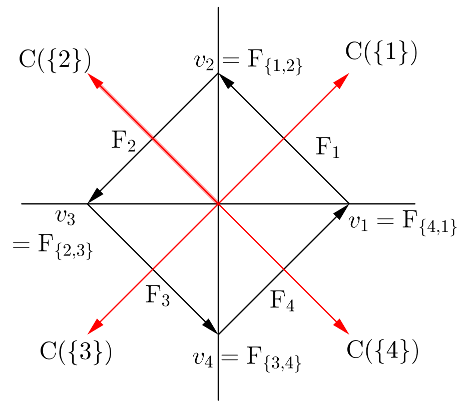


Figure 4.5: Figure

Example 4.3. Let v_1, v_2, v_3, v_4 be vectors above. Then it is also easy to compute

$$\deg(\Delta) = 1 = C(\tilde{\mathcal{P}}),\ \#(\mathcal{P}^\circ) = 1,\ \text{and}\ B(\tilde{\mathcal{P}}) = \#(\partial\mathcal{P}) = 4.$$

Moreover, $A(\tilde{\mathcal{P}}) = \mathrm{vol}(\mathcal{P}) = 4$. So we see that this fits well with our main Theorem.

Example 4.4. Let v_1, v_2, v_3, v_4 be vectors as in Figure 4.6. Then it is also easy to compute

$$\deg(\Delta) = 1 = C(\tilde{\mathcal{P}}),\ \#(\mathcal{P}^\circ) = 1,\ \text{and}\ B(\tilde{\mathcal{P}}) = \#(\partial\mathcal{P}) = 8.$$

Moreover, $A(\tilde{\mathcal{P}}) = \mathrm{vol}(\mathcal{P}) = 4$. So we see that this fits well with our main Theorem 4.2.

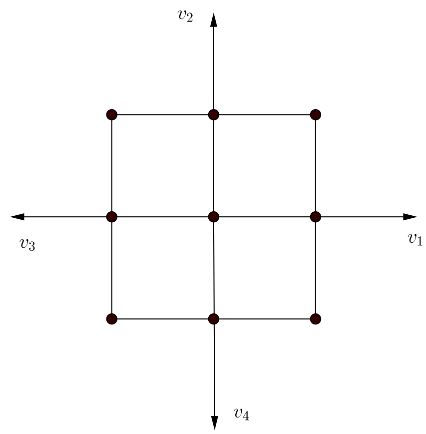


Figure 4.6: Figure

Example 4.5. Take a unimodular sequence

$$\mathcal{P} = (v_1, \dots, v_5, v_1) = ((1,0),(0,1),(-1,0),(0,-1),(-1,-1),(1,0)),$$

see Figure 4.8. Then

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_5 = 1,\ \epsilon_4 = -1 \text{ and } a_1 = a_4 = a_5 = 1, a_2 = a_3 = 0,$$

and the rotation number of $\mathcal{P}$ around the origin is 1. Note also that

$$\sharp\tilde{\mathcal{P}} = 4 \text{ and } A(\tilde{\mathcal{P}}) = \frac{3}{2},$$

and it follows from Theorem 2.1 and Definition 2.3 that

$$B(\tilde{\mathcal{P}}) = 1+1+1-1+1 = 3 \text{ and } C(\tilde{\mathcal{P}}) = \frac{1}{12}(3+3\cdot 4-3) = 1.$$

Further, note also that

$$A(\mathcal{P}) = \frac{3}{2},\ B(\mathcal{P}) = 3,\ \text{and } C(\mathcal{P}) = 1,$$

as expected. Figure 4.7 shows multi-fan for Example 4.7.

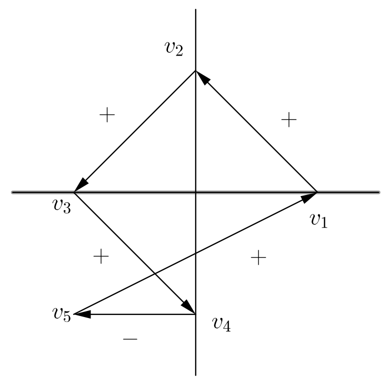


Figure 4.7

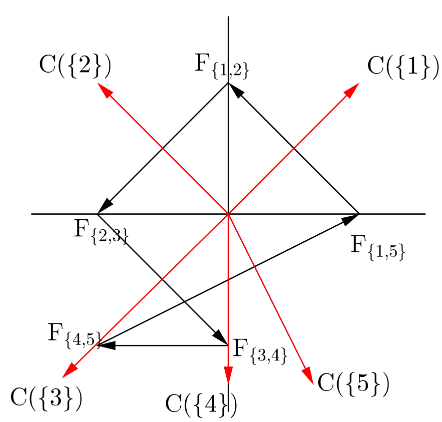


Figure 4.8

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