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## The Existence of Some

Metrics on Riemannian Warped Product Manifolds with Fiber Manifold of Class (B)

조선대학교 교육대학원
수학교육전공
채 송 화

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엽다양체가 $(\mathrm{B})$ 류인 경우의 리만 휜곱다양체 위의 거리의 존재성

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이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.
2015년 10월

조선대학교 교육대학원
수학교육전공
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채송화의 교육학 석사학위 논문을 인준함.

심사위원장 조선대학교 교수 김 남 권 (인)

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2015년 11월

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## 국 문 초 록

# The Existence of Some Metrics on Riemannian Warped Product Manifolds with Fiber Manifold of Class (B) 

- 엽다양체가 (B)류인 경우의 리만 훤곱다양체 위의 거리의 존재성 -

채 송 화<br>지도교수 : 정 윤 태<br>조선대학교 교육대학원 수학교육전공

미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수 에 관한 연구이다.
연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.
Kazdan and Warner ([K.W.1,2,3])의 결과에 의하면 $N$ 위의 함수 $f$ 가 $N$ 위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있는 데 먼저
(A) $N$ 위의 함수 $f$ 가 Riemannian metric의 scalar curvature이면 그 함수 $f$ 가 적당한 점 $x_{0}$ 에서 $f\left(x_{0}\right)<0$ 일 때이다. 즉, $N$ 위에 nagative constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.
(B) $N$ 위의 함수 $f$ 가 Riemannian metric의 scalar curvature이면 그 함수 $f$ 가 항등적으로 $f \equiv 0$ 이거나 모든 점에서 $f(x)<0$ 인 경우이다.
이 경우에는, $N$ 위에서 zero scalar curvature를 갖는 Riemannian metric이

존재하는 경우이다.
(C) $N$ 위의 임의의 함수 $f$ 를 scalar curvature로 갖는 Riemannian metric 이 존재하는 경우이다.
본 논문에서는 엽다양체 $N$ 이 (B)에 속하는 compact Riemannian manifold 일 때, Riemannian warped product manifold인 $M=[a, \infty) \times{ }_{f} N$ 위에 함수 $R(t, x)$ 가 적당한 조건을 만족하면 $R(t, x)$ 가 Riemannian warped product metric의 scalar curvature가 될 수 있는 warping function $f(x)$ 가 존재할 수 있음을 상해•하해 방 법을 이용하여 증명하였다.

## I. INTRODUCTION

One of the basic problems in the differential geometry is studying the set of curvature functions which a given manifold possesses.

The well-known problem in differential geometry is that of whether there exists a warping function of warped metric with some prescribed scalar curvature function. One of the main methods of studying differential geometry is the existence and the nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. In order to study these kinds of problems, we need some analytic methods in differential geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [B.K.], [B.O.], [D.D.], [G.L.], [K.K.P.], [L.M.], [M.M.]), and also in studying $L_{2}$-cohomology (cf. [Z.]).

In a study [L. 1, 2], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and the nonexistence of Riemannian warped metric with some prescribed scalar curvature function.

In this paper, we also study the existence and the nonexistence of Riemannian warped product metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. So, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M=B \times_{f} N$ in terms of its warping function $f$ and the scalar curvatures of $B$ and $N$.

By the results of Kazdan and Warner (cf. [K.W. 1, 2, 3]), if $N$ is a compact Riemannian $n$-manifold without boundary, $n \geq 3$, then $N$ belongs to one of the following three categories:
(A) A smooth function on $N$ is the scalar curvature of some Riemannian metric on $N$ if and only if the function is negative somewhere.
(B) A smooth function on $N$ is the scalar curvature of some Riemannian metric on $N$ if and only if the function is either identically zero or strictly negative everywhere.
(C) Any smooth function on $N$ is the scalar curvature of some Riemannian metric on $N$.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold $N$.

In [K.W. 1, 2, 3], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on open manifold. Results of Gromov and Lawson (cf. [G.L.]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded (cf. [G.L.], [L.M., p.322]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature (cf. [B.K.]). It follows from the results of Aviles and McOwen (cf.

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[A.M.]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In this paper, when $N$ is a compact Riemannian manifold, we discuss the method of using warped products to construct Riemannian metrics on $M=$ $[a, \infty) \times{ }_{f} N$ with specific scalar curvatures, where $a$ is a positive constant. It is shown that if the fiber manifold $N$ belongs to class (B), then $M$ admits a Riemannian metric with some prescribed scalar curvature outside a compact set. That is, suppose that $R(g)=0$. and that $R(t) \in C^{\infty}([a, \infty))$ is a function such that

$$
\frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}}>R(t) \geq-\frac{4 n}{n+1} e^{\alpha t} \quad \text { for } \quad \mathrm{t}>\mathrm{t}_{0}
$$

where $t_{0}>a, \alpha>0$ and $0<c<1$ are constants. Then equation (3.4) has a positive solution on $[a, \infty)$.

These results are extensions of the results in [J.L.K.L.].
Although we will assume throughout this paper that all data ( $M$, metric $g$, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the given data is Hölder continuous.

## II. PRELIMINARIES

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. Let $\mathcal{X}(M)$ denote the set of all smooth vector fields defined on $M$, and let $\Im(M)$ denote the ring of all smooth real-valued functions on $M$. A connection $\nabla$ on a smooth manifold $M$ is a function

$$
\nabla: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)
$$

such that
(D1) $\nabla_{V} W$ is $\Im(M)$-linear in $V$,
(D2) $\nabla_{V} W$ is $\mathbb{R}$-linear in $W$,
(D3) $\nabla_{V}(f W)=(V f) W+f \nabla_{V} W$ for $f \in \Im(M)$.
(D4) $[V, W]=\nabla_{V} W-\nabla_{W} V$, and
(D5) $X<V, W>=<\nabla_{X} V, W>+<V, \nabla_{X} W>$
for all $X, V, W \in \mathcal{X}(M)$.

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If $\nabla$ satisfies axioms (D1) $\sim(\mathrm{D} 3)$, then $\nabla_{V} W$ is called the covariant derivative of $W$ with respect to $V$ for the connection $\nabla$. If $\nabla$ satisfies axioms (D4) $\sim$ (D5), then $\nabla$ is called the Levi - Civita connection of $M$, which is characterized by the Köszulformula (cf. [O.]).

A geodesic $c:(a, b) \rightarrow M$ is a smooth curve of $M$ such that the tangent vector $c^{\prime}$ moves by parallel translation along $c$. In other words, $c$ is a geodesic if

$$
\nabla_{c^{\prime}} c^{\prime}=0 \quad \text { (geodesic } \quad \text { equation) }
$$

A pregeodesic is a smooth curve $c$ which may be reparametrized to be a geodesic. Any parameter for which $c$ is a geodesic is called an affine parameter. If $s$ and $t$ are two affine parameters for the same pregeodesic, then $s=a t+b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c^{\prime} c^{\prime}}=0$ may be expressed as a system of linear differential equations. To this end, we let $\left(U,\left(x^{1}, x^{2}, \ldots, x^{n}\right)\right)$ be local coordinates on $M$ and
let $\frac{\partial}{\partial x^{1}}, \frac{\partial}{\partial x^{2}}, \ldots, \frac{\partial}{\partial x^{n}}$ denote the natural basis with respect to these coordinates. The connection coefficients $\Gamma_{i j}^{k}$ of $\nabla$ with respect to $\left(x^{1}, x^{2}, \ldots, x^{n}\right)$ are defined by

$$
\nabla_{\frac{\partial}{\partial x^{i}}}\left(\frac{\partial}{\partial x^{j}}\right)=\sum_{k=1}^{n} \Gamma_{i j}^{k} \frac{\partial}{\partial x^{k}} \quad \text { (connection } \quad \text { coefficients). }
$$

Using these coefficients, we may write equation as the system

$$
\frac{d^{2} x^{k}}{d t^{2}}+\sum_{k=1}^{n} \Gamma_{i j}^{k} \frac{d x^{i}}{d t} \frac{d x^{j}}{d t}=0 \quad \text { (geodesic equations in coordinates). }
$$

Definition 2.2. The curvature tensor of the connection $\nabla$ is a linear transformation valued tensor $R$ in $\operatorname{Hom}(\mathcal{X}(M), \mathcal{X}(M))$ defined by :

$$
R(X, Y)=\nabla_{X} \nabla_{Y}-\nabla_{Y} \nabla_{X}-\nabla_{[X, Y]} .
$$

Thus, for $Z \in \mathcal{X}(M)$,

$$
R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z .
$$

It is well-known that $R(X, Y) Z$ at $p$ depends only upon the values of $X, Y$ and $Z$ at $p$ (cf. [O.]).

If $\omega \in T_{p}^{*}(M)$ is a cotangent vector at $p$ and $x, y, z \in T_{p}(M)$ are tangent vectors at $p$, then one defines

$$
R(\omega, X, Y, Z)=(\omega, R(X, Y) Z)=\omega(R(X, Y) Z)
$$

for $X, Y$ and $Z$ smooth vector fields extending $x, y$ and $z$, respectively.

The curvature tensor $R$ is a (1,3)-tensor field which is given in local coordinates by

$$
R=\sum_{i, j, k, m=1}^{n} R_{j k m}^{i} \frac{\partial}{\partial x^{i}} \otimes d x^{j} \otimes d x^{k} \otimes d x^{m},
$$

where the curvature components $R_{j k m}^{i}$ are given by

$$
R_{j k m}^{i}=\frac{\partial \Gamma_{m j}^{i}}{\partial x^{k}}-\frac{\partial \Gamma_{k j}^{i}}{\partial x^{m}}+\sum_{a=1}^{n}\left(\Gamma_{m j}^{a} \Gamma_{k a}^{i}-\Gamma_{k j}^{a} \Gamma_{m a}^{i}\right) .
$$

Notice that $R(X, Y) Z=-R(Y, X) Z, \quad R(\omega, X, Y, Z)=-R(\omega, Y, X, Z)$ and $R_{j k m}^{i}=-R_{j m k}^{i}$.

Furthermore, if $X=\sum x^{i} \frac{\partial}{\partial x^{i}}, Y=\sum y^{i} \frac{\partial}{\partial y^{i}}, Z=\sum z^{i} \frac{\partial}{\partial z^{i}}$, and $\omega=\sum \omega_{i} d x^{i}$ then

$$
R(X, Y) Z=\sum_{i, j, k, m=1}^{n} R_{j k m}^{i} z^{j} x^{k} y^{m} \frac{\partial}{\partial x^{i}}
$$

and

$$
R(\omega, X, Y, Z)=\sum_{i, j, k, m=1}^{n} R_{j k m}^{i} z^{j} x^{k} y^{m}
$$

Consequently, one has $R\left(d x^{i}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{m}}, \frac{\partial}{\partial x^{j}}\right)=R_{j k m}^{i}$.

Definition 2.3. From the curvature tensor $R$, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its components are $R_{i j}=\sum_{i=1}^{n} R_{i k j}^{k}$. The Ricci tensor is symmetric and its contraction $S=\sum_{i j=1}^{n} R_{i j} g^{i j}$ is called the scalar curvature (cf. [A.],[B.E.],[B.E.E.]).

Definition 2.4. Suppose $\Omega$ is a smooth, bounded domain in $R^{n}$, and let $g=\Omega \times R \rightarrow R$ be a Caratheodory function. Let $u_{0} \in H_{0}^{1,2}(\Omega)$ be given.

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Consider the equation

$$
\begin{gathered}
\Delta u=g(x, u) \quad \text { in } \quad \Omega \\
u=u_{0} \quad \text { on } \quad \partial \Omega
\end{gathered}
$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $u \leq u_{0} \quad$ on $\quad \partial \Omega \quad$ and

$$
\int_{\Omega} \nabla u \nabla \varphi d x+\int_{\Omega} g(x, u) \varphi d x \leq 0 \quad \text { for } \quad \text { all } \quad \varphi \in C_{0}^{\infty}(\Omega), \quad \varphi \geq 0
$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold. We briefly recall some results on warped product manifolds. Complete details may be found in [B.E.] or [O.]. On a semi-Riemannian product manifold $B \times F$. let $\pi$ and $\sigma$ be the projections of $B \times F$ onto $B$ and $F$, respectively, and let $f>0$ be a smooth function on $B$.

Definition 2.5. The warped product manifold $M=B \times_{f} F$ is the product manifold $M=B \times F$ furnished with metric tensor

$$
g=\pi^{*}\left(g_{B}\right)+(f \circ \pi)^{2} \sigma^{*}\left(g_{F}\right)
$$

where $g_{B}$ and $g_{F}$ are metric tensors of $B$ and $F$, respectively. In other words, if $v$ is tangent to $M$ at $(p, q)$, then

$$
g(v, v)=g_{B}(d \pi(v), d \pi(v))+f^{2}(p) g_{F}(d \sigma(v), d \sigma(v))
$$

Here $B$ is called the base of $M$ and $F$ the fiber ([O.]).

We denote the metric $g$ by $<\quad, \quad>$. In view of Remark 2.13 (1) and Lemma 2.14 we may also denote the metric $g_{B}$ by $<,>$. The metric $g_{F}$ will be denoted by ( , ).

Remark 2.6. Some well known elementary properties of warped product manifold $M=B \times_{f} F$ are as follows :
(1) For each $q \in F$, the map $\left.\pi\right|_{\sigma^{-1}(q)=B \times q}$ is an isometry onto $B$.
(2) For each $p \in B$, the map $\left.\sigma\right|_{\pi^{-1}(q)=p \times F}$ is a positive homothetic map onto $F$ with homothetic factor $\frac{1}{f(p)}$.
(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at $(p, q)$.

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(4) The horizontal leaf $\sigma^{-1}(q)=B \times q$ is a totally geodetic submanifold of $M$ and vertical fiber $\pi^{-1}(q)=p \times F$ is a totally umbilic submanifold of $M$.
(5) If $\phi$ is an isometry of $F$, then $1 \times \phi$ is an isometry of $M$, and if $\psi$ is an isometry of $B$ such that $f=(f \circ \psi)$ then $\psi \times 1$ is an isometry of $M$.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification.

$$
T_{(p, q)}\left(B \times_{f} F\right) \cong T_{(p, q)}(B \times F) \cong T_{p} B \times T_{q} F
$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If $X$ is a vector field on $B$, we define $\bar{X}$ at $(p, q)$ by setting $\bar{X}(p, q)=\left(X_{p}, 0_{q}\right)$. Then $\bar{X}$ is $\pi$-related to $X$ and $\sigma$-related to the zero vector field on $F$. Similarly, If $Y$ is a vector field of $F, \bar{Y}$ is defined by $\bar{Y}(p, q)=\left(0_{p}, Y_{q}\right)$.

Lemma 2.7. If $h$ is a smooth function an $B$, Then the gradient of the lift ( $h \circ \pi$ ) of $h$ to $M$ is the lift to $M$ of gradient of $h$ on $B$.

Proof. We must show that $\operatorname{grad}(h \circ \pi)$ is horizontal and $\pi$-related to $\operatorname{grad}(h)$ on $B$. If $v$ is vertical tangent vector to $M$, then

$$
<\operatorname{grad}(h \circ \pi), v>=v(h \circ \pi)=d \pi(v) h=0, \quad \text { since } \quad d \pi(v)=0 .
$$

Thus $\operatorname{grad}(h \circ \pi)$ is horizonal. If $x$ is horizonal,

$$
\begin{gathered}
<d \pi(\operatorname{grad}(h \circ \pi)), d \pi(x)>=<\operatorname{grad}(h \circ \pi), x>=x(h \circ \pi) \\
=d \pi(x) h<\operatorname{grad}(h), d \pi(x)>.
\end{gathered}
$$

Hence at each point, $d \pi(\operatorname{grad}(h \circ \pi))=\operatorname{grad}(h)$.
In view of Lemma 2.14, we simplify the notations by writing $h$ for ( $h \circ \pi$ ) and $\operatorname{grad}(h)$ for $\operatorname{grad}(h \circ \pi)$. For a covariant tensor $A$ on $B$, its lift $\bar{A}$ to $M$ is just its pullback $\pi^{*}(A)$ under the projection $\pi: M \rightarrow B$ That is, if $A$ is a ( $1, \mathrm{~s}$ )-tensor, and if $v_{1}, v_{2}, \ldots, v_{s} \in T_{(p, q)} M$, then $\bar{A}\left(v_{1}, \ldots, v_{s}\right)=A\left(d \pi\left(v_{1}\right), \ldots, d \pi\left(v_{s}\right)\right) \in T_{p}(B)$. Hence if $v_{k}$ is vertical, then $\bar{A}=0$ on $B$. For example, if $f$ is a smooth function
on $B$, the lift to $M$ of the Hessian of $f$ is also denoted by $H^{f}$. This agrees with the Hessian of the lift $(f \circ \pi)$ generally only on horizontal vector. For detailed computations, see Lemma 5.1 in [B.E.P.].

Now we recall the formula for the Ricci curvature tensor Ric on the warped product maniford $M=B \times{ }_{f} F$. We write Ric $^{B}$ for the pullback by $\pi$ of the Ricci curvature of $B$ and similarly for $R i c^{F}$.

Lemma 2.8. On a warped product maniford $M=B \times{ }_{f} F$ with $n=\operatorname{dim} F>1$ let $X, Y$ be horizontal and $V, W$ vertical.

Then
(1) $\operatorname{Ric}(X, Y)=\operatorname{Ric}^{B}(X, Y)-\frac{n}{f} H^{f}(X, Y)$,
(2) $\operatorname{Ric}(X, Y)=0$,
(3) $\operatorname{Ric}(V, W)=\operatorname{Ric}^{F}(V, W)-<V, W>f^{\sharp}$,

Where $f^{\sharp}=\frac{\Delta f}{f}+(n-1) \frac{\langle\operatorname{grad}(f), \operatorname{grad}(f)>}{f^{2}}$ and $\Delta f=\operatorname{trace}\left(H^{f}\right)$ is the Laplacian on $B$.

Proof. See Corollary 7.43 in (cf. [O.].)

On the given warped product manifold $M=B \times_{f} F$, we also write $S^{B}$ For the pullback by $\pi$ of the scalar curvature $S_{B}$ of $B$ and similarly for $S^{F}$ From now on, we denote $\operatorname{grad}(f)$ by $\Delta f$.

Lemma 2.9. If $S$ is the scalar curvature of $M=B \times_{f} F$ with $n=\operatorname{dim} F>1$, then

$$
\begin{equation*}
S=S^{B}+\frac{S^{F}}{f^{2}}-2 n \frac{\Delta f}{f}-n(n-1) \frac{<\nabla f, \nabla f>}{f^{2}} \tag{2.1}
\end{equation*}
$$

where $\Delta$ is the Laplacian on $B$.

Proof. For each $(p, q) \in M=B \times_{f} F$, let $\left\{e_{i}\right\}$ be an orthonormal basis for $T_{p} B$. Then by the natural isomorphism $\left\{\overline{e_{i}}=\left(e_{i}, 0\right)\right\}$ is an orthonormal set in $T_{(p, q)} M$. We can choose $\left\{d_{j}\right\}$ on $T_{q} F$ such that $\left\{\overline{e_{i}}, \overline{d_{j}}\right\}$ forms an orthonormal basis for $T_{(p, q)} M$. Then

$$
1=<\overline{d_{j}}, \overline{d_{j}}>=f(p)^{2}\left(d_{j}, d_{j}\right)=\left(f(p) d_{j}, f(p) d_{j}\right)
$$

which implies that $\left\{f(p) d_{j}\right\}$ forms an orthonormal basis for $T_{q} F$.
By Lemma 2.8 (1) and (3), for each $i$ and $j$

$$
\operatorname{Ric}\left(\overline{e_{i}}, \overline{e_{i}}\right)=\operatorname{Ric}^{B}\left(\overline{e_{i}}, \overline{e_{i}}\right)-\sum_{i} \frac{n}{f} H^{f}\left(\overline{e_{i}}, \overline{e_{i}}\right)
$$

and

$$
\operatorname{Ric}\left(\overline{d_{j}}, \overline{d_{j}}\right)=\operatorname{Ric}^{F}\left(\overline{d_{j}}, \overline{d_{j}}\right)-f^{2}(p) g_{F}\left(d_{j}, d_{j}\right)\left(\frac{\Delta f}{f}+n(n-1) \frac{<\nabla f, \nabla f>}{f^{2}}\right)
$$

Hence, for $\epsilon_{i}=g\left(\overline{e_{i}}, \overline{e_{i}}\right)$ and $\epsilon_{j}=\left(\overline{d_{j}}, \overline{d_{j}}\right)$

$$
\begin{aligned}
S(p, q) & =\sum_{\alpha} \epsilon_{\alpha} R_{\alpha \alpha} \\
& =\sum_{i} \epsilon_{i} \operatorname{Ric}\left(\overline{e_{i}}, \overline{e_{i}}\right)+\sum_{j} \epsilon_{j} \operatorname{Ric}\left(\overline{d_{j}}, \overline{d_{j}}\right) \\
& =S^{B}(p, q)+\frac{S^{F}(p, q)}{f^{2}}-2 n \frac{\Delta f}{f}-n(n-1) \frac{<\nabla f, \nabla f>}{f^{2}},
\end{aligned}
$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$.

## III. MAIN RESULTS

Let $(N, g)$ be a Riemannian manifold of dimension $n$ and let $f:[a, \infty) \rightarrow R^{+}$ be a smooth function, where $a$ is a positive number. A Riemannian warped product of $N$ and $[a, \infty)$ with warping function $f$ is defined to be the product manifold $\left([a, \infty) \times_{f} N, g^{\prime}\right)$ with

$$
\begin{equation*}
g^{\prime}=d t^{2}+f^{2}(t) g \tag{3.1}
\end{equation*}
$$

Let $R(g)$ be the scalar curvature of $(N, g)$. Then equation (2.1) implies that the scalar curvature $R(t, x)$ of $g^{\prime}$ is given by the equation

$$
\begin{equation*}
R(t, x)=\frac{1}{f^{2}(t)}\left\{R(g)(x)-2 n f(t) f^{\prime \prime}(t)-n(n-1)\left|f^{\prime}(t)\right|^{2}\right\} \tag{3.2}
\end{equation*}
$$

for $t \in[a, \infty)$ and $x \in N$. (For details, cf. [D.D.] or [G.L.]).

Problem : Given a fiber $N$ with constant scalar curvature $R(g)$, can we find a warping function $f>0$ on $B=[a, \infty)$ such that for any smooth function $R(t, x)$, the warped metric $g^{\prime}$ admits $R(t, x)$ as the scalar curvature
on $M=[a, \infty) \times{ }_{f} N$ ?

If we denote

$$
u(t)=f^{\frac{n+1}{2}}(t), \quad t>a,
$$

then equation (3.2) can be changed into

$$
\begin{equation*}
\frac{4 n}{n+1} u^{\prime \prime}(t)+R(t, x) u(t)-R(g)(x) u(t)^{1-\frac{4}{n+1}}=0 . \tag{3.3}
\end{equation*}
$$

In this paper, we assume that the fiber manifold $N$ is nonempty, connected and a compact Riemannian $n$-manifold without boundary.

If $N$ is in class (B), then we assume that $N$ admits a Riemannian metric of zero scalar curvature. In this case, equation (3.3) is changed into

$$
\begin{equation*}
\frac{4 n}{n+1} u^{\prime \prime}(t)+R(t, x) u(t)=0 \tag{3.4}
\end{equation*}
$$

If $N$ admits a Riemannian metric of zero scalar curvature, then we let $u(t)=$ $t^{\alpha}$ in equation(3.4), where $\alpha>1$ is a constant, and we have

$$
R(t, x)=\frac{4 n}{n+1} \alpha(1-\alpha) \frac{1}{t^{2}}<0, \quad t>a
$$

Thus we have the following theorem.

Theorem 3.1. For $n \geq 3$, let $M=[a, \infty) \times{ }_{f} N$ be the Riemannian warped product $(n+1)$-manifold with $N$ compact $n$-manifold. Suppose that $N$ is in class (B), then on $M$ there is a Riemannian metric of negative scalar curvature outside a compact set.

Theorem 3.2. Suppose that $R(g)=0$ and $R(t, x)=R(t) \in C^{\infty}([a, \infty))$. Assume that for $t>t_{0}$, there exist an upper solution $u_{+}(t)$ and a lower solution $u_{-}(t)$ such that $0<u_{-}(t) \leq u_{+}(t)$. Then there exists a solution $u(t)$ of equation (3.4) such that for $t>t_{0}, \quad 0<u_{-}(t) \leq u(t) \leq u_{+}(t)$.

Proof. See Theorem 3.2 in [J.L.K.L.].

Lemma 3.3. On $[a, \infty)$, there does not exist a positive solution $u(t)$ such that

$$
t^{2} u^{\prime \prime}(t)+\frac{c}{4} u(t) \leq 0 \quad \text { for } \quad t \geq t_{0}
$$

where $c>1$ and $t_{0}>a$ are constants.

Proof. See Lemma 3.2 in [C.Y.L.].

Theorem 3.4. If $R(g)=0$, then there is no positive solution to equation (3.4) with

$$
R(t) \geq \frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}} \quad \text { for } \quad t \geq t_{0}
$$

where $c>1$ and $t_{0}>a$ are constants.

Proof. Assume that

$$
R(t) \geq \frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}} \quad \text { for } \quad t \geq t_{0}
$$

with $c>1$. Equation (3.4) gives

$$
t^{2} u^{\prime \prime}(t)+\frac{c}{4} u(t) \leq 0
$$

By Lemma 3.3, we complete the proof.

In particular, if $R(g)=0$, then using Riemannian warped product it is impossible to obtain a Riemannian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t)=t^{\frac{1}{2}}$, or $f(t)=t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

Theorem 3.5. Suppose that $R(g)=0$. Assume that $R(t, x)=R(t) \in$ $C^{\infty}([a, \infty))$ is a function such that

$$
\frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}}>R(t) \geq-\frac{4 n}{n+1} e^{\alpha t} \quad \text { for } \quad \mathrm{t}>\mathrm{t}_{0}
$$

where $t_{0}>a, \alpha>0$ and $0<c<1$ are constants. Then equation (3.4) has a positive solution on $[a, \infty)$.

Proof. Since $R(g)=0$, put $u_{+}(t)=t^{\frac{1}{2}}$. Then $u_{+}^{\prime \prime}(t)=\frac{-1}{4} t^{\frac{1}{2}-2}$. Hence

$$
\begin{aligned}
& \frac{4 n}{n+1} u_{+}^{\prime \prime}(t)+R(t) u_{+}(t) \\
\leq & \frac{4 n}{n+1} u_{+}^{\prime \prime}(t)+\frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}} u_{+}(t) \\
= & \frac{4 n}{n+1} \frac{-1}{4} t^{\frac{1}{2}-2}+\frac{4 n}{n+1} \frac{c}{4} \frac{1}{t^{2}} t^{\frac{1}{2}} \\
\leq & \frac{4 n}{n+1} \frac{1}{4} t^{\frac{1}{2}-2}[-1+c] \\
< & 0 .
\end{aligned}
$$

Therefore $u_{+}(t)$ is our (weak) upper solution.
And put $u_{-}(t)=e^{-e^{\beta t}}$, where $\beta$ is a positive large constant. Then $u_{-}^{\prime \prime}(t)=$ $-\beta^{2} e^{\beta t} e^{-e^{\beta t}}+\beta^{2} e^{2 \beta t} e^{-e^{\beta t}}$. Hence

$$
\begin{aligned}
& \frac{4 n}{n+1} u_{-}^{\prime \prime}(t)+R(t) u_{-}(t) \\
\geq & \frac{4 n}{n+1} u_{-}^{\prime \prime}(t)-\frac{4 n}{n+1} e^{\alpha t} u_{-}(t) \\
= & \frac{4 n}{n+1} e^{-e^{\beta t}}\left[-\beta^{2} e^{\beta t}+\beta^{2} e^{2 \beta t}-e^{\alpha t}\right] \\
= & \frac{4 n}{n+1} e^{-e^{\beta t}}\left[\beta^{2} e^{\beta t}\left(-1+e^{\beta t}\right)-e^{\alpha t}\right] \\
> & 0
\end{aligned}
$$

for large $\beta$. Thus, for large $\beta, u_{-}(t)$ is a (weak) lower solution and $0<u_{-}(t)<$ $u_{+}(t)$. So, by Theorem 3.2, equation (3.4) has a (weak) positive solution $u(t)$ such that $0<u_{-}(t) \leq u(t) \leq u_{+}(t)$ for large $t$.

Remark 3.6. In case that $R(g)=0$, the results in Theorem 3.4 and Theorem 3.5 are almost sharp because if $u(t)=t^{\frac{1}{2}}$, then $R(t)=\frac{4 n}{n+1} \frac{1}{4} \frac{1}{t^{2}}$.

Example 3.7. If $R(g)=0$ and $R(t)=-\frac{4 n}{n+1} \frac{2}{t^{2}}$, then there is a positive solution to equation (3.4). In fact, we have only to solve the following equation.

$$
\begin{equation*}
t^{2} u^{\prime \prime}(t)-2 u(t)=0 \tag{3.5}
\end{equation*}
$$

Applying the method for the Euler-Cauchy equation to (3.5), we put $u(t)=t^{m}$. Then

$$
m(m-1) t^{m-2} t^{2}-2 t^{m}=0
$$

and

$$
\left(m^{2}-m-2\right) t^{m}=0,
$$

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so $m=2,-1$. Thus $u(t)=c_{1} t^{2}+c_{2} t^{-1}$ is solution of equation (3.5), where $c_{1}$ and $c_{2}$ are constants.

Therefore $u(t)=c_{2} t^{-1}$ is our (weak) solution in the sense of Theorem 3.5 such that $0<u_{-}(t) \leq u(t) \leq u_{+}(t)$.

## REFERENCES

[A.] T. Aubin, "Nonlinear analysis on manifolds", Monge-Ampere equations, Springer-verlag New York Heidelberg Berlin, 1982.
[A.M.] P. Aviles and R. McOwen, Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds, Diff. Geom. 27(1998), 225-239.
[B.E.] J.K. Beem and P.E.Ehrlich, "Global Lorentzian Geometry", Pure and Applied Mathematics, Vol 67, Dekker, New York, 1981.
[B.E.E.] J.K. Beem, P.E.Ehrlich and K.L.Easley, "Global Lorentzian Geometry", Pure and Applied Mathematics, Vol. 202, Dekker, New York, 1996.
[B.E.P.] J.K. Beem, P.E.Ehrlich and Th.G.Powell, Warped product manifolds in relativity, Selected Studies (Th.M.Rassias, eds.), North-holland, 1982, 41-56.
[B.K.] J. Bland and M. Kalka, Negative scalar curvature metrics on noncompact manifolds, Trans. Amer. Math. Soc. 316(1989), 433-446.
[B.O.] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145(1969), 1-49.
[C.Y.L.] E-H Choi, Y-H Yang and S-Y Lee, The nonexistence of warping functuins on Riemannian warped product manifolds, J. Chungcheong Math. Soc. 24(2011), no 2, 171-185.
[D.D.] F. Dobarro and E. Lami Dozo, Positive scalar curvature and the Dirac operater on complete Riemannian manifolds, Publ. Math.I.H.E.S. 58(1983), 295-408.
[G.L.] M. Gromov and H.B. Lawson, positive scalar curvature and the Dirac operater on complete Riemannian manifolds, Math. I.H.E.S. 58(1983), 295-408.
[J.L.K.L.] Y-T Jung, G-Y Lee, A-R Kim and S-Y Lee, The existence of warping functions on Riemannian warped product manifolds with fiber manifold of class (B), Honam Math. J. 36(2014), no 3, 597-603.
[K.K.P.] H. Kitabara, H. Kawakami and J.S. Pak, On a construction of completely simply connected Riemmannian manifolds with negative curvature, Nagoya Math. J.113(1980), 7-13.
[K.W.1] J.L. Kazdan and F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Diff. Geo. 10(1975), 113-134.
[K.W.2] J.L. Kazdan and F.W. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, Ann. of Math. 101(1975), 317-331.
[K.W.3] J.L. Kazdan and F.W. Warner, Curvature functions for compact 2manifolds, Ann. of Math. 99(1974), 14-74.
[L.1] M.C. Leung, Conformal scalar curvature equations on complete manifolds, Commum. Partial Diff. Equation 20(1995), 367-417.
[L.2] M.C. Leung, Conformal deformation of warped products and scalar curvature functions on open manifolds, Bulletin des Science Mathematiques. 122(1998), 369-398.
[L.M.] H.B. Lawson and M. Michelsohn, "Spin geometry", Princeton University Press, Princeton, 1989.
[M.M.] X. Ma and R.C. Mcown, The Laplacian on complete manifolds with warped cylindrical ends, Commum. Partial Diff. Equation 16(1991),1583-1614. [O.] B. O'Neill. "Semi-Riemannian geometry with applications to relativity", Academic Press, New York, 1983.
[Z.] S. Zucker, $L_{2}$ cohomology of warped products and arithmetric groups, Invent. Math. 70(1982), 169-218.

