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A criterion for multi-polytopes via Duistermaat-Heckman functions

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조 미 주
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하토리와 마수다에 의해 발견된 다중 팬은 토러스 다양체와 기하학적으로 깊은 관련을 가지고 있다. 다중 팬은 보통의 팬과 여러 가지 다른 성질을 가지고 있는 반면, 또한 비슷한 중요한 성질도 함께 공유하고 있다. 본 논문에서는 하토리와 마수다의 결과를 확장하여, 다중 팬과 다중 폴리토프에 대응하는 듀이스터매트-핵크만 함수를 정의하고 다중 폴리토프가 보통 폴리토프가 될 필요충분조건을 듀이스터매트-핵크만 함수의 값을 이용하여 찾았다. 좀 더 구체적으로, N를 n 차원 격자라 하고 V를 $N \otimes R$이라 할 때, V의 쌍대공간 V^*에 있는 아핀 초평면 F_i의 합집합 $\bigcup_{i=1}^{d} F_i$의 여집합 $V^* - \bigcup_{i=1}^{d} F_i$를 정의역으로 갖는 듀이스터매트-핵크만 함수 DH_P는

$$DH_P = \sum_{I \in \Sigma^{(n)}} (-1)^I \omega(I) \phi_I$$

으로 정의된다. 여기서, $I \in \Sigma^{(n)}$는 다중 팬 Σ의 n차원 골격을 나타내고 $\omega(I)$와 ϕ_I는 각각 I에 대응하는 가중치와 특성함수를 나타낸다. 본 논문에서 다중 폴리토프 P가 보통의 폴리토프이면 듀이스터매트-핵크만 함수 DH_P는 다중 폴리토프의 내부에서 1의 값을 갖고 외부에서는 0의 값을 가지며 그 역도 성립함을 증명하였다.

A criterion for multi-polytopes via Duistermaat-Heckman functions

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Chapter 1

Introduction

In the paper [8], Masuda first introduced the notion of a unitary toric manifold which properly contained a compact non-singular toric variety, and associated with it a combinatorial object, called a multi-fan. It turns out that a multi-fan is a much more general notion than a complete non-singular fan. Shortly after that, a multi-fan as a purely combinatorial object which generalizes an ordinary fan in algebraic geometry has been greatly developed by Hattori and Masuda in [6]. One typical geometric realization of a multi-fan is a torus manifold, while an ordinary fan is associated with a toric variety. Here a *toric variety* means a normal complex algebraic variety of dimension n with a $(\mathbb{C}^*)^n$-action having one unique dense orbit and other orbits of smaller dimensions. It is well known that there is a one-to-one correspondence between toric varieties and fans (see [2], [10], and [1] for more details). Roughly speaking, the fan associated with a toric variety is a collection of cones in $\mathbb{R}^n$ with apex at the origin, and to each orbit of a $(\mathbb{C}^*)^n$-action on a toric variety there corresponds a cone of dimension equal to the codimension of the orbit.

This new notion of a multi-fan shares many important properties with the ordinary fan. On the other hand, there is one important peculiar feature, compared to the ordinary fan, that in case of a multi-fan the union of cones in a multi-fan may overlap several times. Moreover, it is an open and intriguing question whether or not there is a one-to-one correspondence between relevant toric varieties and multi-fans. At the moment, we just know that two different torus manifolds may correspond to the same multi-fan. Nonetheless, many important topological properties of a torus manifold can be detected by its associated multi-fan. Indeed, in [6] Hattori and Masuda provide several combinatorial invariants of a multi-fan which correspond to the ordinary topological invariants of the associated torus manifold.

Associated to an ordinary fan, there is a notion of a convex polytope. Analogously, there is a notion of a multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ associated to a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ (refer to Chapter 2 for more precise definitions and notations). Indeed, let N be a lattice of rank n which is isomorphic to $\mathbb{Z}^n$, and let M be the dual lattice $\mathrm{Hom}(N, \mathbb{Z})$. Let $V := N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. From now on, we also assume that a multi-polytope is simple, which means that the multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is complete and simplicial.

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ then defines an arrangement of affine hyperplanes F_i $(1 \leq i \leq d)$ in V^*, and one can associate with $\mathcal{P}$ a function, called a *Duistermaat-Heckman function*, on V^* minus the affine hyperplanes when $\mathcal{P}$ is simple. It can be shown that the Duistermaat-Heckman function is locally constant, and Guillemin-Lerman-Sternberg formula ([4], [5]) tells us that it agrees with the density function of a Duistermaat-Heckman measure, when $\mathcal{P}$ arises from a moment map. More precisely, the Duistermaat-Heckman function that we are mostly concerned with in this thesis is defined as follow.

Definition 1.1. Let $\Sigma^{(n)}$ denote the n-skeleton of Σ, and let ϕ_I denote the characteristic function defined over a suitably defined convex cone associated with $I \in \Sigma^{(n)}$. We then define a function $\mathrm{DH}_{\mathcal{P}}$ on $V^* \setminus \bigcup_{i=1}^{d} F_i$ by

$$\mathrm{DH}_{\mathcal{P}} := \sum_{I \in \Sigma^{(n)}} (-1)^I \omega(I) \phi_I,$$

and call it the *Duistermaat-Heckman function associated with* $\mathcal{P}$. We refer the reader to Chapter 3 for more details.

The primary aim of this paper is to provide a criterion for a multi-polytope to be an ordinary polytope in terms of the values of the Duistermaat-Heckman function associated with a multi-polytope. To be more precise, our main result is

Theorem 1.2. *A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ is an ordinary polytope if and only if the Duistermaat-Heckman function* $\mathrm{DH}_{\mathcal{P}}$ *defined on $V^* \setminus \bigcup_{i=1}^{d} F_i$ satisfies the following identity:*

$$\mathrm{DH}_{\mathcal{P}}(u) = \begin{cases} 1, & \text{if } u \text{ lies in the interior } \mathcal{P}^\circ \text{ of } \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

There is another locally constant function defined on the complement of the hyperplanes $\{F_i\}$ associated to a multi-polytope $\mathcal{P}$, called the *winding number*. It turns out that the values of Duistermaat-Heckman function is exactly same as those of the winding number (see [6]). Moreover, the winding number also satisfies a wall crossing formula entirely similar to the Duistermaat-Heckman function. Hence Theorem 1.2 can be stated in terms of the winding numbers instead of the Duistermaat-Heckman functions. In the forthcoming thesis [9], Moon will give a direct proof of this fact without using the equivalence of the values of Duistermaat-Heckman functions and winding numbers.

Now we briefly explain the contents of each chapter, as follows. In Chapter 2, we give definitions of a multi-fan and a multi-polytope, and then introduce certain related notions. The completeness of a multi-fan is one of the most important points in this chapter. The definition of the Duistermaat-Heckman function is given in Chapter 3. As briefly mentioned above, a multi-polytope is a pair $\mathcal{P} = (\Delta, \mathcal{F})$ of an n-dimensional complete multi-fan Δ and a arrangement of hyperplanes $\mathcal{F} = \{F_i\}$ in

$H^2(BT;\mathbb{R})$ with the same index set as the set of 1-dimensional cones in Δ. Recall that a multi-polytope is called simple if its associated multi-fan Δ is simplicial. The Duistermaat-Heckmann function $\mathrm{DH}_{\mathcal{P}}$ associated with a simple multi-polytope $\mathcal{P}$ is a locally constant integer-valued function with bounded support defined on the complement of the hyperplanes $\{F_i\}$. The wall crossing formula which describes the difference of the values of the function on adjacent components plays an important role in the proof of our main Theorem 1.2. In addition, several interesting examples will be provided in Chapter 4. Finally, Chapter 5 is devoted to proving our main Theorem 1.2.

Chapter 2

Multi-fans and Multi-polytopes

The aim of this chapter is to set up some basic notations and terminology necessary for the proofs of our main results in Chapter 5. The material of this chapter is largely taken from the excellent paper [6] of Hattori and Masuda (see also [7]).

To do so, let N be a lattice of rank n, which is isomorphic to $\mathbb{Z}^n$. We denote the real vector space $N \otimes \mathbb{R}$ by $N_\mathbb{R}$. A subset σ of $N_\mathbb{R}$ is called a strongly convex rational polyhedral cone (with apex at the origin) if there exits a finite number of vectors $v_1, \ldots, v_m$ in N such that

$$\sigma = \{r_1 v_1 + \cdots + r_m v_m \,|\, r_i \in \mathbb{R} \text{ and } r_i \geq 0 \text{ for all } i\},$$

and $\sigma \cap (-\sigma) = \{0\}$. Here "rational" means that it is generated by vectors in the lattice N, and "strong" convexity means that it contains no line through the origin. We often call a strongly convex rational polyhedral cone in $N_\mathbb{R}$ simply a cone in N. The dimension $\dim \sigma$ of a cone σ is the dimension of the linear space spanned by vectors in σ. A subset τ of σ is called a *face* of σ if there is a linear function

$$l : N_{\mathbb{R}} \longrightarrow \mathbb{R}$$

such that l takes nonnegative values on σ and such that $\tau = l^{-1}(0) \cap \sigma$. A cone shall be regarded as a face of itself, while others are called *proper faces*.

Definition 2.1. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

(1) each face of a cone in Δ is also a cone in Δ, and

(2) the intersection of two cones in Δ is a face of each.

We also need a series of the following definitions.

Definition 2.2. A fan Δ is said to be *complete* if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$.

A cone is called *simplicial* if it is generated by linearly independent vectors. If the generating vector can be taken as a part of a basis of N, then the cone is called *nonsingular*.

Definition 2.3. A fan Δ is said to be *simplicial* (resp. *non-singular*) if every cone in Δ is simplicial (resp. *non-singular*).

Denote by $\mathrm{Cone}\,(N)$ the set of all cones in N. An ordinary fan is a subset of $\mathrm{Cone}\,(N)$. The set $\mathrm{Cone}\,(N)$ has a partial ordering $\prec$ defined by : $\tau \prec \nu$ if and only if τ is a proper face of ν. The cone $\{0\}$ consisting of the origin is the unique minimum element of Cone(N).

Next, let Σ be a partial ordering finite set with a unique minimum element. We denote the strict partial ordering by $<$ and the minimum element by $*$. A typical example of Σ thesis is an abstract simplicial set with an empty set added as a member, which we call an *augmented simplicial set*. In this case, the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1)-simplex. Suppose that there is a map

$$C : \Sigma \to \mathrm{Cone}\,(N)$$

such that

(1) $C(*) = \{0\}$;

(2) If $I < J$ for $I, J \in \Sigma$, then $C(I) < C(J)$;

(3) For any $J \in \Sigma$ the map C restricted on $\{I \in \Sigma \mid I \leq J\}$ is an isomorphism of ordered sets onto $\{K \in \mathrm{Cone}(N) \mid K \leq C(J)\}$.

For an integer m such that $0 \leq m \leq n$, we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \dim C(I) = m\}.$$

One can easily check that $\Sigma^{(m)}$ does not depend on the choice of C satisfying the above three conditions. When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m, that is, I is an $(m-1)$-simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notation $|I|$ for m when $I \in \Sigma^{(m)}$.

The image $C(\Sigma)$ is a finite set of cones in N. We may think of a pair (Σ, C) as a set of cones in N labeled by the ordered set Σ. Cones in an ordinary fan intersect only at their faces, but cones in $C(\Sigma)$ may overlap. Furthermore, it can happen that the same cone may appear repeatedly with different labels. The pair (Σ, C) is almost what we can possibly call a multi-fan, but we need to further incorporate a pair of weight functions on cones in $C(\Sigma)$ of the highest dimension $n = \operatorname{rank} N$. More precisely, we consider two functions

$$\omega^{\pm} : \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}$$

such that $\omega^{+}(I) > 0$ or $\omega^{-}(I) > 0$ for every $I \in \Sigma^{(n)}$.

These two functions $\omega^{\pm}$ actually have its origin from toric geometry. In fact, if M is a torus manifold of dimension $2n$ and if $M_{i_1}, \cdots, M_{i_n}$ are characteristic submanifolds such that their intersection contains at least one T-fixed point, then the intersection $M_1 = \cap_\nu M_{i_\nu}$ consists of a finite number of T-fixed points. At each fixed point $p \in M_I$ the tangent space τ_p has two orientations; one is endowed by the orientation of M and the other comes from the intersection of the oriented submanifolds M_{i_ν}. Denoting the ratio of the above two orientations by ε_p we define the number $\omega^+(I)$ (resp. $\omega^-(I)$) to be the number of points $p \in M_I$ with $\varepsilon_p = +1$ (resp. $\varepsilon_p = -1$).

Now we are ready to give a precise definition of a multi-fan, as follows.

Definition 2.4. We call a triple $\Delta := (\Sigma, C, \omega^\pm)$ a *multi-fan* in N. We define the dimension of Δ to be the rank of N (or the dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of Cone (N), one can view it as a multi-fan by taking $\Sigma = \Delta$, $C =$ the inclusion map, $\omega^+ = 1$, and $\omega^- = 0$. This is the way how to obtain an ordinary fan from a multi-fan. Note that a convex polytope gives rise to a complete fan.

As in the case of ordinary fans, we shall say that a multi-fan $\Delta = (\Sigma, C, \omega^\pm)$ is *simplicial* (resp. *non-singular*) if every cone in $C(\Sigma)$ is *simplicial* (resp. *non-singular*).

Lemma 2.5. *A multi-fan $\Delta = (\Sigma, C, \omega^\pm)$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

The definition of completeness of a multi-fan Δ is rather complicated. A naive definition of the completeness would be that the union of cones in $C(\Sigma)$ covers the entire space $N_{\mathbb{R}}$. Although the two weighted functions $\omega^{\pm}$ are incorporated in the definition of a multi-fan, only the difference

$$\omega := \omega^{+} - \omega^{-}$$

will be important in this thesis.

In order to give a precise definition of completeness of a multi-fan, we first need to introduce the following intermediate notion of pre-completeness. To do so, recall first what a generic vector means: a vector $v \in N_{\mathbb{R}}$ will be called *generic* if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n. For a generic vector v we set

$$d_v = \sum_{\substack{v \in C(I), \\ I \in \Sigma^{(n)}}} \omega(I),$$

where the sum is understood to be zero if there is no such I.

Definition 2.6. We call a multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ of dimension n is *pre-complete* if $\Sigma^{(m)} \neq 0$ and the integer d_v is independent of the choice of generic vectors v. We call this integer the *degree* of Δ and denote it by $\deg(\Delta)$.

We remark that for an ordinary fan, pre-completeness is the same as completeness. See Figure 2.1 for typical examples of an ordinary fan and a multi-fan.

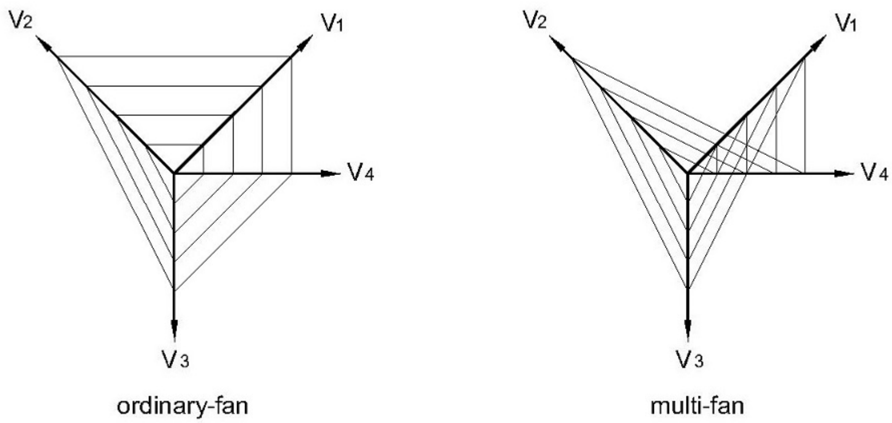

Figure 2.1: Typical examples of an ordinary fan and a multi-fan

Recall that $V = N_{\mathbb{R}}$. A convex polytope P in $V^* = \mathrm{Hom}\,(V, \mathbb{R})$ is the convex hull of a finite set of points in V^*. It is the intersection of a finite number of half space in V^* separated by affine hyperplanes, so there are a finite number of nonzero vectors $v_1, \cdots, v_d$ in V and real numbers $c_1, \cdots, c_d$ such that

$$P = \{u \in V^* \mid \langle u, v_i \rangle \leq c_i \text{ for all } i\},$$

where $\langle\ ,\rangle$ denotes the natural pairing between V^* and V. Note that a convex polytope gives rise to a complete fan.

Next, we begin explaining how to obtain a multi-polytope from a complete multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$. The procedure of obtaining a multi-polytope from a multi-fan is completely analogous. More precisely, let $\mathrm{HP}(V^*)$ be the set of all affine hyperplanes in V^*.

Definition 2.7. Let $\Delta = (\Sigma, C, \omega^{\pm})$ be a complete multi-fan and let

$$\mathcal{F} : \Sigma^{(1)} \longrightarrow \mathrm{HP}(V^*)$$

be a map such that the affine hyperplane $\mathcal{F}(I)$ is perpendicular to the half line $C(I)$ for each $I \in \Sigma^{(1)}$, i.e., an element in $C(I)$ takes a constant on $\mathcal{F}(I)$. We call a pair $(\Delta, \mathcal{F})$ a *multi-polytope* and denote it by $\mathcal{P}$. The *dimension* of a multi-polytope $\mathcal{P}$ is defined to be the dimension of the multi-fan Δ. We say that a multi-polytope $\mathcal{P}$ is *simple* if Δ is simplicial.

Strictly speaking, the completeness assumption for Δ given in Definition 2.7 is not needed for the definition of multi-polytopes. But we incorporated it there because most of our results in this thesis depend on that assumption. We note that the notion of multi-polytopes is a direct generalization of that of twisted polytopes given in [3].

Finally, in order to help readers to better understand the notion of a multi-polytope we give examples of multi-polytopes obtained from the multi-fan given in Figure 2.1.

Example 2.8. If four points $l_1 \cap l_2, l_2 \cap l_3, l_3 \cap l_4$ and $l_4 \cap l_1$ are presumed to be vertices and the others such as $l_2 \cap l_4$ are not, then we can find the figure P in Figure 2.2. But, if different four points $l_1 \cap l_4, l_4 \cap l_2, l_2 \cap l_3$ and $l_3 \cap l_1$ are presumed to be vertices, then we obtain a figure P' shaded in Figure 2.2.

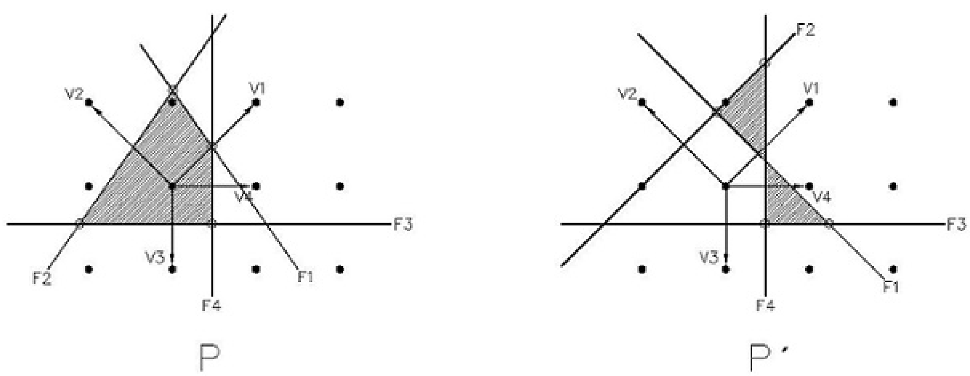

Figure 2.2: Examples of multi-polytopes

Chapter 3

Duistermaat-Heckman Functions

As mentioned in Chapter 2, a multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ defines an arrangement of affine hyperplanes in V^*. We can then associate with $\mathcal{P}$ a function on V^* minus the affine hyperplanes when $\mathcal{P}$ is simple. This function is locally constant and Guillemin-Lerman-Sternberg formula ([4],[5]) tells us that it agrees with the density function of a Duistermaat-Heckman measure when $\mathcal{P}$ arises from a moment map. The aim of this section is to explain how to define the locally constant function from a multi-polytope, relatively in detail.

Hereafter, our multi-polytope $\mathcal{P}$ is assumed to be simple, so that the multi-fan $\Delta = (\Sigma, C, \omega^{\pm})$ is complete and simplicial, unless otherwise stated. As before, we may assume that Σ consists of subsets of $\{1, \dots, d\}$ and $\Sigma^{(1)} = \{\{1\}, \dots, \{d\}\}$, and denote by v_i a nonzero vector in the one-dimensional cone $C(\{i\})$. We denote $\mathcal{F}(\{i\})$ by F_i and set

$$F_I := \bigcap_{i \in I} F_i \quad \text{for} \quad I \in \Sigma.$$

Then F_I is an affine space of dimension $n - |I|$. In particular, if $|I| = n$, i.e., $I \in \Sigma^{(n)}$, then F_I is a point, denoted by u_I.

Suppose that $I \in \Sigma^{(n)}$. Then the set $\{v_i | i \in I\}$ form a basis of V. Denote its dual basis of V^* by $\{u_i^I \,|\, i \in I\}$, i.e., $\langle u_i^I, v_j \rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Take a generic vector $v \in V$. Then we have $\langle u_i^I, v \rangle \neq 0$ for all $I \in \Sigma^{(n)}$ and $i \in I$. Let

$$(-1)^I := (-1)^{\#\{i \in I | \langle u_i^I, v \rangle > 0\}}, \text{ and}$$
$$(u_i^I)^+ := \begin{cases} u_i^I, & \text{if } \langle u_i^I, v \rangle > 0 \\ -u_i^I, & \text{if } \langle u_i^I, v \rangle < 0. \end{cases}$$

We denote by $C^*(I)^+$ the cone in V^* spanned by $(u_i^I)^+$'s $(i \in I)$ with apex at u_I, and by ϕ_I its characteristic function.

Definition 3.1. We define a function $\mathrm{DH}_{\mathcal{P}}$ on $V^* \setminus \bigcup_{i=1}^d F_i$ by

$$\mathrm{DH}_{\mathcal{P}} := \sum_{I \in \Sigma^{(n)}} (-1)^I \omega(I) \phi_I,$$

and call it the *Duistermaat-Heckman function* associated with $\mathcal{P}$.

The following simple one-dimensional example taken from the paper [6] clearly shows how to calculate the Duistermaat-Heckman function for the one-dimensional multi-polytopes.

Example 3.2. Suppose $\dim \mathcal{P} = 1$. We then identify V with $\mathbb{R}$, so that V^* is also identified with $\mathbb{R}$. Let E be the subset of $\{1, \ldots, d\}$ such that $i \in E$ if and only if $C(\{i\})$ is the half line consisting of nonnegative real numbers. Then the completeness of Δ means that

$$\sum_{i \in E} \omega(\{i\}) = \sum_{i \notin E} \omega(\{i\}) = \deg(\Delta). \tag{3.1}$$

Take a nonzero vector v. Since V^* is identified with $\mathbb{R}$, each affine hyperplane F_i is nothing but a real number. Suppose that v is toward the positive direction. Then

$$(-1)^{\{i\}} = \begin{cases} -1, & \text{if } i \in E, \\ 1, & \text{if } i \notin E, \end{cases} \tag{3.2}$$

and the support of the characteristic function $\phi_{\{i\}}$ is the half line given by

$$\{u \in \mathbb{R} \mid F_i \leq u\}.$$

Therefore

$$\mathrm{DH}_{\mathcal{P}}(u) = \sum_{i \in E \;\; s.t. \;\; F_i < u} -w(\{i\}) + \sum_{i \notin E \;\; s.t. \;\; F_i < u} w(\{i\}). \tag{3.3}$$

for $u \in \mathbb{R} \setminus \cup F_i$. If u is sufficiently small, then the sum above is empty, so it is zero. If u is sufficiently large, then the the sum is also zero by (3.1). Hence the support of the function $\mathrm{DH}_{\mathcal{P}}$ is bounded.

Now, suppose that v is toward the negative direction. Then $(-1)^{\{i\}}$ above is multiplied by -1 and the inequality $\leq$ above turns into $\geq$. Therefore

$$\text{(3.4)} \quad \mathrm{DH}_{\mathcal{P}}(u) = \sum_{i \in E \ \ s.t. \ \ u < F_i} w(\{i\}) + \sum_{i \notin E \ \ s.t. \ \ u < F_i} (-w(\{i\})).$$

It follows that

$$\text{R.H.S. of (3.3)} - \text{R.H.S. of (3.4)} = -\sum_{i \in E} w(\{i\}) + \sum_{i \notin E} w(\{i\})$$

which is zero by (3.1). This shows that the function $\mathrm{DH}_{\mathcal{P}}$ is independent of v when $\dim\mathcal{P} = 1$.

Next, assume $n = \dim \Delta > 1$. For each $\{i\} \in \Sigma^{(1)}$, the projected multi-fan $\Delta_{\{i\}} = (\Sigma_{\{i\}}, C_{\{i\}}, \omega^{\pm}_{\{i\}})$, which we abbreviate as $\Delta_i = (\Sigma_i, C_i, \omega^{\pm}_i)$, is defined on the quotient vector space $V \backslash V_i$ of V by the one-dimensional subspace V_i spanned by v_i. Since Δ is complete and simplicial, so is Δ_i. We then identify the dual space $(V/V_i)^*$ with

$$(V^*)_i := \{u \in V^* | \langle u, v_i \rangle = 0\}$$

in a natural way. We choose an element $f_i \in F_i$ arbitrarily and translate F_i onto $(V^*)_i$ by $-f_i$. If $\{i, j\} \in \Sigma^{(2)}$, then F_j intersects F_i and their intersection will be translated into $(V^*)_i$ by $-f_i$. This observation leads us to consider the map

$$F_i : \Sigma_i \to \mathrm{HP}((V^*)_i)$$

sending $\{j\} \in \Sigma_i^{(1)}$ to $F_i \cap F_j$ translated by $-f_i$. The pair $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$ is a multi-polytope in $(V/V_i)^* \cong (V^*)_i$.

Let $I \in \Sigma^{(n)}$ such that $i \in I$. Since $\langle u_i^I, v_i \rangle = \delta_{ij}, u_i^I$ for $j \neq i$ is an element of $(V^*)_i$, which we also regard as an element of $(V/V_i)^*$ through the isomorphism $(V/V_i)^* \cong (V^*)_i$.

We denote the projection image of the generic element $v \in V$ on V/V_i by $\bar{v}$. Then we have $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle$ for $j \neq i$, where u_j^I at the left-hand side is viewed as an element of $(V/V_i)^*$, while the one at the right-hand side is viewed as an element of $(V^*)_i$. Since $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle \neq 0$ for $j \neq i$, we use $\bar{v}$ to define $\mathrm{DH}_{\mathcal{P}_i}$.

Example 3.3. Let $v_1, \ldots, v_5$ be integral vectors, where the dots denote lattice points. The vectors are rotating around the origin twice in counterclockwise. We take

$$\Sigma = \{\phi, \{1\}, \ldots, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\},$$

and define $C : \Sigma \to \mathrm{Cone}(N)$ by

$$C(\{i\}) = \text{the cone spanned by } v_i,$$
$$C(\{i, i+1\}) = \text{the cone spanned by } v_i \text{ and } v_{i+1},$$

where $i = 1, \ldots, 5$ and 6 is understood to be 1, and take $\omega^{\pm}$ such that $\omega = 1$ on every two dimensional cone. Then $\Delta = (\Sigma, C, \omega^{\pm})$ is a complete non-singular two-dimensional multi-fan with $\deg(\Delta) = 2$ by (3.1). Note that the degree $\deg(\Delta)$ can be calculated as follows:

$$\sum_{i \in E} \omega(\{i\}) = 1 + 1 = \deg(\Delta)$$

One obtains the arrangement of lines with a suitable choice of the map $\mathcal{F}$. The pentagon produces the same arrangement of lines and can be viewed as a multi-polytope, but these two multi-polytopes are different because the underlying multi-fan are different. In fact, one is a multi-fan of degree two, while the other is an ordinary fan. Note that we have a star-shaped figure in the former multi-polytope (see Figure 3.1).

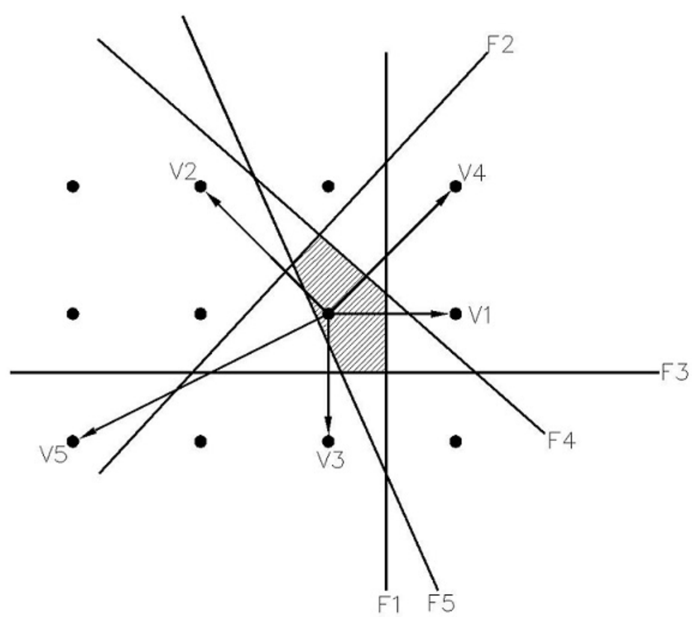

Figure 3.1: Multi-polytope for Example 3.3

Chapter 4

Some Interesting Examples

The aim of this chapter is to give a series of several interesting examples of showing in detail how to calculate the Duistermaat-Heckman functions for various multi-polytopes. They both illustrate and also strongly support our main result (Theorem 1.2).

Example 4.1. We first give an example of a complete non-singular multi-fan of degree two. To do so, let $v_1, \ldots, v_5$ be vectors shown in Figure 4.1, as follows.

$v_1 = (1,0), v_2 = (-1,1), v_3 = (0,-1), v_4 = (1,1), v_5 = (-2,-1)$,

$v = (2,3)$.

Let

$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}$.

It is easy to see that we have

$$\Sigma^{(2)} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\}.$$

Next, we need to consider the following cases:

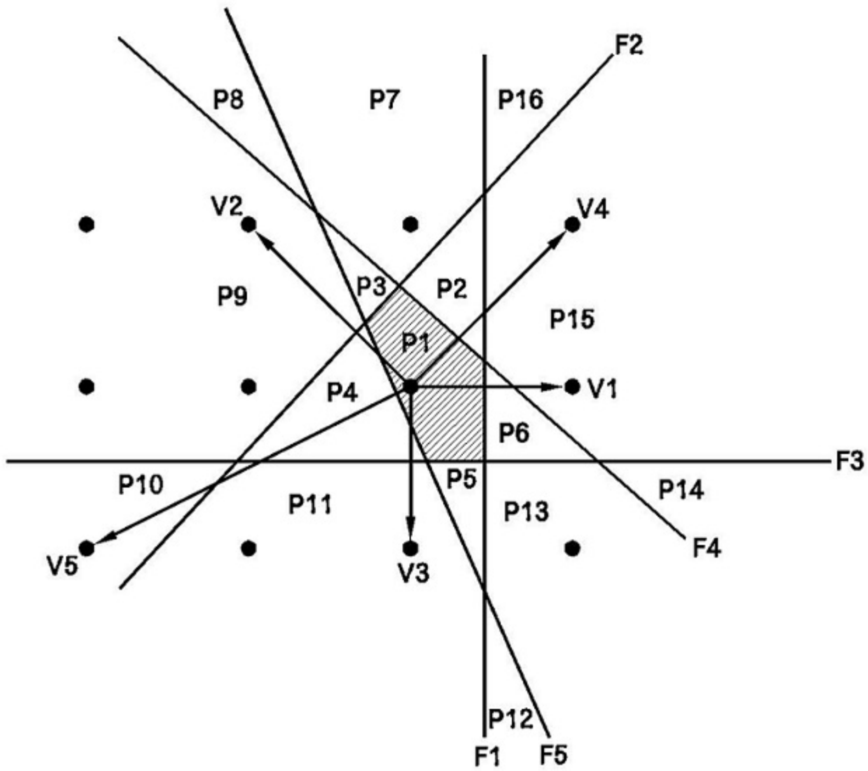

Figure 4.1: A multi-polytope $\mathcal{P}$ of Example 4.1

(1) $I = \{1,2\}$ case ;

$$u_1^{\{1,2\}} = (1,1),\ (u_1^{\{1,2\}})^+ = (1,1),$$
$$u_2^{\{1,2\}} = (0,1),\ (u_2^{\{1,2\}})^+ = (0,1).$$

(2) $I = \{2,3\}$ case;

$$u_1^{\{2,3\}} = (-1,0),\ -(u_1^{\{2,3\}})^+ = (1,0),$$
$$u_2^{\{2,3\}} = (-1,-1),\ -(u_2^{\{2,3\}})^+ = (1,1).$$

(3) $I = \{3,4\}$ case;

$$u_1^{\{3,4\}} = (1,-1),\ -(u_1^{\{3,4\}})^+ = (-1,1),$$
$$u_2^{\{3,4\}} = (1,0),\ (u_2^{\{3,4\}})^+ = (1,0).$$

(4) $I = \{4,5\}$ case ;

$$u_1^{\{4,5\}} = (-1,2),\ (u_1^{\{4,5\}})^+ = (-1,2),$$
$$u_2^{\{4,5\}} = (-1,1),\ (u_2^{\{4,5\}})^+ = (-1,1).$$

(5) $I = \{5,1\}$ case ;

$$u_1^{\{5,1\}} = (0,-1),\ -(u_1^{\{1,2\}})^+ = (0,1),$$
$$u_2^{\{5,1\}} = (1,-2),\ -(u_2^{\{5,1\}})^+ = (-1,2).$$

Therefore, for $u \in \mathcal{P}_1$ we can obtain

$$\begin{aligned}\mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_1}(u) &= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}}(u) + (-1)^{\{2,3\}}\omega(\{2,3\})\phi_{\{2,3\}}(u)\\ &+ (-1)^{\{3,4\}}\omega(\{3,4\})\phi_{\{3,4\}}(u) + (-1)^{\{4,5\}}\omega(\{4,5\})\phi_{\{4,5\}}(u)\\ &+ (-1)^{\{5,1\}}\omega(\{5,1\})\phi_{\{5,1\}}(u)\\ &= (-1)^2 \cdot 1 \cdot 0 + (-1)^0 \cdot 1 \cdot 1 + (-1)^1 \cdot -1 \cdot 0\\ &+ (-1)^2 \cdot -1 \cdot 0 + (-1)^0 \cdot 1 \cdot 1\\ &= 0 + 1 + 0 + 0 + 1 = 2.\end{aligned}$$

$$
\begin{aligned}
\mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_2} &= \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_3} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_4} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_5} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_6} \\
&= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}} + (-1)^{\{2,3\}}\omega(\{2,3\})\phi_{\{2,3\}} \\
&+ (-1)^{\{3,4\}}\omega(\{3,4\})\phi_{\{3,4\}} + (-1)^{\{4,5\}}\omega(\{4,5\})\phi_{\{4,5\}} \\
&+ (-1)^{\{5,1\}}\omega(\{5,1\})\phi_{\{5,1\}} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_7} &= \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_8} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_9} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{10}} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{11}} \\
&= \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{12}} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_7 13} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{14}} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{15}} = \mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}_{16}} \\
&= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}} + (-1)^{\{2,3\}}\omega(\{2,3\})\phi_{\{2,3\}} \\
&+ (-1)^{\{3,4\}}\omega(\{3,4\})\phi_{\{3,4\}} + (-1)^{\{4,5\}}\omega(\{4,5\})\phi_{\{4,5\}} \\
&+ (-1)^{\{5,1\}}\omega(\{5,1\})\phi_{\{5,1\}} \\
&= 0
\end{aligned}
$$

Note that $\Delta = (\Sigma, C, \omega^{\pm})$ is a complete and simplicial multi-fan, but not ordinary-fan.

Example 4.2. Let Σ be an ordinary polytope given by the following simplicial complex.

$$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}.$$

Then define a function $C : \Sigma \longrightarrow \mathrm{Cone(N)}$ by

$$C(\{1\}) = v_1 = (0,1), C(\{2\}) = v_2 = (1,0), C(\{3\}) = v_3 = (-1,-1),$$

and

$$C(\{i, i+1\}) = \text{the cone spanned by } v_i \text{ and } v_{i+1}.$$

Here we assume that $v_4 = v_1$. Let us also take weight functions $\omega^{\pm}$ such that $\omega = 1$ on every two dimensional cone in

$$\Sigma^{(2)} = \{\{1,2\}, \{2,3\}, \{3,1\}\}.$$

Then

$$\Delta = (\Sigma, C, \omega), I \in \Sigma^{(2)}.$$

is a complete non-singular two-dimensional multi-fan with $\deg(\Delta) = 1$. (see Figure 4.2). Next, let us take a generic vector $v = (2,3)$, and we want to calculate the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$, as follows :

(1) $I = \{1,2\}$;

$$u_1^{\{1,2\}} = (0,1), (u_1^{\{1,2\}})^+ = (0,1),$$

$$u_2^{\{1,2\}} = (1,0), (u_2^{\{1,2\}})^+ = (1,0).$$

(2) $I = \{2,3\}$;

$$u_1^{\{2,3\}} = (1,-1), -(u_1^{\{2,3\}})^+ = (-1,1),$$
$$u_2^{\{2,3\}} = (0,-1), -(u_2^{\{2,3\}})^+ = (0,1).$$

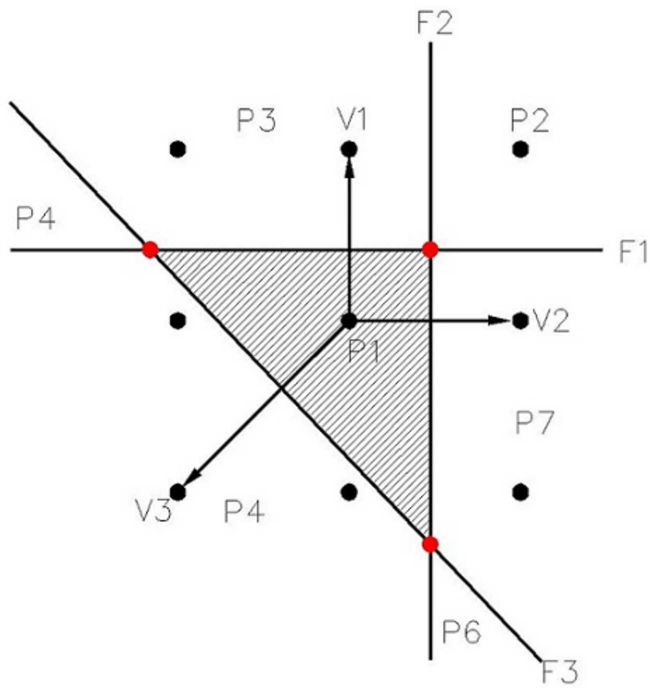

Figure 4.2: An ordinary polytope $\mathcal{P}$ of Example 4.2.

(3) $I = \{3, 1\}$;

$$u_1^{\{3,1\}} = (-1, 0), -(u_1^{\{3,4\}})^+ = (1, 0),$$
$$u_2^{\{3,1\}} = (-1, 1), (u_2^{\{3,4\}})^+ = (-1, 1).$$

Therefore, for u in the bounded region we can obtain

$$\begin{aligned}
\mathrm{DH}_{\mathcal{P}}|_{\mathcal{P}^\circ}(u) &= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}}(u) + (-1)^{\{2,3\}}\omega(\{2,3\})\phi_{\{2,3\}}(u) \\
&+ (-1)^{\{3,1\}}\omega(\{3,1\})\phi_{\{3,1\}}(u) \\
&= (-1)^2 \cdot 1 \cdot 0 + (-1)^0 \cdot 1 \cdot 1 + (-1)^1 \cdot -1 \cdot 0 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathrm{DH}_{\mathcal{P}}|_{otherwisw} &= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}}(u) \\
&+ (-1)^{\{2,3\}}\omega(\{2,3\})\phi_{\{2,3\}}(u) + (-1)^{\{3,1\}}\omega(\{3,1\})\phi_{\{3,1\}}(u) \\
&= 0
\end{aligned}$$

Example 4.3. In this case, four points $l_1 \cap l_4, l_4 \cap l_2, l_2 \cap l_3, l_3 \cap l_1$ are presumed to be vertices. Let $v_1, \ldots, v_4$ be vectors shown in Figure 4.3.

$$v_1 = (1, 1), v_2 = (-1, 1), v_3 = (0, -1), v_4 = (1, 0),$$
$$v = (2, 3),$$
$$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 4\}, \{4, 2\}, \{2, 3\}, \{3, 1\}\}$$
$$\Sigma^{(2)} = \{\{1, 4\}, \{4, 2\}, \{2, 3\}, \{3, 1\}\}.$$

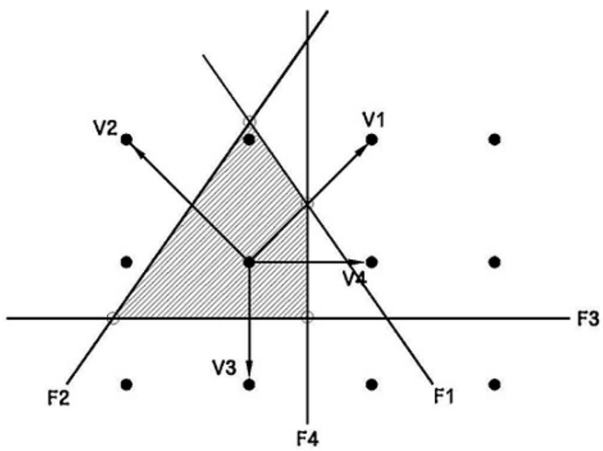

Figure 4.3: A multi-polytope $\mathcal{P}$ of Example 4.3.

(1) $I = \{1, 4\}$;

$$u_1^{\{1,4\}} = (0, 1), (u_1^{\{1,4\}})^+ = (0, 1),$$
$$u_2^{\{1,4\}} = (1, 0), (u_2^{\{1,4\}})^+ = (1, 0).$$

(2) $I = \{4, 2\}$;

$$u_1^{\{4,2\}} = (1, 1), (u_1^{\{4,2\}})^+ = (1, 1),$$
$$u_2^{\{4,2\}} = (0, 1), (u_2^{\{4,2\}})^+ = (0, 1).$$

(3) $I = \{2, 3\}$;

$$u_1^{\{2,3\}} = (-1, 0), -(u_1^{\{2,3\}})^+ = (1, 0),$$
$$u_2^{\{2,3\}} = (-1, -1), -(u_2^{\{2,3\}})^+ = (1, 1).$$

(4) $I = \{3, 1\}$;

$$u_1^{\{3,1\}} = (1, -1), -(u_1^{\{3,1\}})^+ = (-1, 1),$$
$$u_2^{\{3,1\}} = (1, 0), (u_2^{\{3,1\}})^+ = (1, 0).$$

Therefore, for u in the bounded region we can obtain

$$\begin{aligned}
\mathrm{DH}_{\mathcal{P}}(u) &= (-1)^{\{1,4\}}\omega(\{1, 4\})\phi_{\{1,4\}}(u) + (-1)^{\{4,2\}}\omega(\{4, 2\})\phi_{\{4,2\}}(u) \\
&+ (-1)^{\{2,3\}}\omega(\{2, 3\})\phi_{\{2,3\}}(u) + (-1)^{\{3,1\}}\omega(\{3, 1\})\phi_{\{3,1\}}(u) \\
&= (-1)^2 \cdot 1 \cdot 1 + (-1)^2 \cdot -1 \cdot 0 + (-1)^0 \cdot 1 \cdot 1 \\
&+ (-1)^1 \cdot -1 \cdot 0 \\
&= 1 + 0 + 1 + 0 \\
&= 2
\end{aligned}$$

Example 4.4. This time, four points $l_1 \cap l_2, l_3 \cap l_4, l_3 \cap l_1, l_4 \cap l_2$ are presumed to be vertices. Let $v_1, \ldots, v_4$ be vectors shown in Figure 4.4. Notice that the multi-polytope $\mathcal{P}$ consists of two bounded region, even though their intersection is not a vertex of $\mathcal{P}$.

$$v_1 = (1,1), v_2 = (-1,1), v_3 = (0,-1), v_4 = (1,0),$$
$$v = (2,4),$$
$$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{3,4\}, \{3,1\}, \{4,2\}\}$$
$$\Sigma^{(2)} = \{\{1,2\}, \{3,4\}, \{3,1\}, \{4,2\}\},$$

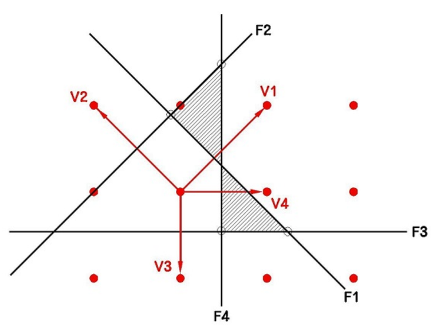

Figure 4.4: A multi-polytope $\mathcal{P}$ of Example 4.4.

(1) $I = \{1,2\}$;

$$u_1^{\{1,2\}} = (\frac{1}{2}, \frac{1}{2}), (u_1^{\{1,2\}})^+ = (\frac{1}{2}, \frac{1}{2}),$$

$$u_2^{\{1,2\}} = (-\frac{1}{2}, \frac{1}{2}), (u_2^{\{1,2\}})^+ = (-\frac{1}{2}, \frac{1}{2}).$$

(2) $I = \{3,4\}$;

$$u_1^{\{3,4\}} = (0,-1), -(u_1^{\{3,4\}})^+ = (0,1),$$

$$u_2^{\{3,4\}} = (1,0), (u_2^{\{3,4\}})^+ = (1,0).$$

(3) $I = \{3,1\}$;

$$u_1^{\{3,1\}} = (1,-1), -(u_1^{\{3,1\}})^+ = (-1,1),$$

$$u_2^{\{3,1\}} = (1,0), (u_2^{\{3,1\}})^+ = (1,0).$$

(4) $I = \{4,2\}$;

$$u_1^{\{4,2\}} = (1,1), (u_1^{\{4,2\}})^+ = (1,1),$$

$$u_2^{\{4,2\}} = (0,1), (u_2^{\{4,2\}})^+ = (0,1).$$

Therefore, for u in one of the bounded region we can obtain

$$\begin{aligned}
\mathrm{DH}_{\mathcal{P}}(u) &= (-1)^{\{1,2\}}\omega(\{1,2\})\phi_{\{1,2\}}(u) + (-1)^{\{3,4\}}\omega(\{3,4\})\phi_{\{3,4\}}(u) \\
&\quad + (-1)^{\{3,1\}}\omega(\{3,1\})\phi_{\{3,1\}}(u) + (-1)^{\{4,2\}}\omega(\{4,2\})\phi_{\{4,2\}}(u) \\
&= (-1)^2 \cdot 1 \cdot 1 + (-1)^1 \cdot 1 \cdot 1 + (-1)^1 \cdot 1 \cdot 0 + (-1)^2 \cdot 1 \cdot 1 \\
&= 1 + (-1) + 0 + 1 \\
&= 1
\end{aligned}$$

Note that for u in other bounded region we have

$$\mathrm{DH}_{\mathcal{P}}(u) = -1.$$

Chapter 5

Main Results: Proof of Theorem 1.2

The aim of this chapter is to give a proof of the following theorem.

Theorem 5.1. *Let $\Delta = (\Sigma, C, \omega^{\pm})$ be a complete and simplicial multi-fan, and let $\mathcal{P}$ be its associated multi-polytope. Then, $\mathcal{P}$ is an ordinary polytope if and only if*

$$\mathrm{DH}_{\mathcal{P}}(u) = \begin{cases} 1, & \textit{if } u \in \mathcal{P}^{\circ}. \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For the proof, we first show that if the multi-polytope is a geometric realization of an ordinary polytope, the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$ defined over $V = M_{\mathbb{R}} \setminus \bigcup_{i=1}^{d} F_i$ satisfies

$$\mathrm{DH}_{\mathcal{P}}(u) = \begin{cases} 1, & \text{if } u \in \mathcal{P} \cap V, \\ 0, & \text{otherwise.} \end{cases}$$

To do so, we want to use the mathematical induction on the dimension $\dim \Delta$ of Δ. So assume that $\dim \Delta$ is equal to one.

In this case, we identify $N_{\mathbb{R}}$ with $\mathbb{R}$, so that $M_{\mathbb{R}}$ is also identified with $\mathbb{R}$. In this case, we have

$$\Sigma = \{\emptyset, \{1\}, \{2\}\}.$$

Let us assume that $C(\{1\})$ is the half line consisting of non-negative real numbers. Then $C(\{2\})$ will be the half line consisting of non-positive real numbers. Note also that the completeness of Δ implies that

$$w(\{1\}) = w(\{2\}) = \deg(\Delta) = 1.$$

Since $M_{\mathbb{R}}$ is identified with $\mathbb{R}$, each affine hyperplane F_i is just a real number. Let us take v to be a generic vector in the positive direction. Then it follows from the definition of the Duistermaat-Heckman function that we have

$$\begin{aligned} \mathrm{DH}_{\mathcal{P}}(u) &= -w(\{1\})\phi_{\{1\}}(u) + w(\{2\})\phi_{\{2\}}(u) \\ &= -\phi_{\{1\}}(u) + \phi_{\{2\}}(u). \end{aligned}$$

It is easy to show that

$$\begin{aligned} C^*(\{1\})^+ &= \{v \in \mathbb{R} \,|\, v \geq F_1\}, \\ C^*(\{2\})^+ &= \{v \in \mathbb{R} \,|\, v \geq F_2\} \end{aligned}$$

(refer to Figure 5.1). Thus, we have

$$\mathrm{DH}_{\mathcal{P}}(u) = \begin{cases} 0, & u < F_2, \\ 1, & F_2 < u < F_1, \\ 0, & u > F_1. \end{cases}$$

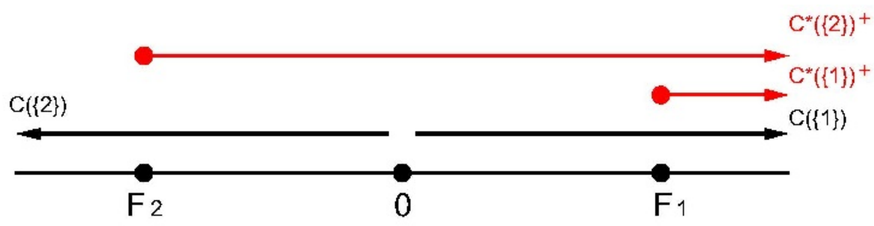

Figure 5.1

This completes the proof for the case of $\dim \Delta = 1$.

Assume now that $\dim \Delta = n > 1$ and that the result holds for any multi-fans with dimension less than n. As above, for each $\{i\} \in \Sigma^{(1)}$, let $\Delta_{\{i\}} = (\Sigma_i, C_i, w_i^{\pm})$ with its associated projected multi-polytope $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$. Note that if Δ is assumed to be a geometric realization of an ordinary fan, then so is Δ_i.

Next, we need to recall the wall crossing formula in ([6], Lemma 5.3).

Theorem 5.2. *Let F denote one of the hyperplanes F_i. Let u_α and u_β be two elements in $M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$ such that the segment from u_α to u_β intersects the wall F transversely at μ, and does not intersect any other $F_j \neq F$. Then, we have*

$$\mathrm{DH}_{\mathcal{P}}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) = \sum_{i \text{ with } F_i = F} \mathrm{sign}\langle u_\beta - u_\alpha, v_i \rangle \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i).$$

In order to use the wall crossing formula (Theorem 5.2), we take $u_\alpha \in \mathcal{P} \cap V$ and $u_\beta \in \mathcal{P}^c \cap V$ such that there is only one wall on the segment between u_α and u_β. Then, since $\mathrm{DH}_{\mathcal{P}}(u_\beta) = 0$ by

[[6], Lemma 5.4] and $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i) = 1$ by induction hypothesis, it follows from Theorem 5.2 that

$$\mathrm{DH}_{\mathcal{P}}(u_\alpha) = \operatorname{sign}\langle u_\beta - u_\alpha, v_i\rangle \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i) = 1$$

(refer to Figure 5.2).

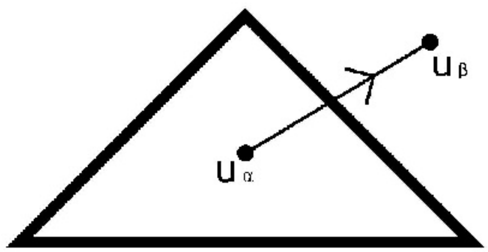

Figure 5.2

Here we also used the fact that $\langle u_\beta - u_\alpha, v_i\rangle > 0$, since v_i is an outward normal vector to the hyperplane F_i. Since u_α can be taken arbitrary in $\mathcal{P} \cap V$, this completes the proof for the necessary condition.

For the converse, we also want to use the mathematical induction on the dimension $\dim \Delta$ of a multi-fan Δ. So assume that $\dim \Delta = 1$. Let E be the subset of $\{1, 2, \cdots, d\}$ such that $i \in E$ if and only if $C(\{i\})$ is the half line consisting of non-negative real numbers. Then the completeness of Δ implies that

$$\sum_{i \in E} w(\{i\}) = \sum_{i \notin E} w(\{i\}) = \deg(\Delta).$$

As before, let us take v to be a generic vector in the positive direction. For simplicity, assume that

$$E = \{j_1, j_2, \cdots, j_l\} \text{ and } E^c = \{i_1, i_2, \cdots, i_k\}$$

such that

$$F_{j_1} < F_{j_2} < \cdots < F_{j_l},\ F_{i_1} < F_{i_2} < \cdots < F_{i_k}, \text{ and } F_{i_k} < 0 < F_{j_1}.$$

Note that

$$\begin{aligned}\mathrm{DH}_{\mathcal{P}}(u) &= \sum_{i\in E} -w(\{i\})\phi_{\{i\}}(u) + \sum_{i\notin E} w(\{i\})\phi_{\{i\}}(u) \\ &= \sum_{i\in E \text{ with } F_i<u} -w(\{i\}) + \sum_{i\notin E \text{ with } F_i<u} w(\{i\})\end{aligned} \tag{5.1}$$

Assume that $l \geq 2$. If u lies between F_{i_k} and F_{j_1} (refer to Figure 5.3), then it follows from (5.1) that we have

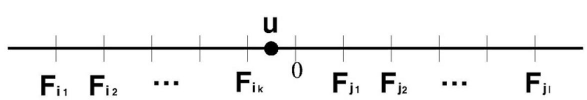

Figure 5.3

$$1 = \mathrm{DH}_{\mathcal{P}}(u) = \sum_{i\notin E \text{ with } F_i<u} w(\{i\}) = \deg(\Delta).$$

On the other hand, if u lies between F_{j_1} and F_{j_2} (refer to Figure 5.4), then we have

$$\begin{aligned}1 = \mathrm{DH}_{\mathcal{P}}(u) &= -w(\{j_1\}) + \sum_{i\notin E \text{ with } F_i<u} w(\{i\}) \\ &= -w(\{j_1\}) + \deg(\Delta) \\ &= -w(\{j_1\}) + 1.\end{aligned}$$

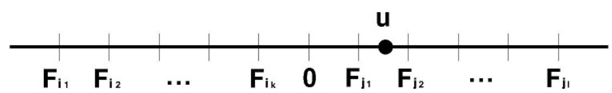

Figure 5.4

Thus $w(\{j_1\}) = 0$. Similarly, it is easy to see that $w(\{j_s\}) = 0$ for all $1 \leq s \leq l$. But this implies that $\deg(\Delta) = \sum_{i \in E} w(\{i\}) = 0$, which is a contradiction.

The case of $k \geq 2$ can be dealt with in a similar way, so that we can conclude that $k = l = 1$. Hence the multi-polytope $\mathcal{P}$ is a geometric realization of an ordinary convex polytope, and so its associated multi-fan Δ is an ordinary fan.

Next we assume that $\dim \Delta = n > 1$ and that the result holds for any multi-fans with dimension less than n. The following lemma plays an important role in the proof.

Lemma 5.3. *For each* $\{i\} \in \Sigma^{(1)}$*, let* $\Delta_{\{i\}} = (\Sigma_i, C_i, w_i^{\pm})$ *with its associated projected multi-polytope* $\mathcal{P}_i = (\Delta_i, \mathcal{F}_i)$*. Then, for* $\mu - f_i \in W := (M_{\mathbb{R}})_i \backslash \bigcup_{j \neq i} \mathcal{F}_i(\{j\})$ *we have*

$$\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i) = \sum_{I \in \Sigma^{(n)} \text{ with } i \in I} (-1)^{I - \{i\}} w_i(I - \{i\}) \phi_I(\mu)$$

$$= \begin{cases} 1, & \mu - f_i \in \mathcal{P}_i \cap W, \\ 0, & \text{otherwise,} \end{cases}$$

where $\phi_I(\mu)$ *is regarded as the value at* μ *of the characteristic function of the cones in* $(M_{\mathbb{R}})_i$ *with apex* u_I *spanned by* $(u_j^I)^+$

for all $j \in I$ such that $j \neq i$.

Proof. For $I \in \Sigma^{(n)}$ with $i \in I$, note that $w(I) = w_i(I - \{i\})$ by definition. Thus we have

$$\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i) = \sum_{I \in \Sigma^{(n)} \text{ with } i \in I} (-1)^{I-\{i\}} w_i(I - \{i\}) \phi_I(\mu),$$

as desired.

For the explicit values of $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i)$, we divide the proof into two subcases. In fact, we can deal with these cases in the same way, but we shall give two different proofs for them, in order to help readers more clearly understand the problems.

First, consider the case where F_i is an affine hyperplane in $\mathrm{HP}(M_{\mathbb{R}})$ which possibly forms an interior wall in the multi-polytope $\mathcal{P}$ and there is an affine hyperplane F_k meeting F_i at a point which is not a vertex of $\mathcal{P}$ (refer to Figure 5.5).

In this case, it follows from the very definition that the affine hyperplane F_k does not contribute to the values of the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i)$. On the other hand, let u_α be an element in the interior of $\mathcal{P}$ and let u_β be an element in the unbounded region in $M_{\mathbb{R}} \backslash \bigcup_{i=1}^{d} F_i$. Assume that the segment from u_α to u_β intersects the wall F_i transversely only once at μ.

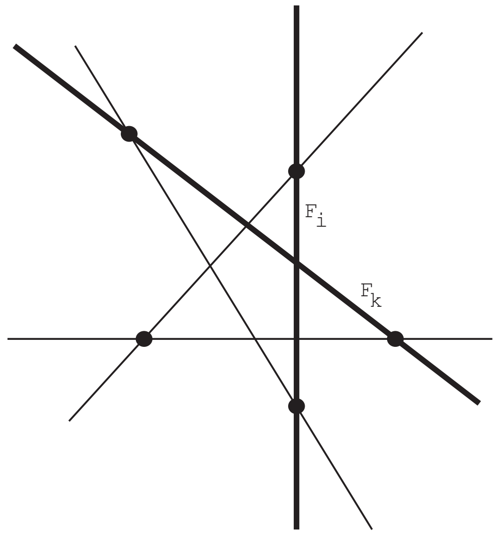

Figure 5.5

Then it follows from Theorem 5.2 that we have

$$\begin{aligned} 1 &= \mathrm{DH}_{\mathcal{P}}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) \\ &= \sum_{I\ \in \Sigma^{(n)} \text{with } i \in I} (-1)^{I-\{i\}} w_i(I - \{i\}) \phi_I(\mu) \\ &= \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i). \end{aligned}$$

Since the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}_i}$ is locally constant on $W = (M_{\mathbb{R}})_i \setminus \cup_{j \neq i} \mathcal{F}_i(\{j\})$, this completes the proof of Lemma 5.3 for this case.

Next we consider the case where F_i is an affine hyperplane in $\mathcal{HP}(M_{\mathbb{R}})$ which possibly forms an interior wall in the multi-polytope $\mathcal{P}$ and there is an affine hyperplane F_k meeting F_i at a point which is also a vertex of $\mathcal{P}$ (refer to Figure 5.6).

In this case, there should be another vertex u_J on F_k in the

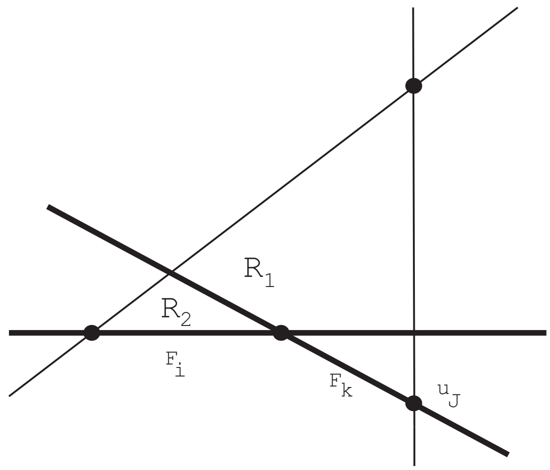

Figure 5.6

direction of the normal vector v_i which contributes the value $+1$ to the Duistermaat-Heckman function $\mathrm{DH}_{\mathcal{P}}$ with respect to a suitably chosen generic vector $v \in N_{\mathbb{R}}$. (Recall that the Duistermaat-Heckman function is independent of the choice of the generic element $v \in N_{\mathbb{R}}$.) Thus, when we remove the vertex u_J together with the affine hyperplanes forming u_J, then we can find two distinct adjacent regions R_1 and R_2 in $\mathcal{P} \cap V$ such that $\mathrm{DH}_{\mathcal{P}}$ over R_1 decreases by one, while $\mathrm{DH}_{\mathcal{P}}$ over R_2 does not change. But, since $\mathrm{DH}_{\mathcal{P}}$ over R_i is equal to one by assumption and the Duistermaat-Heckman function is locally constant on V, we should have

$$1 = \mathrm{DH}_{\mathcal{P}}|_{R_1} \geq \mathrm{DH}_{\mathcal{P}}|_{R_2} + 1 = 2.$$

Clearly it is a contradiction. As a consequence, we can conclude that the second case does not occur.

This completes the proof of Lemma 5.3. □

Finally, we are ready to finish the proof of Theorem 5.1. To do so, as before let F_i be an affine hyperplane in $\mathrm{HP}(M_{\mathbb{R}})$ which possibly forms an interior wall in the multi-polytope $\mathcal{P}$. Then F_i divides the interior of the multi-polytope into two parts. So, let u_α and u_β be two elements in the interior of $\mathcal{P}$ such that the segment from u_α to u_β intersects the wall F_i transversely at μ only once. Then it follows again from Theorem 5.2 that we have

$$\begin{aligned} 0 = 1 - 1 &= \mathrm{DH}_{\mathcal{P}}(u_\alpha) - \mathrm{DH}_{\mathcal{P}}(u_\beta) \\ &= \sum_{I \text{ with } i \in I} (-1)^{I - \{i\}} w_i(I - \{i\}) \phi_I(\mu) \\ &= \mathrm{DH}_{\mathcal{P}_i}(\mu - f_i), \end{aligned}$$

which is a contradiction to Lemma 5.3. This implies that there is no affine hyperplane which forms an interior wall in the multi-polytope $\mathcal{P}$.

There is one more case that we need to consider in order to complete the proof. That is, consider the case of an affine hyperplane F_i which does not divide the interior of the multi-polytope $\mathcal{P}$ into two parts, but does divide $\mathcal{P}$ itself into two parts in such a way that two bounded regions of $\mathcal{P}$ lie on both sides of the hyperplane F_i. In this case, it follows easily from the wall crossing formula (Theorem 5.2) that $\mathrm{DH}_{\mathcal{P}_i}(\mu - f_i)$ should have the value $+1$ and also -1, due to the orientation of the vector v_i normal to F_i. However, clearly this is a contradiction.

In fact, it is also easy to see that this case does not occur by the definition of a multi-polytope.

As a consequence, the multi-polytope $\mathcal{P}$ is actually a geometric realization of an ordinary polytope and, therefore, its associated multi-fan Δ should be an ordinary fan. This completes the proof of Theorem 5.1. $\square$

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