



저작자표시 2.0 대한민국

이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.
- 이차적 저작물을 작성할 수 있습니다.
- 이 저작물을 영리 목적으로 이용할 수 있습니다.

다음과 같은 조건을 따라야 합니다:



저작자표시. 귀하는 원저작자를 표시하여야 합니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 [이용허락규약\(Legal Code\)](#)을 이해하기 쉽게 요약한 것입니다.

[Disclaimer](#) 

2015년 8월

교육학석사(수학)학위논문

Hattori-Masuda multi-polytopes and Generalized Ehrhart polynomials

조선대학교 교육대학

수학교육전공

이 화

Hattori-Masuda multi-polytopes and Generalized Ehrhart polynomials

하토리-마수다 다중 폴리토프와 일반화된 에하트 다항식 연구

2015년 8월

조선대학교 교육대학원

수학교육전공

이 화

Hattori-Masuda multi-polytopes and Generalized Ehrhart polynomials

지도교수 김 진 홍

이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

2015년 4월

조선대학교 교육대학원

수학교육전공

이 화

이 화의 교육학 석사학위 논문을 인준함.

심사위원장 조선대학교 교수 이 관 규 인

심사위원 조선대학교 교수 김 진 홍 인

심사위원 조선대학교 교수 오 동 렬 인

2015년 6월

조선대학교 교육대학원

CONTENTS

국문초록

1. Introduction	2
2. Multi-fans and Multi-polytopes	8
3. Duistermaat-Heckman Functions	13
4. Winding Numbers	19
5. Main Results: Generalized Ehrhart Polynomials	
5.1 Ehrhart polynomials	23
5.2 Generalized Pick's formula	30
5.3 Simple multi-polygons	34
5.4 Generalized twelve-point theorem	41

국 문 초 록

하토리-마수다 다중 폴리토프와 일반화된 에하트 다항식 연구

이 화

지도교수 : 김 진 홍

조선대학교 교육대학원 수학교육전공

하토리-마수다에 의해 발견된 다중 팬은 토러스 다양체와 기하학적인 관련이 있다. 하지만, 토러스 다양체와 관련한 다중 팬이 아닐지라도 흥미로운 문제를 지니고 있다. 본 논문에서는 하토리-마수다의 결과를 통해 단순 격자 다중폴리토프의 새로운 결과를 증명했고, 단순 격자 다중 폴리토프에서

Duistermaat-Heckman 함수 DH_P 을 통해 $\#(P)$ 와 $\#(P^\vee)$ 을 정의하고 $\text{vol}(P)$ 와 $\text{vol}(P^\vee)$ 를 정의하여, 다음과 같은 사실이 성립함을 보였다:

(1) P 가 2차원 단순 격자 다중 폴리토프일 때,

$$\text{vol}(P) = \#(P^\circ) + \frac{1}{2} \#(\partial P) - \deg(\Delta)$$

와 같은 일반화된 Pick's formula가 성립한다.

(2) P 가 2차원 단순 격자 다중 폴리토프이고 P 의 내부에 있는 격자점이 원점만 존재할 때,

$$\text{vol}(P) = \#(P^\circ) + \frac{1}{2} \#(\partial P) - 1$$

이 성립한다.

(3) P 가 2차원 단순 격자 다중 폴리토프이고 단순 폴리곤이며 내부에 격자점이 원점만 존재할 때,

$$\#(\partial P) + \#(\partial P^\vee) = 2(\text{vol}(P) + \text{vol}(P^\vee)) + 2(\deg(\Delta) + \deg(\Delta^\vee)) - 4$$

이 성립한다. 이 등식은 $\#(\partial P) + \#(\partial P^\vee) = 12$ 이므로 twelve-point 정리를 일반화한 것으로 볼 수 있다.

(4) P 가 3차원 단순 격자 다중 폴리토프이고일 때, 에하트 다항식

$$\#(\nu P) = \text{vol}(P)\nu^3 + (\#(P) - \frac{1}{2}\#(\partial P) - \text{vol}(P))\nu^2 +$$

$$\frac{1}{2}(\#(\partial P) - \deg(\Delta))\nu + \deg(\Delta)$$

이 만족된다.

(5) P 가 4차원 단순 격자 다중 폴리토프이고일 때, 에하트 다항식

$$\#(\nu P) = a_4\nu^4 + a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0$$

를 만족하고, 계수들 사이의 관계식

$$a_4 = \text{vol}(P), \quad a_0 = \deg(\Delta), \quad a_1 + a_3 = \frac{1}{2}\#(\partial P)$$

이 성립한다.

Chapter 1

Introduction

A multi-fan, developed first by Hattori and Masuda in their paper [5], is a purely combinatorial object which generalizes an ordinary fan in algebraic geometry. One typical geometric realization of a multi-fan is a torus manifold, while an ordinary fan is associated with a toric variety (refer to [9] and [10]). Recall that a toric variety is a normal complex algebraic variety of dimension n with a $(\mathbb{C}^*)^n$ -action having one unique dense orbit and other orbits of smaller dimensions. It is well known that there is a one-to-one correspondence between toric varieties and fans. The fan associated with a toric variety is a collection of cones in \mathbb{R}^n with apex at the origin, and to each orbit of a $(\mathbb{C}^*)^n$ -action on a toric variety there corresponds a cone of dimension equal to the codimension of the orbit.

Contrary to the case of usual fans, it is unfortunate that two different torus manifolds may correspond to the same multi-fan. Moreover, the union of cones in a multi-fan may overlap several times. Nonetheless, many important topological properties of a torus manifold can be detected by its associated multi-fan. Indeed, in their paper [5] Hattori and Masuda provide several combinatorial invariants of a multi-fan which correspond to the ordinary topological invariants

of the associated torus manifold. However, it is not yet known and also an interesting problem to know whether or not every regular complete multi-fan is realized as a multi-fan associated with a torus manifold.

Associated to an ordinary fan, there is a notion of a convex polytope. Similarly, there is a notion of a multi-polytope associated to a multi-fan. Indeed, let N be a lattice of rank n which is isomorphic to \mathbb{Z}^n , and let M be the dual lattice $\text{Hom}(N, \mathbb{Z})$. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and let $M_{\mathbb{R}} = \text{Hom}(N_{\mathbb{R}}, \mathbb{R})$. A multi-polytope \mathcal{P} is a pair (Δ, \mathcal{F}) of an n -dimensional multi-fan Δ and an arrangement $\mathcal{F} = \{F_i\}$ of affine hyperplanes F_i in the dual space $M_{\mathbb{R}}$ with the same index set as the set of one-dimensional cones in Δ (refer to Chapter 2 for a more precise definition). In their paper [5], Hattori and Masuda also give the definitions of the Duistermaat-Heckman function, the winding number, and the equivariant index in a purely combinatorial manner for multi-fans and multi-polytopes.

A *lattice polytope* P means that each vertex of P lies in the lattice M of $M_{\mathbb{R}}$. For a convex lattice polytope P of dimension n in $M_{\mathbb{R}}$ and a positive integer ν , let νP be

$$\nu P = \{\nu u \mid u \in P\}.$$

Then νP is again a lattice multi-polytope in $M_{\mathbb{R}}$. Let us denote by $\#(\nu P)$ (resp. $\#(\nu P^\circ)$) the number of lattice points in νP (resp. in the interior νP° of νP). Let us also denote by $\#(\partial(\nu P))$ the number of lattice points on the boundary $\partial(\nu P)$ of νP . Then clearly we have

$$\#(\partial(\nu P)) = \#(\nu P) - \#(\nu P^\circ).$$

In this paper, as in [5] we normalize a volume element on $M_{\mathbb{R}}$ so that the volume of the unit cube determined by a basis of M is equal to one. Then $\#(\nu P)$

and $\#(\nu P^\circ)$ are polynomials in ν of degree n , which are called the *Ehrhart polynomials* of P and P° , respectively. Recall that the number of lattice points of a simple, regular convex polytope is equal to its corresponding Riemann-Roch number (refer to, e.g., [3], [7], [4], and [9] for more details).

The main aim of this thesis is to survey some recent results and also show some interesting and new results for simple lattice multi-polytopes. That is, we prove that even for simple lattice multi-polytopes, we can establish similar results such as the generalized Pick's formula and the generalized twelve-point theorem, as follows (see Section 5.2 for precise definitions).

Theorem 1.1. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 2. Then the following identity holds.*

$$\text{vol}(\mathcal{P}) = \#(\mathcal{P}^\circ) + \frac{1}{2}\#(\partial\mathcal{P}) - \deg(\Delta).$$

As an immediate consequence, we can easily obtain the well-known Pick's formula for simple convex polytopes.

Corollary 1.2. *Let P be a simple convex lattice polytope of dimension 2. Then the following identity holds.*

$$\text{vol}(P) = \#(P^\circ) + \frac{1}{2}\#(\partial P) - 1.$$

Now, let \mathcal{P} be a simple lattice multi-polytope whose interior lattice is just the origin. Then the dual multi-polytope \mathcal{P}^\vee is also a simple lattice multi-polytope whose interior lattice point is only the origin. Thus we have

$$\#(\mathcal{P}^\circ) = \#((\mathcal{P}^\vee)^\circ) = 1.$$

By applying the generalized Pick's formula for simple lattice multi-polytopes in Theorem 1.1 to both \mathcal{P} and \mathcal{P}^\vee , it is immediate to obtain the following generalized twelve-point theorem.

Theorem 1.3. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 2 which is a lattice multi-polygon and whose interior lattice point is just the origin. Then we have*

$$\begin{aligned} \#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) \\ = 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)) + 2(\deg(\Delta) + \deg(\Delta^\vee)) - 4. \end{aligned}$$

By applying the well-known twelve-point theorem saying that, for simple lattice polytopes P

$$\#\partial P + \#\partial P^\vee = 12,$$

it is also easy to show the following corollary.

Corollary 1.4. *Let P be a simple convex lattice polytope of dimension 2. Then we have*

$$\text{vol}(P) + \text{vol}(P^\vee) = 6.$$

As a consequence, either $\text{vol}(P)$ and $\text{vol}(P^\vee)$ are 2 and 4, respectively, or $\text{vol}(P)$ and $\text{vol}(P^\vee)$ are 4 and 2, respectively.

Moreover, for simple lattice multi-polytopes \mathcal{P} of dimension 3, we can identify the coefficients of the Ehrhart polynomial $\#(\nu\mathcal{P})$, as follows.

Theorem 1.5. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 3. Then the Ehrhart polynomial $\#(\nu\mathcal{P})$ satisfies*

$$\begin{aligned} \#(\nu\mathcal{P}) = \text{vol}(\mathcal{P})\nu^3 + \left(\#(\mathcal{P}) - \frac{1}{2}\#(\partial\mathcal{P}) - \text{vol}(\mathcal{P}) \right) \nu^2 \\ + \left(\frac{1}{2}\#(\partial\mathcal{P}) - \deg(\Delta) \right) \nu + \deg(\Delta). \end{aligned}$$

Finally, by using a similar method as in Theorems 1.3 and 1.5, we can obtain a weaker result for simple lattice polytopes with the dimension 4, as follows.

Theorem 1.6. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 4, and let us write the Ehrhart polynomial $\#(\nu\mathcal{P})$ of \mathcal{P} , as follows.*

$$\#(\nu\mathcal{P}) = a_4\nu^4 + a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0.$$

Then the following relationship between the coefficients holds:

$$\begin{aligned} a_4 &= \text{vol}(\mathcal{P}), \quad a_0 = \deg(\Delta), \\ a_1 + a_3 &= \frac{1}{2}\#(\partial\mathcal{P}). \end{aligned}$$

Briefly, we now want to explain the contents of each chapter of this thesis. In Chapter 2, we give a definition of a multi-fan and introduce certain related notions which are necessary for later chapters. In Chapter 3, the notion of a multi-polytope and the associated Duistermaat-Heckman function are defined. As explained above, a multi-polytope is a pair $\mathcal{P} = (\Delta, \mathcal{F})$ of an n -dimensional complete multi-fan Δ and a arrangement of hyperplanes $\mathcal{F} = \{F_i\}$ in $H^2(BT; \mathbb{R})$ with the same index set as the set of 1-dimensional cones in Δ . It is called *simple* if the multi-fan Δ is simplicial. The Duistermaat-Heckmann function $\text{DH}_{\mathcal{P}}$ associated with a simple multi-polytope \mathcal{P} is a locally constant integer-valued function with bounded support defined on the complement of the hyperplanes $\{F_i\}$. In Chapter 4, another locally constant function on the complement of the hyperplanes $\{F_i\}$ in a multi-polytope \mathcal{P} , called the *winding number*, is introduced.

If P is a convex lattice polytope and if νP denotes the multiplied polytope by a positive integer ν , then the number of lattice points $\#(\nu P)$ contained in νP ,

called the *Ehrhart polynomial* as in the case of ordinary polytopes, will be shown to be a polynomial in ν whose top coefficient is given by the volume of P and whose constant term is given by the degree of the multi-fan. Chapter 5 will be devoted to a generalization of the Ehrhart polynomial to multi-polytopes. In this section, we will give proofs of our main results stated in Chapter 1. Moreover, we plan to give several applications such as the generalized Pick's theorem and the generalized twelve-point theorem for simple lattice multi-polytopes.

Chapter 2

Multi-fans and Multi-polytopes

The aim of this chapter is to set up some basic notations and terminology necessary for the proofs of our main results in Chapter 5. We remark that Chapters 2, 3, and 4 of this thesis are largely taken from the excellent paper [5].

To do so, let N be a lattice of rank n , which is isomorphic to \mathbb{Z}^n . We denote the real vector space $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with apex at the origin) if there exists a finite number of vectors v_1, \dots, v_m in N such that

$$\sigma = \{r_1 v_1 + \dots + r_m v_m \mid r_i \in \mathbb{R} \text{ and } r_i \geq 0 \text{ for all } i\}, \quad \sigma \cap (-\sigma) = \{0\}.$$

The dimension $\dim \sigma$ of a cone σ is defined to be the dimension of the linear space generated by vectors in σ . A subset τ of σ is called a *face* of σ if there is a linear function

$$l : N_{\mathbb{R}} \longrightarrow \mathbb{R}$$

such that l takes nonnegative values on σ and vanishes on $\tau = l^{-1}(0) \cap \sigma$. A cone is regarded as a face of itself, while others are called *proper* faces.

Definition 2.1. A fan Δ in N is a set of a finite number of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ such that

- (1) each face of a cone in Δ is also a cone in Δ ,
- (2) the intersection of two cones in Δ is a face of each.

We also need the following notion of the completeness of an ordinary fan.

Definition 2.2. A fan Δ is said to be *complete* if the union of cones in Δ covers the entire space $N_{\mathbb{R}}$.

A cone is called *simplicial* if it is generated by linearly independent vectors. If the generating vector can be taken as a part of a basis of N , then the cone is called *nonsingular*.

Definition 2.3. A fan Δ is said to be *simplicial* (resp. *non-singular*) if every cone in Δ is simplicial (resp. non-singular).

Denote by $\text{Cone}(N)$ the set of all cones in N . An ordinary fan is a subset of $\text{Cone}(N)$. The set $\text{Cone}(N)$ has a partial ordering \prec defined by : $\tau \prec \nu$ if and only if τ is a proper face of ν . The cone $\{0\}$ consisting of the origin is the unique minimum element of $\text{Cone}(N)$.

On the other hand, let Σ be a partial ordering finite set with a unique minimum element. We denote the strict partial ordering by $<$ and the minimum element by $*$. An example of Σ used later is an abstract simplicial set with an empty set added as a member, which we call an augmented simplicial set. In this case the partial ordering is defined by the inclusion relation and the empty set is the unique minimum element which may be considered as a (-1) -simplex. Suppose that there is a map

$$C : \Sigma \rightarrow \text{Cone}(N)$$

such that

- (1) $C(*) = \{0\}$,
- (2) If $I < J$ for $I, J \in \Sigma$, then $C(I) < C(J)$,
- (3) For any $J \in \Sigma$ the map C restricted on $\{I \in \Sigma \mid I \leq J\}$ is an isomorphism of ordered sets onto $\{K \in \text{Cone}(N) \mid K \leq C(J)\}$.

For an integer m such that $0 \leq m \leq n$, we set

$$\Sigma^{(m)} := \{I \in \Sigma \mid \dim C(I) = m\}.$$

One can easily check that $\Sigma^{(m)}$ does not depend on C . When Σ is an augmented simplicial set, $I \in \Sigma$ belongs to $\Sigma^{(m)}$ if and only if the cardinality $|I|$ of I is m , namely I is an $(m-1)$ -simplex. Therefore, even if Σ is not an augmented simplicial set, we use the notation $|I|$ for m when $I \in \Sigma^{(m)}$. The image $C(\Sigma)$ is a finite set of cones in N . We may think of a pair (Σ, C) as a set of cones in N labeled by the ordered set Σ . Cones in an ordinary fan intersect only at their faces, but cones in $C(\Sigma)$ may overlap, even the same cone may appear repeatedly with different labels. The pair (Σ, C) is almost what we call a multi-fan, but we incorporate a pair of weight functions on cones in $C(\Sigma)$ of the highest dimension $n = \text{rank } N$. More precisely, we consider two functions

$$\omega^{\pm} : \Sigma^{(n)} \rightarrow \mathbb{Z}_{\geq 0}.$$

We assume that $\omega^+(I) > 0$ or $\omega^-(I) > 0$ for every $I \in \Sigma^{(n)}$. These two functions have its origin from geometry. In fact, if M is a torus manifold of dimension $2n$ and if M_{i_1}, \dots, M_{i_n} are characteristic submanifolds such that their intersection contains at least one T -fixed point, then the intersection $M_1 = \cap_{i_1} M_{i_1}$ consists of a finite number of T -fixed points. At each fixed point $p \in M_I$

the tangent space τ_p has two orientations; one is endowed by the orientation of M and the other comes from the intersection of the oriented submanifolds M_{i_ν} . Denoting the ratio of the above two orientations by ϵ_p we define the number $\omega^+(I)$ to be the number of points $p \in M_I$ with $\epsilon_p = +1$ and similarly for $\omega^-(I)$.

Definition 2.4. We call a triple $\Delta := (\Sigma, C, \omega^\pm)$ a multi-fan in N . We define the *dimension* of Δ to be the rank of N (or dimension of $N_{\mathbb{R}}$).

Since an ordinary fan Δ in N is a subset of $\text{Cone}(N)$, one can view it as a multi-fan by taking $\Sigma = \Delta, C = \text{the inclusion map}, \omega^+ = 1$, and $\omega^- = 0$. In a similar way as in the case of ordinary fans, we say that a multi-fan $\Delta = (\Sigma, C, \omega^\pm)$ is simplicial (resp. non-singular) if every cone in $C(\Sigma)$ is simplicial (resp. non-singular). The following lemma is easy.

Lemma 2.5. *A multi-fan $\Delta = (\Sigma, C, \omega^\pm)$ is simplicial if and only if Σ is isomorphic to an augmented simplicial set as partially ordered sets.*

The definition of completeness of a multi-fan Δ is rather complicated. A naive definition of the completeness would be that the union of cones in $C(\Sigma)$ covers the entire space $N_{\mathbb{R}}$. Although the two weighted functions ω^\pm are incorporated in to definition of a multi-fan, only the difference

$$\omega := \omega^+ - \omega^-$$

is important in this thesis. We shall introduce the following intermediate notion of pre-completeness at first. A vector $v \in N_{\mathbb{R}}$ will be called *generic* if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n . For a generic vector v we set

$$d_v = \sum_{I \in C(I)} \omega(I),$$

where the sum is understood to be zero if there is no such I .

Definition 2.6. We call a multi-fan $\Delta = (\Sigma, C, \omega^\pm)$ of dimension n *pre-complete* if $\Sigma^{(m)} \neq 0$ and the integer d_v is independent of the choice of generic vectors v . We call this integer the degree of Δ and denote it by $\deg(\Delta)$.

We remark that for an ordinary fan, pre-completeness is the same as completeness.

From now on, we set $V = N_{\mathbb{R}}$, unless stated otherwise. A convex polytope P in $V^* = \text{Hom}(V, \mathbb{R})$ is the convex hull of a finite set of points in V^* . It is the intersection of a finite number of half space in V^* separated by affine hyperplanes, so there are a finite number of nonzero vectors v_1, \dots, v_d in V and real numbers c_1, \dots, c_d such that

$$P = \{u \in V^* \mid \langle u, v_i \rangle \leq c_i \text{ for all } i\},$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between V^* and V .

A polytope gives rise to a multi-fan in this way. Note that convex polytope gives rise to a complete fan. Now, we begin with a complete multi-fan $\Delta = (\Sigma, C, \omega^\pm)$. Let $\text{HP}(V^*)$ be the set of all affine hyperplanes in V^* .

Definition 2.7. Let $\Delta = (\Sigma, C, \omega^\pm)$ be a complete multi-fan and let $\mathcal{F} : \Sigma^{(1)} \rightarrow \text{HP}(V^*)$ be a map such that the affine hyperplane $\mathcal{F}(I)$ is perpendicular to the half line $C(I)$. That is, any element in $C(I)$ takes a constant on $\mathcal{F}(I)$. We call a pair (Δ, \mathcal{F}) a *multi-polytope* and denote it by \mathcal{P} . The *dimension* of a multi-polytope \mathcal{P} is defined to be the dimension of the multi-fan Δ . We say that a multi-polytope \mathcal{P} is *simple* if Δ is simplicial.

Chapter 3

Duistermaat-Heckman Functions

A multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ defines an arrangement of affine hyperplanes in V^* . In this chapter, we associate with \mathcal{P} a function on V^* minus the affine hyperplanes when \mathcal{P} is simple. This function is locally constant and Guillemin-Lerman-Sternberg formula tells us that it agrees with the density function of a Duistermaat-Heckman measure when \mathcal{P} arises from a moment map.

From now on, our multi-polytope \mathcal{P} is assumed to be simple, so that the multi-fan $\Delta = (\Sigma, C, w^\pm)$ is complete and simplicial, unless otherwise stated. As before, we may assume that Σ consists of subsets of $\{1, \dots, d\}$ and $\Sigma^{(1)} = \{\{1\}, \dots, \{d\}\}$, and denote by v_i a nonzero vector in the one-dimensional cone $C(\{i\})$. To simplify notation, we denote $\mathcal{F}(\{i\})$ by F_i and set

$$F_I := \cap_{i \in I} F_i \text{ for } I \in \Sigma.$$

Then F_I is an affine space of dimension $n - |I|$. In particular, if $|I| = n$ (i.e., $I \in \Sigma^{(n)}$), then F_I is a point, denoted by u_I . Suppose $I \in \Sigma^{(n)}$. Then the set $\{v_i \mid i \in I\}$ forms a basis of V . Denote its dual basis of V^* by $\{u_i^I \mid i \in I\}$, i.e., $\langle u_i^I, v_j \rangle = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

Next we take a generic vector $v \in V$. Recall that a vector $v \in N_{\mathbb{R}}$ is called

generic if it does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n . Then $\langle u_i^I, v \rangle \neq 0$ for all $I \in \Sigma^{(n)}$ and $i \in I$. Set

$$(-1)^I := (-1)^{\#\{i \in I \mid \langle u_i^I, v \rangle > 0\}},$$

and

$$(u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v \rangle > 0, \\ -u_i^I & \text{if } \langle u_i^I, v \rangle < 0. \end{cases}$$

We denote by $C^*(I)^+$ the cone in V^* spanned by $(u_i^I)^+$'s ($i \in I$) with apex at u_I , and by ϕ_I its characteristic function.

Definition 3.1. We define a function $\text{DH}_{\mathcal{P}}$ on $V^* \setminus \cup_{i=1}^d F_i$ by

$$\text{DH}_{\mathcal{P}} := \sum_{I \in \Sigma^{(n)}} (-1)^I \omega(I) \phi_I,$$

and call it the *Duistermaat-Heckman function* associated with \mathcal{P} .

Remark that the function $\text{DH}_{\mathcal{P}}$ is defined on the whole space V^* and depends on the choice of the generic vector $v \in V$. But it is true ([5], Lemma 5.4 or Theorem 3.3) that it is independent of v on $V^* \setminus \cup F_i$. This is the reason why we restricted the domain of the function to $V^* \setminus \cup F_i$. In the next example, we see that the value of DH function is independent of the choice of v , when $\dim \mathcal{P} = 1$.

Example 3.2. Suppose $\dim \mathcal{P} = 1$. We identify V with \mathbb{R} , so that V^* is also identified with \mathbb{R} . Let E be the subset of $\{1, \dots, d\}$ such that $i \in E$ if and only if $C(\{i\})$ is the half line consisting of non-negative real numbers. Then the completeness of Δ means that

$$(3.1) \quad \sum_{i \in E} w(\{i\}) = \sum_{i \notin E} w(\{i\}) = \deg(\Delta).$$

Take a nonzero vector v . Since V^* is identified with \mathbb{R} , each affine hyperplane F_i is nothing but a real number. Suppose that v is toward the positive direction. Then

$$(3.2) \quad (-1)^{\{i\}} = \begin{cases} -1 & \text{if } i \in E, \\ 1 & \text{if } i \notin E, \end{cases}$$

and the support of the characteristic function $\phi_{\{i\}}$ is the half line given by

$$\{u \in \mathbb{R} \mid F_i \leq u\}$$

Therefore

$$(3.3) \quad \text{DH}_{\mathcal{P}}(u) = \sum_{i \in E \text{ s.t. } F_i < u} -w(\{i\}) + \sum_{i \notin E \text{ s.t. } F_i < u} w(\{i\})$$

for $u \in \mathbb{R} \setminus \cup F_i$. If u is sufficiently small, then the sum above is empty; so it is zero. If u is sufficiently large, then the sum is also zero by (3.1). Hence the support of the function $\text{DH}_{\mathcal{P}}$ is bounded. Now, suppose that v is toward the negative direction. Then $(-1)^{\{i\}}$ above is multiplied by -1 and the inequality \leq above turns into \geq . Therefore, we obtain

$$(3.4) \quad \text{DH}_{\mathcal{P}}(u) = \sum_{i \in E \text{ s.t. } u < F_i} w(\{i\}) + \sum_{i \notin E \text{ s.t. } u < F_i} (-w(\{i\})).$$

It follows that

$$\text{R.H.S. of (3.3)} - \text{R.H.S. of (3.4)} = - \sum_{i \in E} w(\{i\}) + \sum_{i \notin E} w(\{i\})$$

which is zero by (3.1). This shows that the function $\text{DH}_{\mathcal{P}}$ is independent of v when $\dim \mathcal{P} = 1$.

Assume $n = \dim \Delta > 1$. For each $\{i\} \in \Sigma^{(1)}$, the projected multi-fan $\Delta_{\{i\}} = (\Sigma_{\{i\}}, C_{\{i\}}, w_{\{i\}}^{\pm})$, which we abbreviate as $\Delta_i = (\Sigma_{\{i\}}, C_{\{i\}}, w_{\{i\}}^{\pm})$, is defined

on the quotient vector space V/V_i of V by the one-dimensional subspace V_i spanned by v_i . Since Δ is complete and simplicial, so is Δ_i . We identify the dual space $(V/V_i)^*$ with

$$(V^*)_i := \{u \in V^* \mid \langle u, v_i \rangle = 0\}$$

in a natural way. We choose an element $f_i \in F_i$ arbitrarily and translate F_i onto $(V^*)_i$ by $-f_i$. If $\{i, j\} \in \Sigma^{(2)}$, then F_j intersects F_i and their intersection will be translated into $(V^*)_i$ by $-f_i$. This observation leads us to consider the map

$$F_i : \Sigma_i \rightarrow \text{HP}((V^*)_i)$$

sending $\{j\} \in \Sigma_{\{i\}}^{(1)}$ to $F_i \cap F_j$ translated by $-f_i$. The pair $\mathcal{P}_i = (\Delta_i, F_i)$ is a multi-polytope in $(V/V_i)^* \cong (V^*)_i$. Let $I \in \Sigma^{(n)}$ such that $i \in I$. Since $\langle u_j^I, v_i \rangle = \delta_{ij}$, u_j^I for $j \neq i$ is an element of $(V^*)_i$. We denote the projection image of the generic element $v \in V$ on V/V_i by \bar{v} . Then we have $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle$ for $j \neq i$, where u_j^I at the left-hand side is viewed as an element of $(V/V_i)^*$ while the one at the right-hand side is viewed as an element of $(V^*)_i$. Since $\langle \bar{v}, u_j^I \rangle = \langle v, u_j^I \rangle \neq 0$ for $j \neq i$, we use \bar{v} to define $\text{DH}_{\mathcal{P}_i}$.

It turns out that the Duistermaat-Heckman function is bounded, and vanishes outside the bounded region bounded by the hyperplanes in $\text{HP}(V^*)$, as follows.

Theorem 3.3. *The support of the function $\text{DH}_{\mathcal{P}}$ is bounded, and the function is independent of the choice of the generic element $v \in V$.*

We close this chapter with some examples to explain how to calculate the Duistermaat-Heckman function associated to a complete multi-polytope.

Example 3.4. Let Σ be an ordinary polytope given by the following simplicial complex

$$\Sigma = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}.$$

Then define a function $C : \Sigma \rightarrow \text{Cone}(N)$ by

$$C(\{1\}) = v_1 = (0, 1), \quad C(\{2\}) = v_2 = (1, 0), \quad C(\{3\}) = v_3 = (-1, -1),$$

and

$$C(\{i, i+1\}) = \text{the cone spanned by } v_i \text{ and } v_{i+1}.$$

Here we assume that $v_4 = v_1$. Let us also take weight functions w^\pm such that $w = 1$ on every two dimensional cone in

$$\Sigma^{(2)} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}.$$

Then

$$\Delta = (\Sigma, C, \omega), \quad I \in \Sigma^{(2)}$$

is a complete non-singular two-dimensional multi-fan (actually, fan) with $\deg(\Delta) = 1$ (see Figure 3.1).

Next, let us take a generic vector $v = (2, 3)$, and then we want to calculate the Duistermaat-Heckman function $\text{DH}_{\mathcal{P}}$, as follows:

$$\begin{aligned} \text{DH}_{\mathcal{P}} &= (-1)^{\{1,2\}} \omega(\{1, 2\}) \phi_{\{1,2\}} \\ &\quad + (-1)^{\{2,3\}} \omega(\{2, 3\}) \phi_{\{2,3\}} + (-1)^{\{3,1\}} \omega(\{3, 1\}) \phi_{\{3,1\}}. \end{aligned}$$

Hence, we need to consider the following cases:

1) $I = \{1, 2\}$;

$$\begin{aligned} u_1^{\{1,2\}} &= (0, 1), \quad u_2^{\{1,2\}} = (1, 0), \\ (u_1^{\{1,2\}})^+ &= u_1^{\{1,2\}}, \quad (u_2^{\{1,2\}})^+ = u_2^{\{1,2\}}. \end{aligned}$$

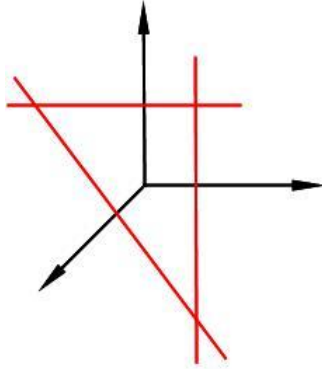


Figure 3.1: ordinary polytope P .

2) $I = \{2, 3\};$

$$u_1^{\{2,3\}} = (1, -1), \quad u_2^{\{2,3\}} = (0, -1),$$

$$(u_1^{\{2,3\}})^+ = -u_1^{\{2,3\}}, \quad (u_2^{\{2,3\}})^+ = -u_2^{\{2,3\}}.$$

3) $I = \{3, 1\};$

$$u_1^{\{3,1\}} = (-1, 0), \quad u_2^{\{3,1\}} = (0, -1),$$

$$(u_1^{\{3,1\}})^+ = -u_1^{\{3,1\}}, \quad (u_2^{\{3,1\}})^+ = -u_2^{\{3,1\}}.$$

Therefore, we can obtain

$$\begin{aligned} \text{DH}_{\mathcal{P}}(u) &= (-1)^2 \cdot 1 \cdot \phi_{\{1,2\}}(u) \\ &\quad + (-1)^0 \cdot 1 \cdot \phi_{\{2,3\}}(u) + (-1)^1 \cdot (-1) \cdot 1 \cdot \phi_{\{3,1\}}(u) \\ &= \begin{cases} 1, & u \in \mathcal{P}^\circ, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Chapter 4

Winding Numbers

The aim of this section is to introduce another locally constant function defined on $V^* \setminus \cup F_i$, called the winding number, which are associated with a simple multi-polytope \mathcal{P} . It turns out that the winding number of a simple multi-polytope \mathcal{P} is identical with the Duistermaat-Heckman function of \mathcal{P} . In this chapter, we quickly review its definition and a few properties. Refer to [5], Section 6 for more details.

As before, let $\mathcal{P} = (\Delta, \mathcal{F})$ be a simple multi-polytope, and let Σ be an augmented simplicial set consisting of subsets of $\{1, 2, \dots, d\}$. Then we fix an orientation on V . Let $I = \{i_1, i_2, \dots, i_n\} \in \Sigma^{(n)}$. Then I is said to have a *positive orientation* if the ordered basis $\{v_{i_1}, v_{i_2}, \dots, v_{i_n}\}$ gives the chosen orientation of V , and is said to have a *negative orientation*, otherwise. We also define

$$\langle I \rangle := \begin{cases} \langle i_1, i_2, \dots, i_n \rangle, & \text{if } \langle i_1, i_2, \dots, i_n \rangle \text{ has a positive orientation,} \\ -\langle i_1, i_2, \dots, i_n \rangle, & \text{if } \langle i_1, i_2, \dots, i_n \rangle \text{ has a negative orientation.} \end{cases}$$

It can be shown that the completeness of Δ implies that

$$\sum_{I \in \Sigma^{(n)}} w(I) \langle I \rangle$$

is a cycle in the chain complex of the simplicial set Σ . In fact, the following lemma holds.

Lemma 4.1. *If a simplicial multi-fan Δ is complete, then $\sum_{I \in \Sigma(n)} w(I) \langle I \rangle$ is a cycle, and, moreover, the converse also holds.*

Then we need a definition, as follows.

Definition 4.2. We shall denote by $[\Delta]$ the homology class that the cycle $\sum_{I \in \Sigma(n)} w(I) \langle I \rangle$ defines in the reduced homology $\tilde{H}_{n-1}(\Sigma; \mathbb{Z})$.

Let S be the realization of the first barycentric subdivision of Σ . For each $i \in \{1, 2, \dots, d\}$, we denote by S_i the union of simplices in S which contains the vertex $\{i\}$, and let $S_I = \cap_{i \in I} S_i$ for $I \in \Sigma$. Note that the boundary ∂S_i of S_i can be identified with the realization of the first barycentric subdivision of Σ_i , where Σ_i is the augmented simplicial set of the projected multi-fan $\Delta_i = (\Sigma_i, C_i, w_i^\pm)$ in $M_{\mathbb{R}}/(M_{\mathbb{R}})_i$. Then, as before the cycle $[\Delta_i]$ defines an element in $\tilde{H}_{n-2}(\Sigma_i, \mathbb{Z}) = \tilde{H}_{n-2}(\partial S_i; \mathbb{Z})$ with respect to the compatible orientation.

The following lemma holds ([5], Lemma 6.1).

Lemma 4.3. *Under the compositions of the following maps*

$$\tilde{H}_{n-2}(\Sigma; \mathbb{Z}) \xrightarrow{i_*} H_{n-1}(S, S \setminus S_i^\circ; \mathbb{Z}) \cong H_{n-1}(S_i, \partial S_i; \mathbb{Z}) \xrightarrow{\partial} \tilde{H}_{n-2}(\partial S_i; \mathbb{Z}),$$

the $(n-1)$ -cycle $[\Delta]$ maps to the $(n-2)$ -cycle $[\Delta_i]$.

We also have the following lemma ([5], Lemma 6.2).

Lemma 4.4. *The following statements hold.*

(a) The multi-polytope \mathcal{P} gives rise to a continuous map

$$\psi : S \rightarrow \cup_{i=1}^d F_i \subset M_{\mathbb{R}}$$

under which S_I is mapped to F_I for each $I \in \Sigma$.

(b) The map ψ induces a homomorphism

$$\psi_* : \tilde{H}_{n-1}(S; \mathbb{Z}) \cong \tilde{H}_{n-1}(\Sigma; \mathbb{Z}) \rightarrow \tilde{H}_{n-1}(M_{\mathbb{R}} - \{u\}; \mathbb{Z})$$

for each $u \in M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$.

We shall denote by $[M_{\mathbb{R}} - \{u\}]$ the fundamental class in $\tilde{H}_{n-1}(M_{\mathbb{R}} - \{u\}; \mathbb{Z})$ for each $u \in M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$.

Definition 4.5. For each $u \in M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$, we define an integer $\text{WN}_{\mathcal{P}}(u)$ by

$$\psi_*([\Delta]) = \text{WN}_{\mathcal{P}}(u)[M_{\mathbb{R}} - \{u\}],$$

and $\text{WN}_{\mathcal{P}}(u)$ is called the *winding number* of the multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$ around u .

Remark 4.6. (a) If u is an element in one of the unbounded regions of $M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$, then $\psi_*([\Delta])$ is homologous to zero. Thus the winding number $\text{WN}_{\mathcal{P}}(u)$ is always equal to zero.

(b) $\text{WN}_{\mathcal{P}}(u)$ is a locally constant function on $M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$, since $[M_{\mathbb{R}} - \{u_0\}]$ is homologous to $[M_{\mathbb{R}} - \{u_1\}]$.

(c) $\text{WN}_{\mathcal{P}}(u)$ is independent of the choice of an orientation of V , since the reversing the orientation of V changes the fundamental classes $[\Delta]$ and $[M_{\mathbb{R}} - \{u\}]$ simultaneously by $-[\Delta]$ and $-[M_{\mathbb{R}} - \{u\}]$.

In fact, it turns out that the winding number $\text{WN}_{\mathcal{P}}(u)$ coincides with the Duistermaat-Heckman function $\text{DH}_{\mathcal{P}}(u)$ for each $u \in M_{\mathbb{R}} \setminus \cup_{i=1}^d F_i$, as follows ([5], Theorem 6.6).

Theorem 4.7. *For any multi-polytope \mathcal{P} , we have $\text{DH}_{\mathcal{P}} = \text{WN}_{\mathcal{P}}$.*

Chapter 5

Main Results: Generalized Ehrhart Polynomials

The aim of this chapter is to give the proofs of our main results stated in Chapter 1.

5.1 Ehrhart polynomials

To do so, we first recall the basic notations, and collect some elementary results regarding the Ehrhart polynomials of the simple lattice multi-polytope.

Let P be a convex lattice polytope of dimension n in V^* , where $V = N_{\mathbb{R}}$. Here what we mean by a lattice polytope is that each vertex of P lies in the lattice $N^* = \text{Hom}(N, \mathbb{Z})$ of $V^* = \text{Hom}(V, \mathbb{R})$. For a positive integer ν , as before let

$$\nu P = \{\nu u \mid u \in P\}.$$

Then it is again a convex lattice polytope in V^* . We denote by $\#(\nu P)$ (resp. $\#(\nu P^\circ)$) the number of lattice points in νP (resp. in the interior of νP). The lattice N^* determines a volume element on V^* by requiring that the volume of the unit cube determined by a basis of N^* is 1. Thus the volume of P , denoted

by $\text{vol}(P)$, is defined.

Theorem 5.1. *Let P be an n -dimensional convex lattice polyope. Then the following statements hold:*

- (1) $\#(\nu P)$ and $\#(\nu P^\circ)$ are polynomials in ν of degree n .
- (2) $\#(\nu P^\circ) = (-1)^n \#(-\nu P)$, where $\#(-\nu P)$ denotes the polynomial $\#(\nu P)$ with ν replaced by $-\nu$.
- (3) The coefficient of ν^n in $\#(\nu P)$ is $\text{vol}(P)$ and the constant term in $\#(\nu P)$ is 1.

The fan Δ associated with P may not be simplicial, but if we subdivide Δ , then we can always take a simplicial fan that is compatible with P . In this chapter, following the paper [5] of Hattori and Masuda we show that the theorem above holds for a simple lattice multi-polytope $\mathcal{P} = (\Delta, \mathcal{F})$.

To do so, we first need to define $\#(\mathcal{P})$ and $\#(\mathcal{P}^\circ)$, and this can be done, as follows. Let v_i ($i = 1, \dots, d$) be a primitive integral vector in the half line $C(\{i\})$. In our convention, v_i is chosen “outward normal” to the face $\mathcal{F}(\{i\})$ when \mathcal{P} arises from a convex polytope. We slightly move $\mathcal{F}(\{i\})$ in the direction of v_i (resp. $-v_i$) for each i , so that we obtain a map \mathcal{F}_+ (resp. \mathcal{F}_-) : $\Sigma^{(1)} \rightarrow HP(V^*)$. We denote the multi-polytopes (Δ, \mathcal{F}_+) and (Δ, \mathcal{F}_-) by \mathcal{P}_+ and \mathcal{P}_- , respectively. Since the affine hyperplanes $\mathcal{F}_\pm(\{i\})$ ’s miss the lattice N^* , the functions $\text{DH}_{\mathcal{P}_\pm}$ and $\text{WN}_{\mathcal{P}_\pm}$ are well defined on N^* .

Definition 5.2. We define

$$\begin{aligned}\#(\mathcal{P}) &:= \sum_{u \in N^*} \text{DH}_{\mathcal{P}_+}(u) = \sum_{u \in N^*} \text{WN}_{\mathcal{P}_+}(u), \\ \#(\mathcal{P}^\circ) &:= \sum_{u \in N^*} \text{DH}_{\mathcal{P}_-}(u) = \sum_{u \in N^*} \text{WN}_{\mathcal{P}_-}(u).\end{aligned}$$

When \mathcal{P} arises from a convex polytope P , $\text{DH}_{\mathcal{P}_+} = \text{WN}_{\mathcal{P}_+}$ (resp. $\text{DH}_{\mathcal{P}_-} = \text{WN}_{\mathcal{P}_-}$) takes 1 on u in P (resp. in the interior of P) and 0, otherwise. Therefore, it is straightforward to see that $\#(\mathcal{P})$ (resp. $\#(\mathcal{P}^\circ)$) agrees with the number of lattice points in P (resp. in the interior of P) in this case.

Next let us denote the volume element on V^* by dV^* , and define the volume $\text{vol}(\mathcal{P})$ of \mathcal{P} by

$$\text{vol}(\mathcal{P}) := \int_{V^*} \text{DH}_{\mathcal{P}} dV^* = \int_{V^*} \text{WN}_{\mathcal{P}} dV^*.$$

When \mathcal{P} arises from a convex polytope P , $\text{vol}(\mathcal{P})$ agrees with the actual volume of P , but otherwise it can be zero or negative. For a (not necessarily positive) integer ν , we denote $(\Delta, \nu\mathcal{F})$ by $\nu\mathcal{P}$, where

$$(\nu\mathcal{F}(\{i\})) := \{u \in V^* \mid \langle u, u_i \rangle = \nu c_i\},$$

when $\mathcal{F}(\{i\}) = \{u \in V^* \mid \langle u, u_i \rangle = c_i\}$ for a constant c_i .

With these understood, we can state and prove the following theorem which is analogous to Theorem 5.1.

Theorem 5.3. *Let $\mathcal{P} = (\Delta, \mathcal{F})$ be a simple lattice multi-polytope of dimension n .*

- (1) $\#(\nu\mathcal{P})$ and $\#(\nu\mathcal{P}^\circ)$ are polynomials in ν of the degree (at most) n .
- (2) $\#(\nu\mathcal{P}^\circ) = (-1)^n \#(-\nu\mathcal{P})$ for any integer ν .
- (3) The coefficient of ν^n in $\#(\nu\mathcal{P})$ is $\text{vol}(\mathcal{P})$ and the constant term in $\#(\nu\mathcal{P})$ is $\deg(\Delta)$.

Since Theorem 5.3 plays an important role in this thesis, here we give its proof, relatively in detail. However, we also remark that this proof is essentially due to that of [5], Theorem 7.2.

To prove the theorem, we first need some more notations and a lemma. Indeed, let $I \in \Sigma^{(n)}$. Although the integral vectors $\{v_i \mid i \in I\}$ are not necessarily a basis of the lattice N , they are linearly independent. Therefore, the sublattice N_I of N generated by v_i 's ($i \in I$) is of the same rank as N , hence N/N_I is a finite group. Note that N/N_I is trivial for any $I \in \Sigma^{(n)}$ if Δ is non-singular. For $u \in N_I^* = \text{Hom}(N_I, \mathbb{Z}) \supset N^*$ and $g \in N/N_I$, we then define

$$(5.1) \quad \chi_I(u, g) := \exp(2\pi\sqrt{-1}\langle u, v_g \rangle),$$

where $v_g \in N$ is a representative of g . The right-hand side does not depend on the choice of the representative v_g , and $\chi(u, \cdot)$ (resp. $\chi_I(\cdot, g)$) is a homomorphism from N/N_I (resp. N_I^*) to \mathbb{C}^* .

Note that $\chi_I(u, \cdot) : N/N_I \rightarrow \mathbb{C}^*$ is trivial if and only if $u \in N^*$. It follows that

$$(5.2) \quad \sum_{g \in N/N_I} \chi_I(u, g) = \begin{cases} |N/N_I| & \text{if } u \in N^*, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.4. *For each $I \in \Sigma^{(n)}$ let u_I be the corresponding vertex of \mathcal{P} and let $\{u_i^I \mid i \in I\}$ be the dual basis of $\{v_i \mid i \in I\}$. Then, for $v \in N$ such that $\langle u_i^I, v \rangle$ is a nonzero integer for any $I \in \Sigma^{(n)}$ and $i \in I$, we have*

$$\sum_{I \in \Sigma^{(n)}} \frac{w(I)z^{\langle v_I, v \rangle}}{|N/N_I|} \sum_{g \in N/N_I} \frac{1}{\prod_{i \in I} (1 - \chi_I(u_i^I, g)z^{-\langle u_i^I, v \rangle})} = \sum_{u \in N^*} \text{DH}_{\mathcal{P}_+}(u)z^{\langle u, v \rangle}$$

as functions of $z \in \mathbb{C}$.

Proof. Note first that the Maclaurin expansion of $1/(1 - az^{-m})$ ($a \in \mathbb{C}^*, m \in \mathbb{Z}$) is given by

$$\begin{cases} -a^{-1}z^m - a^{-2}z^{2m} - \dots, & \text{if } m > 0 \\ 1 + az^{-m} + a^2z^{-2m} + \dots, & \text{if } m < 0. \end{cases}$$

Taking this into account, we can expand the sum

$$S_I := \sum_{g \in N/N_I} \frac{1}{\prod_{i \in I} (1 - \chi_I(u_i^I, g) z^{-\langle u_i^I, v \rangle})}$$

into Maclaurin series, and then we obtain

$$\begin{aligned} S_I &= \sum_{g \in N/N_I} (-1)^I \prod_{i \in I} \sum_{\{b_i\}} (\chi_I(u_i^I, g)^{b_i \langle u_i^I, v \rangle}) \\ &= \sum_{g \in N/N_I} (-1)^I \sum_{\{b_i\}} \chi_I(-\sum_{i \in I} b_i u_i^I, g) z^{\langle \sum_{i \in I} b_i u_i^I, v \rangle}, \end{aligned}$$

where the summation \sum_{b_i} runs over the collection of such $\{b_i \mid i \in I, b_i \in \mathbb{Z}\}$ that

$$(5.3) \quad b_i \geq 1 \text{ for } i \text{ with } \langle u_i^I \rangle > 0 \text{ and } b_i \leq 0 \text{ for } i \text{ with } \langle u_i^I, v \rangle < 0.$$

Since

$$\sum_{g \in N/N_I} \chi_I(-\sum_{i \in I} b_i u_i^I, g) = \begin{cases} |N/N_I| & \text{if } \sum_{i \in I} b_i u_i^I \in N^* \\ 0 & \text{otherwise} \end{cases}$$

by (5.2), the Maclaurin expansion of the left-hand side of the equality in Lemma 5.4 has the form

$$\sum_{w \in N^*} \left(\sum_{I \in \Sigma(n)} (-1)^I w(I) \phi_I^i(u) \right) z^{\langle w, v \rangle},$$

where

$$\phi_I^i(u) = \begin{cases} 1 & \text{if } u = u_I \neq \sum_{i \in I} b_i u_i^I, b_i \text{ are as in (5.3) and } \sum_{i \in I} b_i u_i^I \in N^* \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can easily check that $\sum_{I \in \Sigma(n)} (-1)^I w(I) \phi_I^i(u)$ agrees with $\text{DH}_{\mathcal{P}_+}(u)$, which proves the lemma. \square

Proof of Theorem 5.3. To prove it, we shall prove (2) first. In fact, it suffices to prove

$$\#(\mathcal{P}^\circ) = (-1)^n \#(-\mathcal{P}).$$

Since $\#(\mathcal{P}^\circ) = \sum_{w \in N^*} \text{WN}_{\mathcal{P}_-}(u)$ by the definition of $\text{WN}_{\mathcal{P}_-}(u)$, it suffices to prove that

$$(5.4) \quad \text{WM}_{\mathcal{P}_-}(u) = (-1)^n \text{WN}_{(-\mathcal{P})_+}(u) \text{ for any } u \in N^*.$$

Let $\psi_{\mathcal{P}}$ and $\psi_{(-\mathcal{P})}$, be the maps introduced in Section 6 which are associated with multi-polytopes \mathcal{P}_- and $(-\mathcal{P})_+$ respectively. We note that $\psi_{\mathcal{P}}$ and $-\psi_{(-\mathcal{P})_+}$ considered as maps from S to $V^* \setminus \{u\}$ for $u \in N^*$ are homotopic. Since the multiplication by $-I$ on V^* sends the fundamental class $[V^* \setminus \{u\}]$ to $(-1)^n [V^* \setminus \{u\}]$, we obtain (5.4).

Next, we shall prove (1). Because of (2), it suffices to prove (1) for $\#(\nu\mathcal{P})$. We apply Lemma 7.3 of [5] (or Lemma 5.4) to $\nu\mathcal{P}$ in place of \mathcal{P} (so that u_I is replaced by νu_I), and approach z to 1 in equality. Since the right-hand side approaches $\#(\nu\mathcal{P})$, it suffices to show that the left-hand side approaches a polynomial in ν of degree at most n . When $g \in N/N_I$ is the identity element, $\chi_I(u_i^I, g) = 1$. Therefore, the term in the summand $\sum_{g \in N/N_I}$, in the left-hand side has a pole at $z = 1$ of degree exactly n when g is the identity element, and of degree at most n otherwise. Thus the left-hand side of the equality in Lemma 7.3 of [5] applied to $\nu\mathcal{P}$ can be written as

$$\frac{\sum_{I \in \Sigma(n)} z^{\nu(u_I, v)} h_I(z)}{(1-z)^n f(z)},$$

where $h_I(z)$ and $f(z)$ are polynomials in z and $f(1) \neq 0$. Then the repeated

use of L'Hospital's Theorem implies that when z approached 1, the limit of the above rational function is a polynomial in ν of degree at most n .

Finally, we give a proof of (3). Since

$$\#(\nu\mathcal{P}) = \sum_{u \in H^2(BT)} \text{DH}_{(\nu\mathcal{P})_+}(u) = \sum_{u \in H^2(BT)/\nu} \text{DH}_{\mathcal{P}_+}(u),$$

it follows from the definition of definite integral that

$$\lim_{\nu \rightarrow \infty} \frac{1}{\nu^n} \#(\nu\mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{\nu^n} \sum_{u \in H^2(BT)/\nu} \text{DH}_{\mathcal{P}_+}(u) = \int_{V^*} \text{DH}_{\mathcal{P}} dV^* = \text{vol}(\mathcal{P}),$$

proving that the coefficient of ν^n in $\#(\nu\mathcal{P})$ is $\text{vol}(\mathcal{P})$.

In order to deal with the coefficient of $\#(\nu\mathcal{P})$, we apply Lemma 7.3 of [5] to $0\mathcal{P}$, that is, $\nu\mathcal{P}$ with $\nu = 0$. Then the u_I in the lemma is zero, and $\text{DH}_{(0\mathcal{P})_+}(u) = \text{WN}_{(0\mathcal{P})_+} = 0$ unless $u = 0$ because the origin is the only vertex of $0\mathcal{P}$ so that the vertices of $(0\mathcal{P})_+$ are very close to the origin. Thus the right-hand side of the equality in the lemma applied to $0\mathcal{P}$ is a constant, say c , which is nothing but the constant term in $\#(\nu\mathcal{P})$. Now we approach z to ∞ . Then the equality reduces to

$$\sum_{u \in CI} w(I) = c,$$

because $\langle u_i^I, v \rangle > 0$ for all $i \in I$ if and only if $\nu = \sum_{i \in I} a_i v_i$ with $a_i > 0$ for all $i \in I$, and the latter is equivalent to saying that v belongs to the cone $C(I)$ spanned by v_i 's ($i \in I$). Since $\sum_{u \in C(I)} w(I) = \deg(\Delta)$ by definition, the constant term in $\#(\nu\mathcal{P})$, that is c , agrees with $\deg(\Delta)$.

□

5.2 Generalized Pick's formula for multi-polytopes

In this section, we will provide the proofs of our main results. First, we state and prove the generalized Pick's theorem for simple lattice multi-polytopes, as follows.

Theorem 5.5. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 2. Then the following identity holds.*

$$\text{vol}(\mathcal{P}) = \#(\mathcal{P}^\circ) + \frac{1}{2}\#(\partial\mathcal{P}) - \deg(\Delta).$$

Proof. In order to prove the theorem, we set

$$(5.5) \quad \#(\nu\mathcal{P}) = a_2\nu^2 + a_1\nu + a_0.$$

Then clearly we have

$$(5.6) \quad \#(-\nu\mathcal{P}) = a_2\nu^2 - a_1\nu + a_0.$$

If we add two equations (5.5) and (5.6), we can easily obtain

$$\#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}) = 2a_1\nu.$$

Thus we have

$$(5.7) \quad a_1 = \frac{1}{2}\#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}).$$

Since $\#(-\nu\mathcal{P}) = \#(\nu\mathcal{P}^\circ)$, it follows from (5.7) that we have

$$a_1 = \frac{1}{2}(\#(\nu\mathcal{P}) - \#(\nu\mathcal{P}^\circ)) = \frac{1}{2}\#(\partial\mathcal{P}).$$

Therefore, we have

$$(5.8) \quad \#(\nu\mathcal{P}) = \text{vol}(\mathcal{P})\nu^2 + \frac{1}{2}\#(\partial\mathcal{P})\nu + \deg(\Delta).$$

Finally, if we set $\nu = 1$ in the equation (5.8), we obtain

$$(5.9) \quad \#(\mathcal{P}) = \text{vol}(\mathcal{P}) + \frac{1}{2}\#(\partial\mathcal{P}) + \deg(\Delta).$$

Since

$$\#(\mathcal{P}) = \#(\mathcal{P}^\circ) + \#(\partial\mathcal{P}),$$

it follows from (5.9) that we can also obtain

$$\text{vol}(\mathcal{P}) = \frac{1}{2}\#(\partial\mathcal{P}) + \#(\mathcal{P}^\circ) - \deg(\Delta).$$

This proves Theorem 5.5. □

Next, we deal with the case that \mathcal{P} is a simple lattice multi-polytope of dimension 3.

Theorem 5.6. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 3. Then the Ehrhart polynomial of $\#(\nu\mathcal{P})$ is given by*

$$\begin{aligned} \#(\nu\mathcal{P}) &= \text{vol}(\mathcal{P})\nu^3 + \left(\frac{1}{2}\#(\partial\nu\mathcal{P}) - \deg(\Delta) \right) \nu^2 \\ &\quad + (\#(\mathcal{P}) - \text{vol}(\mathcal{P}) - \frac{1}{2}\#(\partial\mathcal{P}))\nu + \deg(\Delta). \end{aligned}$$

Proof. To prove it, let

$$(5.10) \quad \#(\nu\mathcal{P}) = a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0.$$

Then, clearly we have

$$(5.11) \quad \#(-\nu\mathcal{P}) = -a_3\nu^3 + a_2\nu^2 - a_1\nu + a_0.$$

Thus, by adding two equations (5.10) and (5.11), it is easy to obtain

$$(5.12) \quad \#(\nu\mathcal{P}) + \#(-\nu\mathcal{P}) = 2a_2\nu^2 + 2a_0.$$

Since $\#(-\nu\mathcal{P}) = -\#(\nu\mathcal{P}^\circ)$, it follows from (5.12) that we have

$$\#(\nu\mathcal{P}) - \#(\nu\mathcal{P}^\circ) = 2a_2\nu^2 + 2\deg(\Delta).$$

This implies that

$$a_2\nu^2 = \frac{1}{2}\#(\partial\nu\mathcal{P}) - \deg(\Delta).$$

So, by putting $\nu = 1$ into the previous equation, we have

$$a_2 = \frac{1}{2}\#(\partial\mathcal{P}) - \deg(\Delta).$$

On the other hand, by subtracting the equation (5.11) from (5.10), we see that

$$\#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}) = 2a_3\nu^3 + 2a_1\nu.$$

Since $a_3 = \text{vol}(\mathcal{P})$ and $a_0 = \deg(\Delta)$ by Theorem 5.3, it is easy to obtain

$$\begin{aligned} \#(\nu\mathcal{P}) &= \text{vol}(\mathcal{P})\nu^3 + \left(\frac{1}{2}\#(\partial\nu\mathcal{P}) - \deg(\Delta)\right)\nu^2 \\ &\quad + \left(\#(\mathcal{P}) - \text{vol}(\mathcal{P}) - \frac{1}{2}\#(\partial\mathcal{P})\right)\nu + \deg(\Delta), \end{aligned}$$

as required. \square

Finally, by using a similar method as in Theorems 5.5 and 5.6, we can deal with the case of the dimension 4, as follows.

Theorem 5.7. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 4, and let us write the Ehrhart polynomial of $\#(\nu\mathcal{P})$ as follows.*

$$(5.13) \quad \#(\nu\mathcal{P}) = a_4\nu^4 + a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0.$$

Then the following relationship between the coefficients holds:

$$\begin{aligned} a_4 &= \text{vol}(\mathcal{P}), \quad a_0 = \deg(\Delta), \\ a_1 + a_3 &= \frac{1}{2}\#(\partial\mathcal{P}). \end{aligned}$$

Proof. As before, since

$$(5.14) \quad \#(\nu\mathcal{P}) = a_4\nu^4 + a_3\nu^3 + a_2\nu^2 + a_1\nu + a_0,$$

it is easy to obtain

$$\#(\nu\mathcal{P}) + \#(-\nu\mathcal{P}) = 2a_4\nu^4 + 2a_2\nu^2 + 2a_0.$$

Thus, we have

$$2\#(\nu\mathcal{P}) = 2\text{vol}(\mathcal{P})\nu^4 + 2a_2\nu^2 + 2\deg(\Delta).$$

That is, we have

$$\#(\nu\mathcal{P}) = \text{vol}(\mathcal{P})\nu^4 + a_2\nu^2 + \deg(\Delta).$$

On the other hand, by subtracting (5.14) from (5.13), we get

$$\#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}) = 2a_3\nu^3 + 2a_1\nu.$$

Since in this case $\#(-\nu\mathcal{P}) = \#(-\nu\mathcal{P}^\circ)$, we can also obtain

$$\begin{aligned} \#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}) &= \#(\nu\mathcal{P}) - \#(-\nu\mathcal{P}^\circ) \\ &= \#(\partial\nu\mathcal{P}) = 2a_3\nu^3 + 2a_1\nu. \end{aligned}$$

By putting $\nu = 1$ into the previous equation, we obtain

$$\#(\partial\mathcal{P}) = 2a_3 + 2a_1.$$

In other words, we can obtain

$$a_1 + a_3 = \frac{1}{2}\#(\partial\mathcal{P}),$$

as required. □

5.3 Simple multi-polygons

In this section, we review some well-know facts regarding simple multi-polygons necessary for the proof of the generalized twelve-point theorem given in Section 5.4. Refer to the paper [6] of for more details.

Roughly speaking, a *lattice multi-polygon* is a piecewise linear loop with vertices in \mathbb{Z}^2 together with a sign function which assigns either $+$ or $-$ to each side and satisfies some mild condition. The piecewise linear loop may have a self-intersection and we think of it as a sequence of points in \mathbb{Z}^2 . A lattice polygon can naturally be regarded as a lattice multi-polygon. The generalized Pick's formula holds for lattice multi-polynomial of a lattice multi-polygon is of degree at most two. The constant term is the rotation number of normal vectors to sides of the multi-polygon and not necessarily 1 unlike ordinary Ehrhart polynomials. The other coefficients have similar geometrical meaning to the ordinary ones but they can be zero or negative unlike the ordinary ones. The family of lattice multi-polygons has some natural subfamilies, e.g., the Ehrhart polynomials of not only all lattice multi-polygons but also some natural subfamilies.

We say that a sequence of vectors v_1, \dots, v_d in $\mathbb{Z}^2 (d \geq 2)$ is *unimodular* if each triangle with vertices O, v_i and v_{i+1} contains no lattice point except the vertices, where $O = (0, 0)$ and $v_{d+1} = v_1$. The vectors in the sequence are not necessarily counterclockwise or clockwise. Then any vectors can go back and forth.

We set

$$(5.15) \quad \epsilon_i = \det(v_i, v_{i+1}) \text{ for } i = 1, \dots, d.$$

In other words, $\epsilon_i = 1$ if the rotation from v_i to v_{i+1} (with angle less than π) is

counterclockwise and $\epsilon_i = -1$ otherwise. Since each successive pair (v_j, v_{j+1}) is a basis of \mathbb{Z}^2 for $j = 1, \dots, d$ for $i = 1, \dots, d$, one has

$$(v_i, v_{i+1}) = (v_{i-1}, v_i) \begin{pmatrix} 0 & -\epsilon_{i-1}\epsilon_i \\ 1 & -\epsilon_i a_i \end{pmatrix}$$

with a unique integer a_i for each i . This is equivalent to

$$(5.16) \quad \epsilon_{i-1}v_{i-1} + \epsilon_i v_{i+1} + a_i v_i = 0$$

Note that $|a_i|$ is twice the area of the triangle with vertices O , v_{i-1} and v_{i+1} .

Lemma 5.8. *For a simple lattice multi-polygon $\mathcal{P} = (v_1, \dots, v_d)$, we set*

$$w_i = \frac{v_i - v_{i-1}}{\det(v_{i-1}, v_i)}$$

for $i = 1, \dots, d$, where $v_0 = v_d$. Then w_i is integral and primitive.

With the help of Lemma 5.8, we define the dual multi-polygon \mathcal{P}^\vee of \mathcal{P} by

$$\mathcal{P}^\vee = (w_1, \dots, w_d).$$

Next, we give an example to show how to find the dual multi-polygon of a given simple lattice multi-polygon.

Example 5.9. Let \mathcal{P} be a simple lattice multi-polygon (or multi-polytope) given by

$$\mathcal{P} = ((1, 1), (-1, 1), (-1, -1), (1, -1)).$$

Then it is easy to calculate the following

$$\begin{aligned} w_1 &= \frac{v_1 - v_0}{\det(v_0, v_1)} = (0, 1), \quad w_2 = \frac{v_2 - v_1}{\det(v_1, v_2)} = (-1, 0), \\ w_3 &= \frac{v_3 - v_2}{\det(v_2, v_3)} = (0, -1), \quad w_4 = \frac{v_4 - v_3}{\det(v_3, v_4)} = (1, 0). \end{aligned}$$

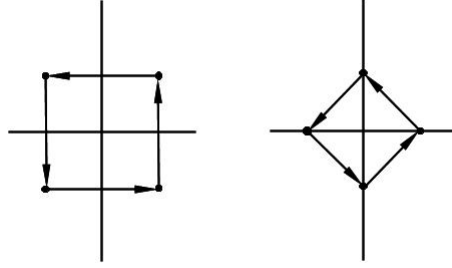


Figure 5.1: \mathcal{P} and \mathcal{P}^\vee .

(see Figure 5.1). Thus, we can obtain

$$\mathcal{P}^\vee = ((0, 1), (-1, 0), (0, -1), (1, 0)).$$

Note that the following identity hold:

$$\#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) = 8 + 4 = 12,$$

$$2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)) = 2(4 + 2) = 12,$$

$$\#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) = 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)).$$

Example 5.10. This time, let \mathcal{P} be a simple lattice multi-polygon (or multi-polytope) given by

$$\mathcal{P} = ((-1, -2), (0, 1), (1, -2), (0, -1))$$

with

$$v_1 = (-1, -2), \quad v_2 = (0, 1), \quad v_3 = (1, -2), \quad v_4 = (0, -1).$$

Then it follows from Lemma 5.8 that we have

$$\begin{aligned} w_1 &= \frac{v_1 - v_0}{\det(v_0, v_1)} = (1, 1), & w_2 &= \frac{v_2 - v_1}{\det(v_1, v_2)} = (-1, -3), \\ w_3 &= \frac{v_3 - v_2}{\det(v_2, v_3)} = (-1, 3), & w_4 &= \frac{v_4 - v_3}{\det(v_3, v_4)} = (1, -1). \end{aligned}$$

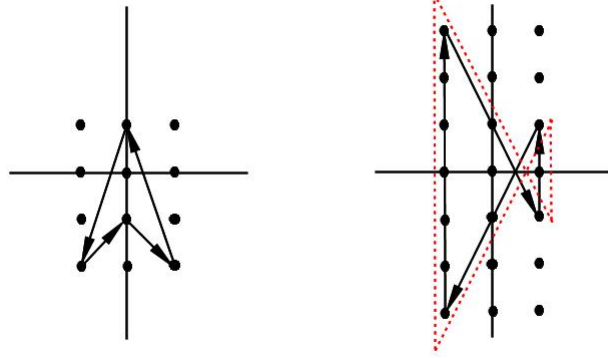


Figure 5.2: \mathcal{P} , \mathcal{P}^\vee , and $(\mathcal{P}^\vee)_+$ (red dotted multi-polygon).

That is, the dual simple lattice multi-polygon \mathcal{P}^\vee is given by

$$\mathcal{P}^\vee = ((1, 1), (-1, -3), (-1, 3), (1, -1)).$$

Now, by using the simple multi-polygon $(\mathcal{P}^\vee)_+$ together with the weight function w equal to the value 1 (see Figure 5.2), it is easy to calculate the following:

$$\#(\partial\mathcal{P}) = 4, \quad \#(\partial\mathcal{P}^\vee) = 10,$$

$$\text{vol}(\mathcal{P}) = 2, \quad \text{vol}(\mathcal{P}^\vee) = 5.$$

Therefore, once again we see that the following identity holds:

$$\begin{aligned} \#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) &= 14 = 2(2 + 5) \\ &= 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)). \end{aligned}$$

The following formula for the rotation number is known from the paper [6].

Theorem 5.11. *The rotation number of a unimodular sequence v_i, \dots, v_d ($d \geq 2$) around the origin is given by*

$$\frac{1}{12} \left(\sum_{i=1}^d a_i + 3 \sum_{i=1}^d \epsilon_i \right),$$

where ϵ_i and a_i are the integers defined in (5.15) and (5.16).

If \mathcal{P} is a convex lattice polygon whose only interior lattice point is the origin and v_i, \dots, v_d are the vertices of \mathcal{P} arranged counterclockwise, then every v_i is primitive and the triangle with the vertices O, v_i and v_{i+1} has no lattice point in the interior for each i , where $v_{d+1} = v_1$ as usual. This observation motivates the following definition.

Definition 5.12. A sequence of vectors $\mathcal{P} = (v_1, \dots, v_d)$, where v_1, \dots, v_d are in \mathbb{Z}^2 and $d \geq 2$, is called a *legal loop* if every v_i is primitive and whenever $v_i \neq v_{i+1}$, v_i and v_{i+1} are linearly independent (i.e. $v_i \neq -v_{i+1}$) and the triangle with the vertices O, v_i and v_{i+1} has no lattice point in the interior. We say that a legal loop is *reduced* if $v_i \neq v_{i+1}$ for any i . A (non-reduced) legal loop \mathcal{P} naturally determines a reduced legal loop, denoted \mathcal{P}_{red} , by dropping all the redundant points. We define the winding number of a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ to be the rotation number of the vectors v_1, \dots, v_d around the origin.

Joining successive points in a legal loop $\mathcal{P} = (v_1, \dots, v_d)$ by straight lines forms a lattice polygon which may have a self-intersection. A unimodular sequence v_1, \dots, v_d determines a reduced legal loop. Conversely, a reduced legal loop $\mathcal{P} = (v_1, \dots, v_d)$ determines a unimodular sequence by adding all the lattice points on the line segment $v_i v_{i+1}$ (called a side of \mathcal{P}) connecting v_i and v_{i+1} for every i . To each side $v_i v_{i+1}$ with $v_i \neq v_{i+1}$, we assign the sign of $\det(v_i, v_{i+1})$, denoted $\text{sgn}(v_i, v_{i+1})$.

Definition 5.13. Let $|v_i v_{i+1}|$ be the number of lattice points on the side $v_i v_{i+1}$ minus 1, so $|v_i v_{i+1}| = 0$ when $v_i = v_{i+1}$. Then we define

$$B(\mathcal{P}) = \sum_{i=1}^d \text{sgn}(v_i, v_{i+1}) |v_i v_{i+1}|$$

Clearly, $B(\mathcal{P}) = B(\mathcal{P}_{red})$.

Theorem 5.14 (Generalized twelve-point theorem). *Let \mathcal{P} be a legal loop and let r be the winding number of \mathcal{P} . then $B(\mathcal{P}) + B(\mathcal{P}^v) = 12r$.*

Let M be a unitary toric manifold of real dimension 4 whose simple multi-polytope Δ is given by a simple lattice multi-polygon \mathcal{P} with v_1, \dots, v_d . Then it turns out that the number of singed boundary lattice points of \mathcal{P} is equal to $\#(\partial\mathcal{P})$ that is defined as

$$\#(\partial\mathcal{P}) := \#(\mathcal{P}) - \#(\mathcal{P}^\circ)$$

in Section 5.2, as follows.

Theorem 5.15. *Let M be a unitary toric manifold of real dimension 4 whose simple multi-polytope Δ is given by a simple lattice multi-polygon \mathcal{P} . Then we have*

$$B(\mathcal{P}) = \#(\partial\mathcal{P}).$$

Proof. In order to prove it, we first need to recall some well-known facts. Indeed, let $\mathcal{T}d(\Delta) = 1 + \mathcal{T}d_1(\Delta) + \mathcal{T}d_2(\Delta) + \dots$ denote the Todd class of Δ . Then, it follows from [5], Theorem 8.5 that we have

$$\#(\mathcal{P}) = \int_{\Delta} e^{c_1(\mathcal{P})} \mathcal{T}d(\Delta)$$

Here, $c_1(\mathcal{P}) = \sum_{i=1}^d c_i \bar{x}_i$, where \bar{x}_i denotes the image of $x_i \in H_T^*(\Delta)$ in $H^*(\Delta)$ (refer to [5], Section 8 for more details), and \mathcal{P} is a simple lattice multi-polytope.

Thus, when $\dim \Delta = 2$, we have

$$\begin{aligned} \#(\mathcal{P}) &= \int_{\Delta} (1 + c_1(\mathcal{P}) + \frac{1}{2!} c_1(\mathcal{P})^2 + \dots) \cup (1 + \mathcal{T}d_1(\Delta) + \mathcal{T}d_2(\Delta) + \dots) \\ &= \frac{1}{2} \int_{\Delta} c_1(\mathcal{P})^2 + \int_{\Delta} c_1(\mathcal{P}) \cup \mathcal{T}d_1(\Delta) + \int_{\Delta} \mathcal{T}d_2(\Delta). \end{aligned}$$

We then claim that

$$\int_{\Delta} \mathcal{T}d_2(\Delta) = \deg(\Delta).$$

To see it, recall that

$$\#(\nu\mathcal{P}) = \int_{\Delta} e^{c_1(\nu\mathcal{P})} \mathcal{T}d(\Delta), \quad \nu \in \mathbb{Z}_{\geq 0}.$$

Thus, by putting $\nu = 0$, we get

$$\begin{aligned} \deg(\Delta) &= \#(0\mathcal{P}) = \int_{\Delta} \mathcal{T}d(\Delta), \\ &= \int_{\Delta} \mathcal{T}d_2(\Delta), \quad (\dim \Delta = 2). \end{aligned}$$

Moreover, it is known from [5], Lemma 8.6 that

$$\frac{1}{2} \int_{\Delta} c_1(\mathcal{P})^2 = \text{vol}(\mathcal{P}).$$

Thus, we have

$$\#(\mathcal{P}) = \text{vol}(\mathcal{P}) + \int_{\Delta} c_1(\mathcal{P}) \cup \mathcal{T}d_1(\Delta) + \deg(\Delta).$$

But, we have already known that

$$\#(\mathcal{P}) = \text{vol}(\mathcal{P}) + \frac{1}{2} \#(\partial\mathcal{P}) + \deg(\Delta).$$

Thus, we have

$$\#(\mathcal{P}) = 2 \int_{\Delta} c_1(\mathcal{P}) \cup \mathcal{T}d_1(\Delta).$$

Finally, for a simple lattice multi-polygon \mathcal{P} which is associated with a unitary toric manifold M , note that

$$c_1(\mathcal{P}) = c_1(L) \quad \text{and} \quad \mathcal{T}d_1(\Delta) = \frac{1}{2}c_1(M).$$

Here, $c_1(L) = \sum_{i=1}^d c_i \bar{x}_i$. Hence,

$$\#(\partial \mathcal{P}) = \int_M c_1(L) \cup c_1(M).$$

However, it is also known from [5], p. 263 that

$$\int_M c_1(L) \cup c_1(M) = B(\mathcal{P}).$$

Therefore, we have

$$\#(\partial \mathcal{P}) = B(\mathcal{P}),$$

as desired. □

5.4 Generalized twelve-point theorem

In this section, we give some interesting applications of results obtained in the previous section.

To do so, throughout this section let us assume that the dimension of a simple lattice polytope is equal to 2. As before, let \mathcal{P} be a simple lattice multi-polytope which is a lattice multi-polygon and whose only interior lattice point is the origin. Let \mathcal{P}^\vee be the dual of \mathcal{P} . Then, \mathcal{P}^\vee is also a simple lattice multi-polytope whose only interior lattice point is the origin. So, $\#(\mathcal{P}^\circ) = \#((\mathcal{P}^\vee)^\circ) = 1$. Applying the generalized Pick's formula to \mathcal{P} , we get

$$(5.17) \quad \begin{aligned} \#(\mathcal{P}) - \frac{1}{2}\#(\partial \mathcal{P}) &= \text{vol}(\mathcal{P}) + \deg(\Delta), \\ \#(\mathcal{P}^\vee) - \frac{1}{2}\#(\partial \mathcal{P}^\vee) &= \text{vol}(\mathcal{P}^\vee) + \deg(\Delta). \end{aligned}$$

By adding the two equations of (5.17), we get

$$\begin{aligned} \#(\mathcal{P}) + \#(\mathcal{P}^\vee) - \frac{1}{2}(\#(\partial \mathcal{P}) + \#(\partial \mathcal{P}^\vee)) \\ = \text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee) + \deg(\Delta) + \deg(\Delta^\vee). \end{aligned}$$

Thus, since $\#(\mathcal{P}^\circ) = \#((\mathcal{P}^\vee)^\circ) = 1$, some simple computations give the following generalized twelve-point theorem

$$\begin{aligned} \#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) \\ = 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)) + 2(\deg(\Delta) + \deg(\Delta^\vee)) - 4. \end{aligned}$$

Theorem 5.16. *Let \mathcal{P} be a simple lattice multi-polytope of dimension 2 which is a lattice multi-polygon and whose interior lattice point is the origin. Then the following identity holds:*

$$\#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) = 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)) + 2(\deg(\Delta) + \deg(\Delta^\vee)) - 4.$$

As an immediate corollary, we can state the following

Corollary 5.17. *Let \mathcal{P} be a simple lattice polytope of dimension 2 whose interior lattice point is the origin. Then the following identity holds:*

$$\#(\partial\mathcal{P}) + \#(\partial\mathcal{P}^\vee) = 2(\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee)) = 12.$$

Proof. It immediately follows from Theorem 5.11, since we have

$$\deg(\Delta) = \deg(\Delta^\vee) = 1.$$

□

As a consequence, we also have the following interesting relationship between the volumes of \mathcal{P} and \mathcal{P}^\vee :

$$\text{vol}(\mathcal{P}) + \text{vol}(\mathcal{P}^\vee) = 6.$$

Bibliography

- [1] M. Brion and M. Vergne, *Lattice points in simple polytopes*, J. Amer. Math. Soc. **10** (1997), 371–392.
- [2] M. Brion and M. Vergne, *An equivariant Riemann-Roch theorem for complete, simplicial toric varieties*, J. reine angew. Math. **482** (1997), 67–92.
- [3] W. Fulton, *Introduction to toric varieties*, Ann. Math. Studies, **131**, Princeton Univ., 1993.
- [4] M. Grossberg and Y. Karshon, *Equivariant index and the moment map for completely integrable torus actions*, Adv. in Math. **133** (1998), 185–223.
- [5] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. **40** (2003), 1–68.
- [6] A. Higashitani and M. Masuda, *Lattice multi-polygons*, preprint (2012); arXiv:1204.0088.
- [7] Y. Karshon and S. Tolman, *The moment map and line bundles over presymplectic toric manifolds*, J. Diff. Geom. **38** (1993), 465–484.

- [8] A. G. Khovanskii and A. V. Pukhlikov, *A Riemann-Roch theorem for integrals and sums of quasipolynomials over virtual polytopes*, St. Petersburg Math. J. **4** (1993), 789–812.
- [9] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. **51** (1999), 237–265.
- [10] T. Oda, *Convex bodies and algebraic geometry*, Springer-Verlag, 1988.