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The Existence of Warping Functions on Riemannian Warped Product Manifolds

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리만 휜곱다양체 위의 휜함수의 존재성

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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다.

연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

Kazdan and Warner ([10], [11], [12])의 결과에 의하면 N 위의 함수 f 가 N 위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있는데 먼저

(A) N 위의 함수 f 가 Riemannian metric의 scalar curvature이면 그 함수 f 가 적당한 점에서 $f(x_0) < 0$ 일 때이다. 즉, N 위에 negative constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(B) N 위의 함수 f 가 Riemannian metric의 scalar curvature이면 그 함수 f 가 항등적으로 $f \equiv 0$ 이거나 모든 점에서 $f(x) < 0$ 인 경우이다. 즉, N 위에서 zero scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(C) N 위의 어떤 f 라도 positive constant curvature를 갖는 Riemannian metric 이 존재하는 경우이다.

본 논문에서는 엽다양체 N 이 (A)에 속하는 compact Riemannian manifold일 때, Riemannian warped product manifold인 $M=[a,\infty)\times_f N$ 위에 함수 $R(t,x)$ 가 어떤 조건을 만족하면 $R(t,x)$ 가 Riemannian warped product metric의 scalar curvature가 될 수 있는 warping function $f(x)$ 가 존재할 수 있음을 상해·하해 방법을 이용하여 증명하였다.

1. INTRODUCTION

One of the basic problems in the differential geometry is to study the set of curvature functions over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwise conformal to some metric with constant scalar curvature or not.

In a recent study ([9]), Jung and Kim have studied the problem of scalar curvature functions on Lorentzian warped product manifolds and obtaind partial results about the existence and nonexistence of Lorentzian warped metric with some prescribed scalar curvature function.

In this paper, we study also the existence and nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds.

By the results of Kazdan and Warner ([10], [11], [12]), if N is a compact Riemannian $n-$manifold without boundary $n \geq 3$, then N belongs to one of the following three categories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [10], [11] and [12], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

In [13] and [14], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric.

In this paper, when N is a compact Riemannian manifold, we consider the existence of warping functions on a warped product manifold $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. That is, it is shown that if the fiber manifold N belongs to class (A) then M admits a Riemannian metric with some negative scalar curvature outside a compact set.

2. PRELIMINARIES

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. *Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\Im(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(D1) $\nabla_V W$ *is* $\Im$ *-linear in* V,

(D2) $\nabla_V W$ *is* $\mathbb{R}$*-linear in* W,

(D3) $\nabla_V(fW) = (Vf)W + f\nabla_v W \quad$ *for* $f \in \Im(M)$,

(D4) $[V, W] = \nabla_V W - \nabla_W V$, *and*

(D5) $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$ *for all* $X, V, W \in \mathfrak{X}(M)$.

If ∇ satisfies axioms (D1)$\sim$(D3), then $\nabla_V W$ is called the *covariant derivative* of W with respect to V for the connection ∇. If ∇ satisfies axioms (D4)$\sim$ (D5), then ∇ is called the *Levi - Civita connection* of M, which is characterized by the *Koszul formula* ([15]).

A geodesic $c : (a, b) \to M$ is a smooth curve of M such that the tangent vector $c^{'}$ moves by parallel translation along c. In order words, c is a geodesic if

$$\nabla_{c'} c^{'} = 0 \quad \text{(geodesic equation)}.$$

A pregeodesic is a smooth curve c which may be reparametrized to be a geodesic. Any parameter for which c is a geodesic is called an affine parameter. If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c'} c^{'} = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U, (x^1, x^2, ..., x^n))$ be local coordinates on M and let $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ denote the natural basis with respect to these coordinates.

The connection coefficients Γ^k_{ij} of ∇ with respect to $(x^1, x^2, ..., x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \text{(connection coefficients)}.$$

Using these coefficients, we may write equation as the system

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \quad \text{(geodesic equations in coordinates)}.$$

Definition 2.2. *The curvature tensor of the connection ∇ is a linear transformation valued tensor R in $Hom(\mathfrak{X}(M), \mathfrak{X}(M))$ defined by :*

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathfrak{X}(M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well-known that $R(X,Y)Z$ at p depends only upon the values of X, Y and Z at p ([15]).

If $w \in T_p^*(M)$ is a cotangent vector at p and $x, y, z \in T_p(M)$ are tangent vectors at p, then one defines

$$R(\omega, X, Y, Z) = (\omega, R(X,Y)Z) = \omega(R(X,Y)Z)$$

for X, Y and Z smooth vector fields extending x, y and z, respectively.

The curvature tensor R is a (1,3)-tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj} \Gamma^i_{ka} - \Gamma^a_{kj} \Gamma^i_{ma}).$$

Notice that $R(X,Y)Z = -R(Y,X)Z$, $R(\omega, X, Y, Z) = -R(\omega, Y, X, Z)$ and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \sum x^i \frac{\partial}{\partial x^i}$, $Y = \sum y^i \frac{\partial}{\partial x^i}$, $Z = \sum z^i \frac{\partial}{\partial x^i}$ and $\omega = \sum \omega_i dx^i$ then

$$R(X,Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} z^j x^k y^m \frac{\partial}{\partial x^i}$$

and

$$R(w, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} w_i z^j x^k y^m.$$

Consequently, one has $R\left(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}\right) = R^i_{jkm}$.

Definition 2.3. *From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its components are $R_{ij} = \sum_{k=1}^{n} R^k_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{ij=1}^{n} R_{ij} g^{ij}$ is called the scalar curvature ([1], [2], [3]).*

Definition 2.4. *Suppose Ω is a smooth, bounded domain in R^n, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation*

$$\Delta u = g(x,u) \quad in \quad \Omega$$

$$u = u_0 \quad on \quad \partial\Omega$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $u \le u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega g(x,u)\varphi dx \le 0 \quad \textit{for all} \quad \varphi \in C_0^\infty(\Omega), \quad \varphi \ge 0.$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold.

We briefly recall some results on warped product manifolds. Complete details may be found in ([2]) or ([15]). On a semi-Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.5. *The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^* g_F$$

where g_B and g_F are metric tensors of B and F, respectively. In order words, if v is tangent to M at (p,q), then

$$g(v,v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber ([15]).

We denote the metric g by $\langle \quad , \quad \rangle$. In view of Remark 2.13 (1) and Lemma 2.14, we may also denote the metric g_B by $\langle \quad , \quad \rangle$. The metric g_F will be denoted by $\quad (\quad , \quad)$.

Remark 2.6. *Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows :*

(1) For each $q \in F$, the map $\pi \mid_{\sigma^{-1}(q) = B \times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma \mid_{\pi^{-1}(p) = p \times F}$ is a positive homothetic map onto F with homothetic factor $\frac{1}{f(p)}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_p B \times T_q F.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p, q) by setting $\overline{X}(p, q) = (X_p, 0_q)$. Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F. Similarly, if Y is a vector field of F, $\overline{Y}$ is defined by $\overline{Y}(p, q) = (0_p, Y_q)$.

Lemma 2.7. *If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B.*

Proof. We must show that $grad(h \circ \pi)$ is horizonal and π-related to $grad(h)$ on B. If v is vertical tangent vector to M, then

$$\langle grad(h \circ \pi), v\rangle = v(h \circ \pi) = d\pi(v)h = 0, \quad \text{since} \quad d\pi(v) = 0.$$

Thus $grad(h \circ \pi)$ is horizonal. If x is horizonal,

$$\langle d\pi(grad(h \circ \pi), d\pi(x)\rangle = \langle grad(h \circ \pi), x\rangle = x(h \circ \pi) = d\pi(x)h$$

$$= \langle grad(h), d\pi(x)\rangle.$$

Hence at each point, $d\pi(grad(h \circ \pi)) = grad(h)$. ■

In view of Lemma 2.14, we simplify the notations by writing h for $h \circ \pi$ and $grad(h)$ for $grad(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a $(1, s)$-tensor, and if $v_1, v_2, ..., v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, ..., v_s) = A(d\pi(v_1), ..., d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in ([4]).

Now we recall the formula for the Ricci curvature tensor Ric on the warped product manifold $M = B \times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.8. *On a warped product manifold $M = B \times_f F$ with $n = dimF > 1$, let X, Y be horizontal and V, W vertical.*

Then

(1) $Ric(X, Y) = Ric^B(X, Y) - \frac{n}{f} H^f(X, Y)$,

(2) $Ric(X, V) = 0$,

(3) $Ric(V, W) = Ric^F(V, W) - \langle V, W \rangle f^\sharp$,

where $f^\sharp = \dfrac{\Delta f}{f} + (n-1)\dfrac{\langle grad(f), grad(f)\rangle}{f^2}$ *and* $\Delta f = trace(H^f)$ *is the Laplacian on* B.

Proof. See Corollary 7.43 in ([15]). ■

On the given warped product manifold $M = B \times_f F$, we also write S^B for the pullback by π of the scalar curvature S_B of B and similarly for S^F. From now on, we denote $grad(f)$ by ∇f.

Lemma 2.9. *If S is the scalar curvature of $M = B \times_f F$ with $n = dimF > 1$, then*

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2} \tag{2.1}$$

where Δ is the Laplacian on B.

Proof. For each $(p, q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)}M$. We can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 = \langle \overline{d_j}, \overline{d_j} \rangle = f(p)^2(d_j, d_j) = (f(p)d_j, f(p)d_j)$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF. By Lemma 2.8 (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i}),$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(p) g_F(d_j, d_j) \left(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2} \right).$$

Hence, for $\epsilon_\alpha = g(\epsilon_\alpha, \epsilon_\alpha)$

$$\begin{aligned}
S(p,q) &= \sum_\alpha \epsilon_\alpha R_{\alpha\alpha} \\
&= \sum_i \epsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_j \epsilon_j Ric(\overline{d_j}, \overline{d_j}) \\
&= S^B(p,q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}
\end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$.

■

3. Main Results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to \mathbb{R}^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, \tilde{g})$ with

$$\tilde{g} = dt^2 + f^2(t)g. \tag{3.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of $\tilde{g}$ is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}[R(g)(x) - 2nf(t)f^{''}(t) - n(n-1)|f^{'}(t)|^2] \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$ (For details, [5] or [7]).

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u^{''}(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary.

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^\alpha$ in (3.3), where $\alpha > 1$ is a constant. We have

$$R(t,x) \leq \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > \alpha.$$

Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [7], we have the following theorem.

Theorem 3.1. *For $n \geq 3$, let $M = [a,\infty) \times_f N$ be the Riemannian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a Riemannian metric of negative scalar curvature outside a compact set.*

Proposition 3.2. *Suppose that* $R(g) = -\frac{4n}{n+1}k^2$ *and* $R(t,x) = R(t) \in C^\infty([a,\infty))$. *Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (3.3) such that for $t > t_0, 0 < u_-(t) \leq u(t) \leq u_+(t)$.*

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty)$, $\tilde{u}_-(t) \le \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a, \infty))$, there exists a positive constant b such that $|R(t)| \le \frac{4n}{n+1} b^2$ for $t \in [a, t_0]$. Since

$$\begin{aligned}
&\frac{4n}{n+1} u_+''(t) + R(t) u_+(t) + \frac{4n}{n+1} k^2 u_+(t)^{1-\frac{4}{n+1}} \\
&\le \frac{4n}{n+1} (u_+''(t) + b^2 u_+(t) + k^2 u_+(t)^{1-\frac{4}{n+1}}),
\end{aligned}$$

if we divide the given interval $[a, t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_{i+}(t)$ by parallel transporting $\cos Bt$ such that $0 < \frac{1}{\sqrt{2}} \le u_{i+}(t) \le 1$. That is to say, for each interval I_i,

$$\begin{aligned}
&\frac{4n}{n+1} u_{i+}''(t) + R(t) u_{i+}(t) + \frac{4n}{n+1} k^2 u_{i+}(t)^{1-\frac{4}{n+1}} \\
&\le \frac{4n}{n+1} (u_{i+}''(t) + b^2 u_{i+}(t) + k^2 u_{i+}(t)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1} (-B^2 \cos Bt + b^2 \cos Bt + k^2 (\cos Bt)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1} \cos Bt (-B^2 + b^2 + k^2 (\cos Bt)^{-\frac{4}{n+1}}) \\
&\le \frac{4n}{n+1} \cos Bt (-B^2 + b^2 + k^2 2^{\frac{2}{n+1}}) \\
&\le 0
\end{aligned}$$

for large B, which means that $u_{i+}(t)$ is an (weak) upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_{i+}(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $\frac{1}{\sqrt{2}} \le \tilde{u}_+(t) \le 1$ for all $t \in [a, t_0]$.

Put $\tilde{u}_-(t) = \frac{1}{\sqrt{2}}e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$,

$$\begin{aligned}
&\frac{4n}{n+1}u''_{i-}(t) + R(t)u_{i-}(t) + \frac{4n}{n+1}k^2 u_{i-}(t)^{1-\frac{4}{n+1}} \\
&\geq \frac{4n}{n+1}(u''_{i-}(t) - b^2 u_{i-}(t)) \\
&= \frac{4n}{n+1}\frac{1}{\sqrt{2}}e^{-\alpha t}(\alpha^2 - b^2) \\
&\geq 0
\end{aligned}$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$, $0 < \tilde{u}_-(t) \leq \tilde{u}_+(t)$. ■

In [5], the authors consider the nonexistence of warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ when N belongs to class (A) with $R(g) = -\frac{4n}{n+1}k^2$.

Proposition 3.3. *Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n(\geq 3)$. We assume that $R(g) = -\frac{4n}{n+1}k^2$, where k is a positive constant. On $M = [a, \infty) \times_f N$, there does not exist a Riemannian warped product metric*

$$\tilde{g} = dt^2 + f^2(t)g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

Proof. In [5], p.179, Theorem 2.5 ■

However, in this paper, when N is a compact Riemannian manifold of class (A), we consider the existence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures. In particular, if $R(t, x)$ is also the function of only t-variable, then we have the following theorems.

Theorem 3.4. *Suppose that $R(g) = -\frac{4n}{n+1}k^2$. Assume that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a negative function such that*

$$-bt^s \leq R(t) \leq -\frac{4n}{n+1}\frac{C}{t^\alpha}, \quad for \quad t \geq t_0,$$

where $t_0 > a, \alpha < 2, C$ and b are positive constants, and s is a positive integer. Then equation (3.3) has a positive solution on $[a, \infty)$.

Proof. We let $u_+(t) = t^m$, where m is some positive number. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m[\frac{m(m-1)}{t^2} + \frac{k^2}{t^{\frac{4}{n+1}m}} - \frac{C}{t^\alpha}] \\
&\leq 0, \quad t \geq t_0,
\end{aligned}$$

for some large t_0, which is possible for large fixed m since $\alpha < 2$. Hence, $u_+(t)$ is an upper solution. Now put $u_-(t) = e^{-\beta t}$, where β is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - bt^s u_-(t) \\
&= e^{-\beta t}[\frac{4n}{n+1}\beta^2 + \frac{4n}{n+1}k^2 e^{\beta t[\frac{4}{n+1}]} - bt^s] \\
&\geq 0, \quad t \geq t_0
\end{aligned}$$

for some large t_0, which means that $u_-(t)$ is a lower solution. And we can take β so large that $0 < u_-(t) < u_+(t)$. So by Theorem 3.2, we obtain a positive solution. ■

The above theorem implies that if $R(t)$ is not rapidly decreasing and less than some negative function, then equation (3.3) has a positive solution.

Theorem 3.5. *Suppose that* $R(g) = -\frac{4n}{n+1}k^2$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a negative function such that*

$$-bt^s \le R(t) \le -\frac{C}{t^2}, \quad for \quad t \ge t_0$$

where $t_0 > a, b$ *and* C *are positive constants, and* s *is a positive integer. If* $C > n(n-1)$, *then equation (3.3) has a positive solution on* $[a,\infty)$.

Proof. In case that $C > n(n-1)$, we may take $u_+(t) = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\le C_+\frac{4n}{n+1}t^{\frac{n-3}{2}}[\frac{n^2-1}{4} + k^2 C_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C] \\
&\le 0,
\end{aligned}$$

which is possible if we take C_+ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C < 0$. Thus $u_+(t)$ is an upper solution. And we take $u_-(t)$ as in Theorem 3.4. In this case, we also obtain a positive solution. ∎

Remark 3.6. *The results in Theorem 3.4, and Theorem 3.5 are almost sharp since we can get as close to* $-\frac{n(n-1)}{t^2}$ *as possible. For example, let* $R(g) = -\frac{4n}{n+1}k^2$ *and* $f(t) = t\ln t$ *for* $t > a$. *Then we have*

$$R = -\frac{1}{t^2}[\frac{4n}{n+1}\frac{k^2}{(\ln t)^2} + \frac{2n}{\ln t} + n(n-1)(1+\frac{1}{\ln t})^2],$$

which converges to $-\frac{n(n-1)}{t^2}$ *as* t *goes to* ∞.

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저작물 이용 허락서

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논문제목	한글 : 리만 휜곱 다양체 위의 휜함수의 존재성 영어 : The existence of warping functions on Riemannian warped product manifolds				

본인이 저작한 위의 저작물에 대하여 다음과 같은 조건아래 조선대학교가 저작물을 이용할 수 있도록 허락하고 동의합니다.

\- 다 음 -

1. 저작물의 DB구축 및 인터넷을 포함한 정보통신망에의 공개를 위한 저작물의 복제, 기억장치에의 저장, 전송 등을 허락함
2. 위의 목적을 위하여 필요한 범위 내에서의 편집・형식상의 변경을 허락함. 다만, 저작물의 내용변경은 금지함.
3. 배포・전송된 저작물의 영리적 목적을 위한 복제, 저장, 전송 등은 금지함.
4. 저작물에 대한 이용기간은 5년으로 하고, 기간종료 3개월 이내에 별도의 의사 표시가 없을 경우에는 저작물의 이용기간을 계속 연장함.
5. 해당 저작물의 저작권을 타인에게 양도하거나 또는 출판을 허락을 하였을 경우에는 1개월 이내에 대학에 이를 통보함.
6. 조선대학교는 저작물의 이용허락 이후 해당 저작물로 인하여 발생하는 타인에 의한 권리 침해에 대하여 일체의 법적 책임을 지지 않음
7. 소속대학의 협정기관에 저작물의 제공 및 인터넷 등 정보통신망을 이용한 저작물의 전송・출력을 허락함.

동의여부 : 동의(O) 반대( )

2013 년 12 월 20 일

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