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The Nonexistence of Conformal Deformations on Riemannian Warped Product Manifolds

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리만 휜곱다양체 위의 등각변형의 비존재성

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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률함수에 관한 연구이다. 특히, 어떤 함수가 주어진 미분다양체의 곡률함수가 되는 문제는 해석적인 방법을 적용하여 주어진 다양체 위에서의 편미분방정식의 해의 존재성의 문제로 바꿀 수 있다.

Kazdan and Warner([13], [14], [15])의 결과에 의하면 N 위의 함수 f 가 N 위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있다.

(A) N 위의 함수 f 가 Riemannian metric의 scalar curvature이면 그 함수 f 가 적당한 점에서 $f(x_0) < 0$ 일 때이다. 즉 N 위에 negative constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(B) N 위의 함수 f 가 Riemannian metric의 scalar curvature이면 그 함수 f 가 항등적으로 $f \equiv 0$ 이거나 모든 점에서 $f(x) < 0$ 인 경우이다. 즉, N 위에서 zero scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(C) N 위의 어떤 f 라도 positive constant scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

본 논문에서는 엽다양체 N이 (C)에 속하는 compact Riemannian manifold 일 때, Riemannian warped product manifold인 $M=(a,\infty)\times_f N$ 위에 함수 $R(t,x)$가 Riemannian warped product metric의 scalar curvature가 되도록 하는 conformal deformation이 존재할 수 없음을 증명하였다.

1. INTRODUCTION

One of the basic problems in the differential geometry is to study the set of curvature function over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwisely conformal to some metric with constant scalar curvature or not.

In a recent study([10]), Jung and Kim have studied the problem of scalar curvature functions on Lorentzian warped product manifolds and obtained partial results about the existence and nonexistence of Lorentzian warped metric with some prescribed scalar curvature function. In this paper, we study also the existence and nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds.

By the results of Kazdan and Warner([13], [14], [15]), if N is a compact Riemannian $n-$ manifold without boundary $n \geq 3$, then N belongs to one of the following three catagories :

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some

Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [13], [14] and [15] Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson([9]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded([9], [16, p.322]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature([6]). It follows from the results of Aviles and McOwen([1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In [16] and [17], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric.

In this paper, when N is a compact Riemannian manifold, we consider

the nonexistence of warping functions on a warped product manifold $M=(a,\infty)\times_f N$ with specific scalar curvatures, where is a positive constant. That is, it is shown that if the fiber manifold N belongs to class (C) then M does not admit a Riemannian metric with some positive scalar curvature near the end outside a compact set.

2. PRELIMINARIES

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\mathfrak{I}(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(D1) $\nabla_V W$ is $\mathfrak{I}$ -linear in V,

(D2) $\nabla_V W$ is R -linear in W,

(D3) $\nabla_V (fW) = (Vf)W + f\nabla_V W$ for $f \in \mathfrak{I}(M)$,

(D4) $[V, W] = \nabla_V W - \nabla_W V$, and

(D5) $X < V, W > = < \nabla_X V, W > + < V, \nabla_X W >$ for all $X, V, W \in \mathfrak{X}(M)$.

If ∇ satisfies axioms (D1)~(D3), then $\nabla_V W$ is called the covariant derivative of W with respect to V for the connection ∇. If ∇ satisfies

axioms (D4)~ (D5), then ∇ is called the $Levi-Civita\,connection$ of M, which is characterized by the Koszul formula([17]).

A geodesic $c:(a,b)\to M$ is a smooth curve of M such that the tangent vector c' moves by parallel translation along c. In order words, c is a geodesic if

$$\nabla_{c'}c' = 0 \quad \text{(geodesic equation).}$$

A pregeodesic is a smooth curve c which may be reparametrized to be a geodesic. Any parameter for which c is a geodesic is called an affine parameter. If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the para metrization on is all of $\mathbb{R}$.

The equation $\nabla_{c'}c' = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U,(x^1, x^2, \cdots, x^n))$ be local coordinates on M and let $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}$ denote the natural basis with respect to these coordinates.

The connection coefficients Γ_{ij}^k of ∇ with respect to $(x^1, x^2, \cdots, x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{(connection coefficients).}$$

Using these coefficients, we may write the geodesic equation as the

following system

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \qquad \text{(geodesic equations in coordinates).}$$

Definition 2.2. The curvature tensor of the connection ∇ is a linear transformation valued tensor R in Hom $(\mathcal{H}(M), \mathcal{H}(M))$ defined by :

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Thus, for $Z \in \mathcal{H}(M)$,

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z .$$

It is well-known that $R(X, Y)Z$ at p depends only upon the values of X , Y and Z at p ([21]).

If $w \in T_P^*(M)$ is a cotangent vector at p and $x, y, z \in T_P(M)$ are tangent vectors at p, then one defines

$$R(w, X, Y, Z) = (w, R(X, Y) Z) = w(R(X, Y) Z)$$

for X , Y and Z smooth vector fields extending x, y and z, respectively.

The curvature tensor R is a $(1,3)$ - tensor field which is given in local coordinates by

$$R = \sum_{i, j, k, m=1}^{n} R_{jkm}^{i} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m ,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

Notice that

$$R(X, Y)Z = -R(Y, X)Z, \; R(w, X, Y, Z) = -R(w, Y, X, Z)$$

and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \sum x^i \frac{\partial}{\partial x^i}$, $Y = \sum y^i \frac{\partial}{\partial x^i}$, $Z = \sum z^i \frac{\partial}{\partial x^i}$ and $w = \sum w_i dx^i$,

then we have

$$R(X, Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} z^j x^k y^m \frac{\partial}{\partial x^i}$$

and

$$R(w, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} w_i z^j x^k y^m.$$

Consequently, one has $R(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}) = R^i_{jkm}$.

Definition 2.3. From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its

components are $R_{ij} = \sum_{k=1}^{n} R^{k}_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{ij=1}^{n} R_{ij} g^{ij}$ is called the scalar curvature ([2], [3], [4]).

Definition 2.4. Suppose Ω is a smooth, bounded domain in R^n, and let $g : \Omega \times R \to R$ be a Caratheodory function.
Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation

$$\triangle u = g(x, u) \quad \text{in} \quad \Omega$$

$$u = u_0 \quad \text{on} \quad \partial\Omega$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $u \leq u_0$ on $\partial\Omega$ and

$$\int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} g(x,u)\varphi \, dx \leq 0 \qquad \text{for all } \varphi \in C_0^{\infty}(\Omega),\ \varphi \geq 0 .$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold.

We briefly recall some results on warped product manifolds. Complete details may be found in [3] or [21]. On a semi-Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.5. The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^* g_F$$

where g_B and g_F are metric tensors of B and F, respectively. In order words, if v is tangent to M at (p, q), then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber([21]).

We denote the metric g by $< \ , \ >$. In view of Remark 2.6. (1) and Lemma 2.7., we may also denote the metric g_B by $< \ , \ >$. The metric g_F will be denoted by $(\ , \ )$.

Remark 2.6. Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows :

(1) For each $q \in F$, the map $\pi|_{\sigma^{n-1}(q) = B \times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(p) = p \times F}$ is a positive homothetic map onto F with homothetic factor $\dfrac{1}{f(p)}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_p B \times T_q F.$$

The decomposition of vectors into horizontal and vertical parts play a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p, q) by setting $\overline{X}(p, q) = (X_p, O_q)$. Then $\overline{X}$ and $\sigma-$related to the zero vector field on F. Similarly, if Y is a vector field of F, $\overline{Y}$ is defined by $\overline{Y}(p, q) = (O_p, Y_q)$.

Lemma 2.7. If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M gradient of h on B.

Proof. We must show that $\mathrm{grad}(h \circ \pi)$ is horizontal and $\pi-$related to $\mathrm{grad}(h)$ on B. If v is vertical tangent vector to M, then

$$\langle grad(h \circ \pi), v \rangle = v(h \circ \pi) = d\pi(v)h = 0,$$

since $d\pi(v) = 0$. Thus grad $(h \circ \pi)$ is horizontal. If x is horizontal,

$$\begin{aligned}\langle d\pi(grad(h \circ \pi), d\pi(x) \rangle &= \langle grad(h \circ \pi), x = x(h \circ \pi) = d\pi(x)h \\ &= \langle grad(h), d\pi(x) \rangle.\end{aligned}$$

Hence at each point, $d\pi(grad(h \circ \pi)) = grad(h)$. □

In view of Lemma 2.7, we simplify the notations by writing h for $h \circ \pi$ and $grad(h)$ for $grad(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a $(1, s)-$ tensor, and if $v_1, v_2, \cdots, v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, \cdots, v_s) = A(d\pi(v_1), \cdots, d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, than $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in [5]. □

Now we recall the formula for the Ricci curvature tensor Ric on the warped product manifold $M = B \times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.8. On a warped product manifold $M = B \times_f F$ with $n = \dim F > 1$, let X, Y be horizontal and V, W vertical.

Then

(1) $Ric(X, Y) = Ric^B(X, Y) - \frac{n}{f} H^f(X, Y)$,

(2) $Ric(X, V) = 0$,

(3) $Ric(V, W) = Ric^F(V, W) - \langle V, W \rangle f^{\#}$

where $f^{\#} = \frac{\triangle f}{f} + (n-1)\frac{\langle grad(f), grad(f) \rangle}{f^2}$ and $\triangle f = trace\ (H^f)$ is the Laplacian on B.

Proof. See Corollary 7. 43. in [17]. □

On the given warped product manifold $M = B \times_f F$, we also write S^B for the pullback by π of the scalar curvature S_B of B and similarly for S^F. From now on, we denote $grad(f)$ by ∇f.

Lemma 2.9. If S is the scalar curvature of $M = B \times_f F$ with $n = \dim F > 1$, then

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\triangle f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2} \tag{2.1}$$

where $\triangle$ is the Laplacian on B.

Proof. For each $(p, q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for $T_p B$. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)} M$. We can choose $\{d_j\}$ on $T_q F$ such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)} M$. Then

$$1 = \langle \overline{d_j}, \overline{d_j} \rangle = f(p)^2 (d_j, d_j) = (f(p) d_j, f(p) d_j)$$

which implies that $\{f(p) d_j\}$ forms an orthonormal basis for $T_q F$. By Lemma 2.8. (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i}),$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(p)_{gF}(d_j, d_j)\left(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}\right).$$

Hence, for $\epsilon_\alpha = g(\epsilon_\alpha, \epsilon_\alpha)$

$$\begin{aligned} S(p, q) &= \sum_\alpha \epsilon_\alpha R_{\alpha\alpha} \\ &= \sum_i^\alpha \epsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_j \epsilon_j Ric(\overline{d_j}, \overline{d_j}) \\ &= S^B(p, q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2} \end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$. □

3. Main Results

M.C. Leung ([16], [17], [18]), has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of positive solution to the equation

$$\triangle_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}}, \tag{3.1}$$

where $\triangle_{g_0}$ is the Laplacian operator for an $n-$dimensional Riemannian manifold (N, g_0) and $d_n = \dfrac{n-2}{4(n-1)}$. Equation (3.1) is derived from the conformal deformation of Riemannian metric ([1], [12], [13], [17], [18]).

Similarly, let (N, g_0) be a compact Riemannian $n-$dimensional manifold. We consider the $(n+1)-$dimensional Riemannian warped product manifold $M = (a, \infty) \times_f N$ with the metric $g = dt^2 + f(t)^2 g_0$, where f is a positive function on (a, ∞). Let $u(t, x)$ be a positive smooth function on M and let g have a scalar curvature equal to $r(t, x)$. If the conformal metric $g_c = u(t, x)^{\frac{4}{n-1}} g$ has a prescribed function $R(t, x)$ as a scalar curvature, then it is well known that $u(t, x)$ satisfies equation

$$(3.2) \qquad \frac{4n}{n-1}\Box_g u(t,x) - r(t,x)u(t,x) + R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0,$$

where $\Box_g$ is the d'Alembertian for a Riemannian warped manifold $M = (a, \infty) \times_f N$.

In this paper, we study the nonexistence of a positive solution to equation (3.2). This paper contains the results of Riemannian version of [20].

First of all, in order to prove the nonexistence of solutions of some partial differential equations, we need brief results about Young's inequality. The following proposition is well known(Theorem 1 in [20], p.48).

Proposition 3.1. Let f be a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then

$$\int_0^a f(t)\,dt + \int_0^b f^{-1}(t)\,dt \geq ab,$$

where f^{-1} is the inverse function of f. Equality holds if and only if $b = f(a)$.

Corollary 3.2. Let $a, b \geq 0$ and $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $\epsilon > 0$,

$$\epsilon \frac{1}{p} a^p + \frac{1}{\epsilon^{\frac{1}{p-1}}} \frac{1}{q} b^q \geq ab .$$

Proof. In Proposition 1, we choose $f(t) = \epsilon t^{p-1}$ for $p > 1$.

From now on, we let (N, g_0) be a compact Riemannian $n-$dimensional manifold with $n \geq 3$ and without boundary. The following proposition is also well known(Theorem 5.4, [5]).

Proposition 3.3. Let $M = (a, \infty) \times_f N$ have a Riemannian warped product metric $g = dt^2 + f(t)^2 g_0$. Then the Laplacian $\Box_g$ is given by

$$\Box_g = \frac{\partial^2}{\partial t^2} + \frac{nf'(t)}{f(t)} \frac{\partial}{\partial t} + \frac{1}{f(t)^2} \triangle_x ,$$

where $\triangle_x$ is the Laplacian on fiber manifold N.

By Proposition 3.3, equation (3.2) is changed into the following equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)} \frac{\partial u(t,x)}{\partial t} + \frac{1}{f(t)^2} \triangle_x u(t,x) - \frac{n-1}{4n} r(t,x) u(t,x) + \frac{n-1}{4n} R(t,x) u(t,x)^{\frac{n+3}{n-1}} = 0 . \tag{3.3}$$

If $u(t,x) = u(t)$ is a positive function with one variable t and if $R(t,x) = R(t)$ and $r(t,x) = r(t)$ are also functions of one variable t, then equation (3.3) becomes

$$u''(t) + \frac{nf'(t)}{f(t)} u'(t) = h(t)\,u(t) - H(t)\,u(t)^{\frac{n+3}{n+1}}, \tag{3.4}$$

where $h(t) = \frac{n-1}{4n} r(t)$ and $H(t) = \frac{n-1}{4n} R(t)$.

The proof of the following theorem is similar to that of Theorem 2 in [12].

Theorem 3.4. Let $u(t)$ be a positive solution of equation (3.4). And let $h(t)$, $H(t)$ satisfy the following condition : $h(t) \geq 0$ and $H(t) \leq -c_1$, where c_1 is positive constant. Assume that there exist positive constants t_0 and C_0 such that $\left| \frac{f'(t)}{f(t)} \right| \leq C_0$ for all $t > t_0$. Then $u(t)$ is bounded from above.

Proof. From equation (3.4) we have

$$\frac{(f^n u')'}{f^n} = h(t)\,u - H(t)\,u^{\frac{n+3}{n-1}}. \tag{3.5}$$

Let $\chi \in C_0^\infty((a,\infty))$ be a cut-off function. Multiplying both sides of equation (3.5) by $\chi^{n+1} u$ and then using integration by parts we obtain

$$(3.6)\quad -\int_a^\infty (f^n u')\left(\frac{\chi^{n+1}u}{f^n}\right)' dt = \int_a^\infty h(t)\chi^{n+1}u^2 dt - \int_a^\infty H(t)\chi^{n+1}u^{\frac{2n+2}{n-1}} dt.$$

Since $h(t) \geq 0$ and $H(t) \leq -c_1$, where c_1 is a positive constant, equation (3.6) implies

$$-\int_a^\infty (f^n u')\left(\frac{\chi^{n+1}u}{f^n}\right)' dt \geq c_1 \int_a^\infty \chi^{n+1} u^{\frac{2n+2}{n-1}} dt.$$

From the left side of the above equation, we have

$$-(f^n u')\left(\frac{\chi^{n+1}u}{f^n}\right)' = -(n+1)\chi^n u\chi' u' - \chi^{n+1}|u'|^2 + n\chi^{n+1}uu'\frac{f'}{f}.$$

Applying the Cauchy inequality, we get

$$\begin{aligned} -(n+1)\chi^n u\chi' u' &= -2\left((n+1)\chi^{\frac{n+1}{2}-1}u\chi'\right)\left(\frac{1}{2}\chi^{\frac{n+1}{2}}u'\right) \\ &\leq (n+1)^2\chi^{n-1}u^2|\chi'|^2 + \frac{1}{4}\chi^{n+1}|u'|^2 \end{aligned}$$

and

$$\begin{aligned} n\chi^{n+1}uu'\frac{f'}{f} &= 2\left(n\chi^{\frac{n+1}{2}}u\frac{f'}{f}\right)\left(\frac{1}{2}\chi^{\frac{n+1}{2}}u'\right) \\ &\leq n^2\chi^{n+1}u^2\left(\frac{f'}{f}\right)^2 + \frac{1}{4}\chi^{n+1}|u'|^2. \end{aligned}$$

Together with the above equations, we obtain

$$n^2\int_a^\infty (\frac{f'}{f})^2\chi^{n+1}u^2dt + (n+1)^2\int_a^\infty \chi^{n-1}u^2|\chi'|^2dt$$

$$\geq c_1\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt + \frac{1}{2}\int_a^\infty \chi^{n+1}|u'|^2dt.$$

Applying Corollary 2 and using the bound $\left|\frac{f'}{f}\right| \leq C_0$, we have

$$\int_a^\infty \chi^{n+1}u^2dt = \int_a^\infty \chi^2\chi^{n-1}u^2dt$$

$$\leq \epsilon\frac{2}{n+1}\int_a^\infty \chi^{n+1}dt + \frac{1}{\epsilon^{\frac{2}{n-1}}}\frac{n-1}{n+1}\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt$$

and

$$\int_a^\infty \chi^{n-1}u^2|\chi'|^2dt = \int_a^\infty |\chi'|^2\chi^{n-1}u^2dt$$

$$\leq \epsilon\frac{2}{n+1}\int_a^\infty |\chi'|^{n+1}dt + \frac{1}{\epsilon^{\frac{2}{n-1}}}\frac{n-1}{n+1}\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt.$$

For large $\epsilon > 0$, we obtain

$$C'\int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}dt + \frac{1}{2}\int_a^\infty \chi^{n+1}|u'|^2dt \tag{3.7}$$

$$\leq C''\int_a^\infty (|\chi'|^{n+1} + \chi^{n+1})dt,$$

where C', C'' are positive constants. Let $\chi \equiv 0$ on $(a, r] \cup [r+3, \infty]$ with $r > t_0$ and $\chi \equiv 1$ on $[r+1, r+2]$, $\chi \geq 0$ on $[a, \infty]$ and $|\chi'| \leq 1$. From equation (3.7) we have

$$C' \int_{r+1}^{r+2} u^{\frac{2n+2}{n-1}} dt + \frac{1}{2} \int_{r+1}^{r+2} |u'|^2 dt \leq C'''$$

for all $r > t_0$, where C''' is a constant independent on r.

Therefore u is bounded from above. □

Theorem 3.5. Let (M, g) be a Riemannian manifold with scalar curvature equal to $h(t)$. Assume that there exist positive constants t_0 and C_0 such that $\left|\frac{f'(t)}{f(t)}\right| \leq C_0$ for all $t > t_0$. For a smooth function $H(t)$, let $h(t)$, $H(t)$ satisfy the following condition : $h(t) \geq 0$, $H(t) \leq -c_1$, where c_1 is positive constant. Then equation (3.4) has no positive solution.

Proof. If $u = u(t)$ is a positive solution of equation (3.4), then by Theorem 4 $u(t)$ is bounded from above on (a, ∞). Then, by Omori-Yau maximum principle(c.f. [22]), there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} u(t_k) = |u'(t_k)| \leq \frac{1}{k}$ and $u''(t_k) \leq \frac{1}{k}$. Since $_{t \in (a, \infty)} u(t) = c_2 > 0$, there exist a number $\epsilon > 0$ and K such that

$$h(t_k)u(t_k) - H(t_k)u(t_k)^{\frac{n+3}{n-1}} > \epsilon$$

for all $k > K$, which is a contradiction to the fact that

$$u''(t_k) + \frac{nf'(t_k)}{f(t_k)}u'(t_k) \leq \frac{1 + nC_0}{k}$$

for all $k > K$. Therefore equation (3.4) has no positive solution. □

The following corollary is easily derived from the previous theorems.

Corollary 3.6. Let $(M, g) = ((a, \infty) \times_f N, g)$ be a Riemannian manifold with scalar curvature equal to $r(t)$. Assume that there exist positive constants t_0 and C_0 such that $\left|\frac{f'(t)}{f(t)}\right| \leq C_0$ for all $t > t_0$. For a smooth function $R(t)$, let $r(t)$, $R(t)$ satisfy the following condition: $r(t) \geq 0$, $R(t) \leq -c_1$ where c_1 is positive constant. Then there does not exist a conformal deformation on M with scalar curvature $R(t)$.

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저작물 이용 허락서

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논문제목	한글 : 리만 휜곱다양체 위의 등각변형의 비존재성 영어 : The nonexistence of conformal deformations on Riemannian warped product manifolds				

본인이 저작한 위의 저작물에 대하여 다음과 같은 조건아래 조선대학교가 저작물을 이용할 수 있도록 허락하고 동의합니다.

- 다 음 -

1. 저작물의 DB구축 및 인터넷을 포함한 정보통신망에의 공개를 위한 저작물의 복제, 기억장치에의 저장, 전송 등을 허락함
2. 위의 목적을 위하여 필요한 범위 내에서의 편집・형식상의 변경을 허락함. 다만, 저작물의 내용변경은 금지함.
3. 배포・전송된 저작물의 영리적 목적을 위한 복제, 저장, 전송 등은 금지함.
4. 저작물에 대한 이용기간은 5년으로 하고, 기간종료 3개월 이내에 별도의 의사표시가 없을 경우에는 저작물의 이용기간을 계속 연장함.
5. 해당 저작물의 저작권을 타인에게 양도하거나 또는 출판을 허락을 하였을 경우에는 1개월 이내에 대학에 이를 통보함.
6. 조선대학교는 저작물의 이용허락 이후 해당 저작물로 인하여 발생하는 타인에 의한 권리 침해에 대하여 일체의 법적 책임을 지지 않음
7. 소속대학의 협정기관에 저작물의 제공 및 인터넷 등 정보통신망을 이용한 저작물의 전송・출력을 허락함.

동의여부 : 동의(O) 반대( )

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