[UCI]I804:24011-200000263929

2013년 8월
교육학석사(수학교육전공)학위논문

Some Estimate for Bochner-Riesz type multipliers

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이 논문을 교육학석사(수학교육)학위 청구논문으로 제출함.

2013년 4월

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채원재의 교육학 석사학위 논문을 인준함.

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2013년 6월

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CONTENT

ABSTRACT

어떤 Bochner-Riesz 류의 승수계산

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우리는 이 논문에서 Bochner-Riesz type 승수 즉, 구와 콘 승수를 수반한 합성곱 연산자에 대하여 L^p유계성을 계측한다. 첫째로 구 승수를 수반한 합성곱 연산자 S^δ가 $\delta > \frac{n-1}{2}$이고 $1 \le p \le \infty$ 일 때, $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ 유계됨을 재 증명한다. 두 번째로 콘 승수를 수반한 합성곱 연산자 T^δ가 $\delta > \frac{n-1}{2}$이고 $1 < p < \infty$ 일 때, $L^p(\mathbb{R}^{n+1}) \to L^p(\mathbb{R}^{n+1})$ 유계됨을 증명한다.

1. Introduction

1.1 We define Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all indefinitely differentiable functions f on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n}\left|x^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}f(x)\right|<\infty \ ,$$

for every multi-index α and β. In other words, f and all its derivatives are required to be rapidly decreasing.

Example 1. An example of a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$ is the n-dimensional Gaussian given by $e^{-\pi|x|^2}$.

The Fourier transform of a Schwartz function f is defined by

(1.1) $$\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)e^{-2\pi ix\cdot\xi}\,dx,\quad \text{for } \xi\in\mathbb{R}^n.$$

Example 2. Let f and g be the functions defined by

$$f(x)=\chi_{[-1,1]}(x)=\begin{cases}1 & \text{if } |x|\le 1\\ 0 & \text{otherwise},\end{cases}\qquad \text{and}\qquad g(x)=\begin{cases}1-|x| & \text{if } |x|\le 1\\ 0 & \text{otherwise}.\end{cases}$$

Although f is not continuous, the integral defining its Fourier transform still makes sense.

Show that

$$\hat{f}(\xi)=\frac{\sin 2\pi\xi}{\pi\xi}\quad \text{and}\quad \hat{g}(\xi)=\left(\frac{\sin\pi\xi}{\pi\xi}\right)^2$$

with the understanding that $\hat{f}(0)=2$ and $\hat{g}(0)=1$.

We also define inverse Fourier transform by

(1.2) $$\check{f}(x)=\int_{\mathbb{R}^n}\hat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi.$$

Example 3. The inverse Fourier transform of $e^{-\pi|x|^2}$ in $\mathbb{R}^2$ is as follows :

$$\int_{\mathbb{R}^2} e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} dx = \int_{\mathbb{R}} e^{-\pi x_2^2} e^{-2\pi i x_2\cdot\xi_2}\left(\int_{\mathbb{R}} e^{-\pi x_1^2} e^{-2\pi i x_1\cdot\xi_1} dx_1\right) dx_2$$

$$= \int_{\mathbb{R}} e^{-\pi x_2^2} e^{-2\pi i x_2\cdot\xi_2} e^{-\pi\xi_1^2} dx_2$$

$$= e^{-\pi\xi_1^2} e^{-\pi\xi_2^2}$$

$$= e^{-\pi|\xi|^2}.$$

Let $n \geq 2$. For $f \in \varsigma(\mathbb{R}^n)$ we consider the convolution operators

$$\widehat{S^\delta} f(\xi) = \left(1 - |\xi'|^2\right)_+^\delta \hat{f}(\xi'), \quad \xi' \in \mathbb{R}^n .$$

and for $f \in \varsigma(\mathbb{R}^{n+1})$

$$\widehat{T^\delta} f(\xi) = \left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta \hat{f}(\xi', \xi_{n+1}), \quad \xi = (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R},$$

where $s_+^\delta = s^\delta$ for $s > 0$, and $s_+^\delta = 0$ otherwise.

In this thesis, we study L^p-boundedness for convolution operators S^δ and T^δ, when $1 < p < \infty$ and $\delta > \dfrac{n-1}{2}$ for $n \geq 2$.

The main results are as follows :

Theorem 1.1 Let $\delta > 0$. The convolution operator S^δ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and $\delta > \dfrac{n-1}{2}$.

Theorem 1.2 Let $\delta > 0$. The convolution operator T^δ is bounded from $L^p(\mathbb{R}^{n+1})$ to $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$ and $\delta > \dfrac{n-1}{2}$.

Remark . If $\delta \leq \dfrac{n-1}{2}$, then the convolution operator S^δ and T^δ is unbounded on $L^1(\mathbb{R}^n)$ and $L^1(\mathbb{R}^{n+1})$, respectively.

2. Preliminaries

In this section we study some properties of a Schwartz function, Fourier, inverse Fourier transform, and convolution (see [SS]).

Proposition 2.1 Let $f \in \varsigma(\mathbb{R}^n)$.

(1) $f(x+h) \to \hat{f}(\xi) e^{2\pi i \xi \cdot h}$ whenever $h \in \mathbb{R}^n$.

(2) $f(x) e^{-2\pi i x \cdot h} \to \hat{f}(\xi + h)$ whenever $h \in \mathbb{R}^n$.

(3) $f(\delta x) \to \delta^{-1} \hat{f}(\delta^{-1} \xi)$ whenever $\delta > 0$.

(4) $(\frac{\partial}{\partial x})^{\alpha} f(x) \to (2\pi i \xi)^{\alpha} \hat{f}(\xi)$.

(5) $(-2\pi i \xi)^{\alpha} f(x) \to (\frac{\partial}{\partial \xi})^{\alpha} \hat{f}(\xi)$.

(6) $f(Rx) \to \hat{f}(R\xi)$ whenever R is a rotation.

Proof. Property (1) is an immediate consequence of the translation invariance of the integral. Property (2) follows from the definition (1.1),

$$\int_{\mathbb{R}^n} f(x) e^{-2\pi i x . h} e^{-2\pi i x . \xi} \, dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x . (h+\xi)} \, dx = \hat{f}(\xi + h).$$

Also, if $\delta > 0$, then $\delta \int_{\mathbb{R}^n} f(\delta x) dx = \int_{\mathbb{R}^n} f(x) dx$ establishes property (3).

Integrating by parts gives

$$\int_{-N}^{N} f'(x) e^{-2\pi i x \cdot \xi} dx = [f(x) e^{-2\pi i x . \xi}]_{-N}^{N} + 2\pi i x \cdot \xi \int_{-N}^{N} f(x) e^{-2\pi i x \cdot \xi} dx.$$

If we repeat this process δ times, and let N go to infinity, we get (4).

To verify the last property, simply change variables $y = Rx$ in the integral. Then, recall that $|\det(R)| = 1$, and $R^{-1} y \cdot \xi = y \cdot R\xi$, because R is a rotation.

Corollary 2.2 The Fourier transform maps $\varsigma(\mathbb{R}^n)$ to itself.

Remark. (i) As an example we consider Gaussian $e^{-\pi|x|^2}$. Also, we observe that when $n=1$, the radial functions are precisely the even functions, that is, those for which $f(x)=f(-x)$.

(ii) As for the Fourier transform of radial function, we refer the appendix to the interested readers.

Definition 2.3 Given two integrable functions f and g on $\mathbb{R}^n$, we define their convolution $f*g$ by

$$(f*g)(x)=\int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

The above integral makes sense for each x, since the product of two integrable functions is again integrable.

Proposition 2.4 Suppose that f, g, and h are 2π-periodic integrable functions. Then the following holds :

(1) $f*(g+h)=(f*g)+(f*h)$.

(2) $(cf)*g=c(f*g)=f*(cg)$ for any $c\in\mathbb{C}$.

(3) $f*g=g*f$

(4) $(f*g)*h=f*(g*h)$.

(5) $f*g$ is continuous.

(6) $\widehat{f*g}(n)=\hat{f}(n)\hat{g}(n)$.

proof. Since we have

$$\begin{aligned} f*(g+h) &= \int_{\mathbb{R}^n} f(y)(g+h)(x-y)\,dy \\ &= \int_{\mathbb{R}^n} f(y)g(x-y)\,dy + \int_{\mathbb{R}^n} f(y)h(x-y)\,dy \\ &= (f*g)(x)+(f*h)(x)\ , \end{aligned}$$

(1) follows at once. As for (2), we have

$$
\begin{aligned}
(cf) * g &= \int (cf)(y) g(x-y)\, dy \\
&= c\int f(y) g(x-y)\, dy \qquad (= c(f * g)\ ) \\
&= \int f(y)(cg)(x-y)\, dy \\
&= f * (cg).
\end{aligned}
$$

We turn to (3). If F is continuous and periodic, then

$$
\int_{\mathbb{R}^n} F(y) dy = \int_{\mathbb{R}^n} F(x-y) dy \qquad \text{for any } x \in \mathbb{R},
$$

because of a change of variables y by $-y$, followed by a translation from y to $y-x$. Then, if one takes $F(y) = f(y) g(x-y)$, we obtain the desired.

In order to obtain (4), we consider

$$
\begin{aligned}
(f * g) * h &= \int_{\mathbb{R}^n} (f * g)(x-y) h(y)\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-z) g(z) h(y)\, dz\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-z) g(x-y) h(y)\, dx\, dy \\
&= \int_{\mathbb{R}^n} f(x-z)(g * h)(x)\, dz \\
&= f * (g * h).
\end{aligned}
$$

We proceed to (5). We show that if f and g are continuous, then $f * g$ is continuous. First, we may write

$$
(f * g)(x_1) - (f * g)(x_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left[g(x_1 - y) - g(x_2 - y)\right] dy
$$

Since g is continuous it must be uniformly continuous on any closed and bounded interval. But g is also periodic, so it must be uniformly continuous on all of $\mathbb{R}$; given $\epsilon > 0$ there exists $\delta > 0$ so that $|g(s) - g(t)| < \epsilon$ whenever $|s-t| < \delta$. Then, $|x_1 - x_2| < \delta$ implies $|(x_1 - y) - (x_2 - y)| < \delta$ for any y, hence

$$
\begin{aligned}
|(f * g)(x_1) - (f * g)(x_2)| &\le \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(y)\,[g(x_1-y)-g(x_2-y)]\,dy\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)||g(x_1-y)-g(x_2-y)|\,dy \\
&\le \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |f(y)|\,dy \\
&\le \frac{\epsilon}{2\pi} 2\pi B,
\end{aligned}
$$

where B is chosen so that $|f(x)| \le B$ for all x. As a result, we conclude that $f * g$ is continuous, and the proposition is proved, at least when f and g are continuous.

Finally, we show (6). The Fourier transform of $\widehat{f * g}$ is

$$
\begin{aligned}
\widehat{f * g}(n) &= \int_{\mathbb{R}^n} (f * g)(x) e^{-2\pi i n \cdot x} dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x-y)\,dy\right) e^{-2\pi i n \cdot x}\,dx \\
&= \int_{\mathbb{R}^n} f(y) e^{-2\pi i n \cdot y}\left(\int_{\mathbb{R}^n} g(x-y) e^{-2\pi i n \cdot (x-y)}\,dx\right) dy \\
&= \int_{\mathbb{R}^n} f(y) e^{-2\pi i n \cdot .y}\left(\int_{\mathbb{R}^n} g(x) e^{-2\pi i n \cdot x} dx\right) dy \\
&= \hat{f}(n)\hat{g}(n).
\end{aligned}
$$

We finish the proof of Proposition 3.1. □

Definition 2.5 A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if it satisfies the following properties :

(1) For all $n \ge 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x) dx = 1.$$

(2) There exists $M > 0$ such that for all $n \ge 1$,

$$\int_{-\pi}^{\pi} |K_n(x)| dx \le M.$$

(3) For every $\delta > 0$,

$$\int_{\delta \le |x| \le \pi} |K_n(x)| dx \to 0, \quad \text{as } n \to \infty$$

The following theorem is called Plancherel Theorem.

Theorem 2.6 Suppose $f \in \varsigma(\mathbb{R}^n)$. Then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Moreover

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

Proof. Step1. The Fourier transform of $e^{-\pi|x|^2}$ is $e^{-\pi|\xi|^2}$. To prove this, notice that the properties of the exponential functions imply that

$$e^{-\pi|x|^2} = e^{-\pi x_1^2} \cdots e^{-\pi x_n^2} \quad \text{and} \quad e^{-2\pi i x . \xi} = e^{-2\pi i x_1 \cdot \xi_1} \cdots e^{-2\pi i x_n \cdot \xi_n}.$$

Thus the integrand in the Fourier transform is a product of n functions, each depending on the variable x_j $(1 \le j \le n)$ only. Thus the assertion follows by writing the integral over $\mathbb{R}^n$ as a series of repeated integrals, each taken over $\mathbb{R}$.

Step2. The family $K_\delta(x) = \delta^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{\delta}}$ is a family of good kernels in $\mathbb{R}^n$. By this we mean that

(1) $\int_{\mathbb{R}^n} K_\delta(x) dx = 1$,

(2) $\int_{\mathbb{R}^n} |K_\delta(x)|\, dx \le M$ (in fact $K_\delta(x) \ge 0$),

(3) For every $\eta > 0$, $\int_{|x| \ge \eta} |K_\delta(x)| dx \to 0$ as $\delta \to 0$.

The proofs of these assertions are almost identical to the case $n = 1$.

As a result

$$\int_{\mathbb{R}^n} K_\delta(x)F(x)\,dx \to F(0) \text{ as } \delta \to 0$$

when F is a Schwartz function, or more generally when F is bounded and continuous at the origin.

Step3. The multiplication formula

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(y)g(y)\,dy$$

holds whenever f and g are in ς. The proof requires the evaluation of the integral of $f(x)g(y)e^{-2\pi ix\cdot y}$ over $(x,y)\in\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ as a repeated integral, with each separate integration taken over $\mathbb{R}^n$.

The Fourier inversion is then a simple consequence of the multiplication formula and the family of good kernels K_δ. It also follows that the Fourier transform $\hat{f}$ is a bijective map of $\varsigma(\mathbb{R}^n)$ to itself, whose inverse is

$$\check{g}(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi ix\cdot\xi}d\xi$$

Step4. Next we turn to the convolution, defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy, \quad f,g\in\varsigma$$

We have that $f,g\in\varsigma(\mathbb{R}^n)$, $f*g=g*f$, and $(\widehat{f*g})(\xi)=\hat{f}(\xi)\hat{g}(\xi)$. The argument is similar to that in one-dimension. The calculation of the Fourier transform of $f*g$ involves an integration of $f(y)g(x-y)e^{-2\pi ix\cdot\xi}$(over $\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$) expressed as a repeated integral.

This completes the proof of Theorem 2.5. $\square$

Example 2.7 This example is the application of Theorem 2.5, which shows that Plancherel Theorem holds for n-dimensional Gaussian $e^{-\pi|x|^2}$.

Theorem 2.8 Chebyshev's Inequality. If $f \in L^p$ $(0 < p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\| f \|_p}{\alpha}\right)^p.$$

proof. Let $E_\alpha = \{x : |f(x)| > \alpha\}$. Then

$$\| f \|_p^p = \int |f|^p \geq \int_{E_\alpha} |f|^p \geq \alpha^p \int_{E_\alpha} 1 = \alpha^p \mu(E_\alpha). \qquad \square$$

2.1 L^p and Weak L^p spaces . In this section we consider the definitions of L^p and Weak L^p spaces and study some properties.

Let X be a measure space and let μ be a positive, not necessarily finite, measure on X. For $0 < p < \infty$, $L^p(X,\mu)$ will denote the set of all complex-valued μ-measurable functions on X whose modulus to the pth power is integrable. $L^\infty(X,\mu)$ will be the set of all complex-valued μ-measurable functions f on X such that for some $B > 0$, the set $\{x : |f(x)| > B\}$ has μ-measure zero. Two functions in $L^p(X,\mu)$ will be considered equal if they are equal μ-almost everywhere. The notation $L^p(\mathbb{R}^n)$ will be reserved for the space $L^p(\mathbb{R}^n, |\cdot|)$, where $|\cdot|$ denotes n-dimensional Lebesgue measure. Lebesgue measure on $\mathbb{R}^n$ will also be denoted by dx. Within context and in the lack of ambiguity, $L^p(X,\mu)$ will simply be L^p. The space $L^p(\mathbb{Z})$ equipped with counting measure will be denoted by $l^p(\mathbb{Z})$ or simply l^p.

Definition 2.9 For $0 < p < \infty$, we define the L^p quasi-norm of a function f by

$$\| f \|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \tag{2.2.1}$$

and for $p=\infty$ by

$$\| f \|_{L^{\infty}(X,\mu)} = \inf \{B>0 : \mu(\{x : |f(x)| > B\}) = 0\}. \tag{2.2.2}$$

It is well known that Minkowski's (or the triangle) inequality

$$\| f+g \|_{L^p(X,\mu)} \leq \| f \|_{L^p(X,\mu)} + \| g \|_{L^p(X,\mu)} \tag{2.2.3}$$

holds for all f, g in $L^p = L^p(X,\mu)$, whenever $1 \leq p \leq \infty$. Since in addition $\| f \|_{L^p(X,\mu)} = 0$ implies that $f=0$ ($\mu-$a.e.), the L^p spaces are normed linear spaces or $1 \leq p \leq \infty$. For $0<p<1$, inequality (2.2.3) is reversed when $f, g \geq 0$. However, the following substitute of (2.2.3) holds:

$$\| f+g \|_{L^p(X,\mu)} \leq 2^{\frac{1-p}{p}} \left(\| f \|_{L^p(X,\mu)} + \| g \|_{L^p(X,\mu)} \right) \tag{2.2.4}$$

and thus the spaces $L^p(X,\mu)$ are quasi-normed linear spaces.

Definition 2.10 For f a measurable function on X, the distribution function of f is the function d_f defined on $[0,\infty)$ as follows:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}).$$

Example 2.11 The simple functions are finite linear combinations of characteristic functions of sets of finite measure

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where the sets E_j are pairwise disjoint and $a_1 > \cdots > a_N > 0$. If $\alpha \geq a_1$, then clearly $d_f(\alpha)=0$. However if $a_2 < \alpha < a_1$ then $|f(x)| > \alpha$ precisely when $x \in E_1$ and, in general,

if $a_{j+1} \leq \alpha < a_j$, then $|f(x)| > \alpha$ precisely when $x \in E_1 \cup \cdots \cup E_j$.

Setting

$$B_j = \sum_{k=1}^{j} \mu(E_k)$$

we have

$$d_f(\alpha) = \sum_{j=1}^{N} B_j \chi_{[a_{j+1}, a_j)}(\alpha),$$

where $a_{N+1} = 0$.

We now state a few simple facts about the distribution function d_f. We have

Proposition 2.12 Let f and g be measurable functions on (X,μ). Then for all $\alpha, \beta > 0$ we have

(1) $|g| \le |f|$ $\mu - a.e.$ implies that $d_g \le d_f$

(2) $d_{cf}(\alpha) = d_f(\frac{\alpha}{|c|})$, for all $c \in C \setminus \{0\}$

(3) $d_{f+g}(\alpha + \beta) \le d_f(\alpha) + d_g(\beta)$

(4) $d_{fg}(\alpha\beta) \le d_f(\alpha) + d_g(\beta)$.

Proposition 2.13 For f in $L^p(X,\mu)$, $0 < p < \infty$, we have

$$\| f \|_{L^p}^p = p \int_0^{\infty} \alpha^{p-1} d_f(\alpha)\, d\alpha.$$

proof. We use Fubini's theorem to obtain the second equality below

$$\begin{aligned} p \int_0^{\infty} \alpha^{p-1} d_f(\alpha)\, d\alpha &= p \int_0^{\infty} \alpha^{p-1} \int_X \chi_{\{x : |f(x)| > \alpha\}} d\mu(x)\, d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha\, d\mu(x) \end{aligned}$$

$$= \int_X |f(x)|^p \, d\mu(x)$$

$$= \| f \|_{L^p}^p.$$

This completes the proof. $\square$

Definition 2.14 For $0 < p < \infty$, the space weak $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\| f \|_{L^{p,\infty}} = \inf \left\{ C > 0 : d_f(\alpha) \le \frac{C^p}{\alpha^p} \quad \text{for all} \quad \alpha > 0 \right\}$$

$$= \sup \left\{ \gamma \, d_f(\gamma)^{\frac{1}{p}} : \gamma > 0 \right\}$$

is finite. The space weak-$L^\infty(X,\mu)$ is by definition $L^\infty(X,\mu)$.

The weak L^p spaces will also be denoted by $L^{p,\infty}(X,\mu)$. Two functions in $L^{p,\infty}(X,\mu)$ will be considered equal if they are equal μ-a.e.. The notation $L^{p,\infty}(\mathbb{R}^n)$ is reserved for $L^{p,\infty}(\mathbb{R}^n, |\cdot|)$. Using Proposition 2.12 (2), we can easily show that

$$\| kf \|_{L^{p,\infty}} = |k| \, \| f \|_{L^{p,\infty}},$$

for any complex nonzero constant k. The analogue of Proposition 2.12 is

$$\| f + g \|_{L^{p,\infty}} \le c_p (\| f \|_{L^{p,\infty}} + \| g \|_{L^{p,\infty}}),$$

where $c_p = \max(2, 2^{\frac{1}{p}})$ a fact that follows from Proposition 2.12 (3) with $\alpha_1 = \alpha_2 = \dfrac{\alpha}{2}$.

We also have that

$$\| f \|_{L^{p,\infty}} = 0 \Rightarrow f = 0 \qquad \mu\text{-a.e.}$$

So that, $L^{p,\infty}$ is a quasi-normed linear space for $0 < p < \infty$.

The weak L^p spaces are larger than the usual L^p spaces. We have the following:

Proposition 2.15 For any $0<p<\infty$, and any f in $L^p(X,\mu)$ we have $\| f \|_{L^{p,\infty}} \leq \| f \|_{L^p}$; hence $L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu)$.

proof. This is just a trivial consequence of Chebychev's inquality:

$$\alpha^p d_f(\alpha) \leq \int_{\{x : |f(x)| > \alpha\}} |f(x)|^p \, d\mu(x). \tag{2.2.5}$$

As the integral in (2.2.5) is at most $\| f \|_{L^p}^p$, using sup $\left\{ \gamma \, d_f(\gamma)^{\frac{1}{p}} \; : \; \gamma > 0 \right\}$ we obtain that

$$\| f \|_{L^{p,\infty}} \leq \| f \|_{L^p}. \qquad \square$$

Remark. The inclusion $L^p \subseteq L^{p,\infty}$ is strict. For example, on $\mathbb{R}^n$ with the usual Lebesgue measure, let $h(x) = |x|^{-\frac{n}{p}}$. Obviously, h is not in $L^p(\mathbb{R}^n)$ but h is in $L^{p,\infty}(\mathbb{R}^n)$ and we may check easily that $\| h \|_{L^{p,\infty}(\mathbb{R}^n)}$ is the measure of the unit ball of $\mathbb{R}^n$.

2.2 The Marcinkiewicz Interpolation Theorem (Real method).

Let T be an operator defined on a linear subspace of the space of all complex-valued measurable functions on a measure space (X,μ) and taking values in the set of all complex-valued measurable functions on a measure space (Y,ν).

Definition 2.16

(1) T is called linear if for all f, g and all $\lambda \in \mathbb{C}$, we have

$$T(f+g) = T(f) + T(g) \quad \text{and} \quad T(\lambda f) = \lambda T(f).$$

(2) T is called sublinear if for all f, g and all $\lambda \in \mathbb{C}$, we have

$$|T(f+g)| \le |T(f)| + |T(g)| \quad \text{and} \quad |T(\lambda f)| = |\lambda||T(f)|.$$

(3) T is called quasi-linear if for all f, g and all $\lambda \in \mathbb{C}$, we have

$$|T(f+g)| \le K(|T(f)| + |T(g)|) \quad \text{and} \quad |T(\lambda f)| = |\lambda||T(f)|$$

for some constant $K > 0$. Sublinearity is a special case of quasi-linearity.

We begin with our first interpolation theorem.

Theorem 2.17 Let (X, μ) and (Y, ν) be two measure space and let $0 < p_0 < p_1 \le \infty$. Let T be a sublinear operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that there exist two positive constants A_0 and A_1 such that

$$\| T(f) \|_{L^{p_0,\infty}(Y)} \le A_0 \| f \|_{L^{p_0}(X)} \qquad \text{for all } f \in L^{p_0}(X),$$

$$\| T(f) \|_{L^{p_1,\infty}(Y)} \le A_1 \| f \|_{L^{p_1}(X)} \qquad \text{for all } f \in L^{p_1}(X).$$

Then for all $p_0 < p < p_1$ and for all f in $L^p(X)$ we have the estimate

(2.3.1) $$\| T(f) \|_{L^p(Y)} \le A \| f \|_{L^p(X)},$$

where

$$A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1 - p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

Proof. See p.32-35 in [LG]. □

3. Calderón-Zygmund decomposition

To prove that singular integrals are of weak type (1,1) we will need to introduce the Calderón-Zygmund decomposition. This is a powerful stopping time construction.

Theorem 3.1 Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then there exist functions g and b on $\mathbb{R}^n$ such that

(1) $f = g + b$

(2) $\| g \|_{L^1} \le \| f \|_{L^1}$ and $\| g \|_{L^\infty} \le 2^n \alpha$.

(3) $b = \sum_j b_j$, where each b_j is supported in a dyadic cube Q_j. Furthermore, the cubes Q_k and Q_j have disjoint interiors when $j \neq k$.

(4) $\int_{Q_j} b_j(x)dx = 0$.

(5) $\| b_j \|_{L^1} \le 2^{n+1} \alpha |Q_j|$.

(6) $\sum_j |Q_j| \le \alpha^{-1} \| f \|_{L^1}$.

Remark. This is called the Calderón-Zygmund decomposition of f at height α. The function g is called the good function of the decomposition since it is both integrable and bounded; hence the letter g. The function b is called the bad function since it contains the singular part of f (hence the letter b), but it is carefully chosen to have mean value zero. It follows from (5) and (6) that the bad function b is integrable and

$$\| b \|_{L^1} \le \sum_j \| b \|_{L^1} \le 2^{n+1} \alpha \sum_j |Q_j| \le 2^{n+1} \| f \|_{L^1}$$

By (2) the good function is integrable and bounded; hence it is in all the L^p spaces for $1 \le p \le \infty$. More specifically, we have the following estimate:

$$\| g \|_{L^p} \le \| g \|_{L^1}^{\frac{1}{p}} \| g \|_{L^\infty}^{1-\frac{1}{p}} \le \| f \|_{L^1}^{\frac{1}{p}} (2^n \alpha)^{1-\frac{1}{p}} = 2^{\frac{n}{p'}} \alpha^{\frac{1}{p'}} \| f \|_{L^1}^{\frac{1}{p}}$$

Proof. Recall that a dyadic cube in $\mathbb{R}^n$ is a cube of the form

$$[2^k m_1, 2^k (m_1+1)) \times \cdots \times [2^k m_n, 2^k (m_n+1)),$$

where k, $m_1, \cdots, m_n$ are integers. Decompose $\mathbb{R}^n$ into a mesh of equal size disjoint dyadic cubes so that

$$|Q| \ge \frac{1}{\alpha} \| f \|_{L^1}$$

for every cube Q in the mesh. Subdivide each cube in the mesh into 2^n congruent cubes by bisecting each of its sides. We now have a new mesh of dyadic cubes. Select a cube in the new mesh if

$$\frac{1}{|Q|} \int_Q |f(x)| dx > \alpha.$$

Let S be the set of all selected cubes. Now subdivide each nonselected cube into 2^n congruent subcubes by bisecting each side ad before. Then select one of these new cubes if $\frac{1}{|Q|} \int_Q |f(x)| dx > \alpha$ holds. Put all selected cubes of this generation into the set S. Repeat this procedure indefinitely.

The set of all selected cubes S is exactly the set of the cubes Q_j proclaimed in the proposition. Let us observe that these cubes are disjoint, for otherwise some Q_k would be a proper subset of some Q_j, which is impossible since the selected cube Q_j was never subdivided. Now define

$$b_j = \left(f - \frac{1}{|Q_j|} \int_{Q_j} f \, dx \right) \chi_{Q_j},$$

$b = \sum_j b_j$ and $g = f - b$.

For a selected cube Q_j there exists a unique nonselected cube Q' with twice its side length that contains Q_j. Let us call this cube the parent of Q_j. Since its parent Q' was not selected, we have $|Q'|^{-1}\int_{Q'}|f|\,dx \le \alpha$. Then

$$\frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,dx = \frac{2^n}{|Q'|}\int_{Q'}|f(x)| \le 2^n\alpha.$$

Consequently,

$$\int_{Q_j}|b_j|\,dx \le \int_{Q_j}|f|\,dx + |Q_j|\left|\frac{1}{|Q_j|}\int_{Q_j}f\,dx\right| \le 2\int_{Q_j}|f|\,dx \le 2^{n+1}\alpha|Q_j|,$$

which proves (5). To prove (6), simply observe that

$$\sum_j|Q_j| \le \frac{1}{\alpha}\sum_j\int_{Q_j}|f|\,dx = \frac{1}{\alpha}\int_{\cup_j Q_j}|f|\,dx \le \frac{1}{\alpha}\parallel f\parallel_{L^1}.$$

Next we need to obtain the estimates on g. Write $\mathbb{R}^n = \cup_j Q_j \cup F$, where F is a closed set. Since $b=0$ on F and $f-b_j = |Q_j|^{-1}\int_{Q_j}f\,dx$, we have

$$g = \begin{cases} f & \text{on } F, \\ \dfrac{1}{|Q_j|\int_{Q_j}f\,dx} & \text{on } Q_j. \end{cases} \tag{3.1}$$

On the cube Q_j, g is equal to the constant $|Q_j|^{-1}\int_{Q_j}f\,dx$, and this is bounded by $2^n\alpha$.

It suffices to show that g is bounded on the set F. Given $x\in F$, we have that x does not belong to any selected cube. Therefore, there exists a sequence of cubes $Q^{(k)}$ whose closures contain x and whose side lengths tend to zero as $k\to\infty$. Since the cubes $Q^{(k)}$ were never selected, we have

$$\left|\frac{1}{|Q^{(k)}|}\int_{Q^{(k)}}f\,dx\right| \le \frac{1}{|Q^{(k)}|}\int_{Q^{(k)}}|f|\,dx \le \alpha.$$

The balls are replaced with cubes, we conclude that

$$|f(x)| = \left|\lim_{k\to\infty}\frac{1}{|Q^{(k)}|}\int_{Q^{(k)}} f\,dx\right| \leq \alpha$$

whenever $x \in F$. But since $g = f$ a.e. on F, if follows that g is bounded by α on F. Finally, it follows (3.1) that $\| g \|_{L^1} \leq \| f \|_{L^1}$. □

4. Proof of Theorems 1.1 and 1.2

For the kernel estimates, we shall use the idea of Müller and Seeger in [MS]. They used dyadic decomposition of Bessel function to prove local smoothing conjecture for spherically symmetric initial data including endpoint results.

4.1 Dyadic decomposition of Bessel function. Let $\eta \in C_0^\infty(\mathbb{R})$ be supported in $(-\frac{1}{2}, 2)$ and equal to 1 in $(-\frac{1}{4}, \frac{1}{4})$. For $m = 0, 1, 2, \cdots$ we set

$$\eta_m(\sigma, \nu) = \begin{cases} \eta(\nu(1-\sigma^2)) & \text{if } m = 0 \\ \eta(2^{-m}\nu(1-\sigma^2)) - \eta(2^{-m+1}\nu(1-\sigma^2)) & \text{if } m > 0 \end{cases}$$

and

$$J_\mu^m(uv) = A_\mu (uv)^\mu \int_{-1}^{1} e^{i(uv)\sigma} (1-\sigma^2)^{\mu - \frac{1}{2}} \eta_m(\sigma, \nu)\, d\sigma.$$

For a positive integer M we define

$$\phi_{m\nu}(\sigma) = \begin{cases} (1-\sigma^2)^{\mu - \frac{1}{2}} \eta_m(\sigma, \nu) & \text{if } m = 0 \\ \left(\frac{1}{iuv}\right)^M \left(\frac{d}{d\sigma}\right)^M [\eta_m(\sigma, \nu)(1-\sigma^2)^{\mu - \frac{1}{2}}] & \text{if } m > 0. \end{cases}$$

Then by integration by parts if $m > 0$ we have

$$J_\mu^m(uv) = A_\mu (uv)^\mu \int_{-1}^{1} e^{i(uv)\sigma} \phi_{m\nu}(\sigma)\, d\sigma. \tag{4.0}$$

We note that the integrand in (4.1) has the following upper bound:

$$|\phi_{m\nu}(\sigma)| \le Cu^{-M} 2^{-mM} (2^m \nu^{-1})^{\mu - \frac{1}{2}}$$

and that $\phi_{m\nu}$ vanishes unless either $1-\sigma^2 \approx 2^m \nu^{-1}$ for $m > 0$, or $1-\sigma^2 \le \nu^{-1}$ for $m = 0$ so if σ is in the support of $\phi_{m\nu}$ then either $|\nu - \nu\sigma| \le 2^m$ or $|\nu + \nu\sigma| \le 2^m$. (see [MS, p.5])

Consider the family of Fourier multipliers

$$m^{\delta}(\xi) = (1-|\xi|^2)_+^{\delta}, \qquad \xi \in \mathbb{R}^n$$

with $m^{\delta}(\xi) = 0$ when $|\xi| > 1$. Then define convolution operators S^{δ} by

$$\widehat{S^{\delta} f}(\xi) = m^{\delta}(\xi)\hat{f}(\xi).$$

Let $\phi \in C_0^{\infty}(\mathbb{R})$ be supported in $\left(\frac{1}{2}, 2\right)$ such that $\sum_{k \geq 1} \phi(2^k s) = 1$ for $0 < s < 1$. Fix k.

We shall need point estimates for the kernels of

$$S_k^{\delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} H_k^{\delta}(x-y) f(y)\, dy$$

where

$$H_k^{\delta}(x) = \int_{\mathbb{R}^n} \varphi(2^k(1-|\xi'|^2))(1-|\xi'|^2)_+^{\delta} e^{i<x,\xi>} d\xi. \tag{4.1}$$

We write $\sum_{k \geq 1} H_k^{\delta} = H^{\delta}$ and $\sum_{k \geq 1} S_k^{\delta} = S^{\delta}$.

We let $|x| = r$ and define $H_k(x) = L_k(|x|)$. By Bochner's formula (see Appendix) and change of variables, we have

$$L_k(r) = r^{-(n-2)/2} \int_0^1 J_{\frac{n-2}{2}}(\rho r) \varphi(2^k(1-\rho^2))(1-\rho^2)^{\delta} \rho^{n/2} d\rho. \tag{4.2}$$

Here J_{μ} is the Bessel function of order $\mu > -\frac{1}{2}$ defined by

$$J_{\mu}(t) = A_{\mu} t^{\mu} \int_{-1}^{1} e^{is\sigma}(1-\sigma^2)^{\mu - \frac{1}{2}} d\sigma \tag{4.3}$$

where $A_{\mu} = \left[2^{\mu}\Gamma(2\mu+1)\Gamma(\frac{1}{2})\right]^{-1}$.

For the following lemma, we use dyadic decompositions of Bessel functions (see Appendix) following the article by Müller and Seeger [MS].

Lemma 4.2 Suppose that $|x|>2$. Then for each k there is an estimate as follows :

(4.4) $$|L_k(|x|)| \le C2^{-k(\delta+1)}|x|^{-(n-1)/2}\min\{1,(2^{-k}|x|)^{-N}\}.$$

Therefore,

(4.5) $$|H^\delta(x)| \le \sum_{k\ge 1} H_k \le C\frac{1}{(1+|x|)^{\delta+(n+1)/2}}.$$

Proof. Fix k and $v=r$ in subsection 4.1. We may decompose the kernel (4.2) as

$$L_{k,0}=\sum_{m=0} L_{k,0}^m$$

where

(4.6) $$L_{k,0}^m(r)=r^{-(n-2)/2}\int_{\mathbb{R}}\int_0^1 J_{\frac{n-2}{2},k}^m(\rho r)\varphi(2^k(1-\rho^2))(1-\rho^2)^\delta\rho^{n/2}\,d\rho.$$

Formula (4.6) and straightforward computation imply that

(4.7) $$L_{k,0}^m(r)=A_{\frac{n-2}{2}}\int_{-1}^1\phi_{mkr}(\sigma)\int_0^1\varphi(2^k(1-\rho^2))(1-\rho^2)^\delta\rho^{n-1}e^{ipr\delta}d\rho d\sigma.$$

We integrate by parts with respect to ρ and in (4,7) and by Fubini's theorem

(4.8) $$\left|L_{k,0}^m(r)\right| \le C2^{-k(n-1)/2}\int_{-1}^1\int_0^1|\phi_{mkr}(\sigma)|(1+|\sigma r|)^{-N}$$
$$\times\left|\left(\frac{\partial}{\partial\rho}\right)^N\varphi(2^k(1-\rho^2))(1-\rho^2)^\delta\rho^{(n-1)/2}\right|d\rho\, d\sigma.$$

Note that

(4.9) $$|\phi_{mkr}(\sigma)| \le C2^{-mM}(2^{m+k}r^{-1})^{(n-3)/2}.$$

Moreover, $\phi_{mk\nu}$ vanishes unless either $1-\sigma^2\approx 2^{m+k}r^{-1}$ for $m>0$, or $1-\sigma^2\le 2^k r^{-1}$ for $m=0$. Hence if σ is in the support of ϕ_{mkr} then either

$|r-r\sigma|\leq 2^{m+k}$ or $|r+r\sigma|\leq 2^{m+k}$.

Then using the estimates (4.9), the integrand of (4.8) is bounded by

$$C2^{-k\delta}|\phi_{mkr}(\sigma)|\frac{1}{(1+2^{-k}|\sigma r|^{N_1})}$$

$$\leq C2^{k\{(n-3)/2-\delta\}}2^{m((n-3)/2+N-M)}r^{-(n-3)/2}\frac{1}{(1+2^{-k}r)^N}.$$

If we integrate over the support of $\varphi(2^k(1-\rho^2))\otimes\phi_{mkr}$ for $m\geq 0$ in (4.8), we gain an additional factor of $C2^m r^{-1}$. Since $M>N+(n-1)/2$, we may sum over m and the desired estimates (4.4) follow from (4.8). Hence we obtain

$$\left\{C\sum_{r\leq 2^k}2^{-k(\delta+1)}r^{-(n-1)/2}+C\sum_{r>2^k}2^{-k(\delta+1-N)}r^{-(n-1)/2-N}\right\},$$

and thus (4.5) is established.

Proof of Theorem 1.

For $\delta>\dfrac{n-1}{2}$ we use Lemma 4.2 to have

$$\begin{aligned}\| S^\delta f\|_{L^1(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}|S^\delta f(x)|\,dx\\&=\int_{\mathbb{R}^n}|(H^\delta * f)(x)|\,dx\\&\leq C\| f\|_{L^1(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{1}{(1+|x|)^{\delta+(n+1)/2}}dx\\&\leq C\| f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

For L^2-bound we apply Plancherel Theorem in Section 2 to obtain

$$\| S^\delta f\|_{L^2(\mathbb{R}^n)}=\|\widehat{S^\delta f}\|_{L^2(\mathbb{R}^n)}.$$

Since the multiplier $m^\delta(\xi)=(1-|\xi|^2)^\delta_+$ is bounded by 1 for $\delta>0$, we have

$$\| S^\delta f \|_{L^2(\mathbb{R}^n)} = \| m^\delta \hat{f} \|_{L^2(\mathbb{R}^n)}$$
$$\leq \| \hat{f} \|_{L^2(\mathbb{R}^n)} = \| f \|_{L^2(\mathbb{R}^n)}.$$

We now interpolate with real method between the results on $L^1(\mathbb{R}^n)$ for $\delta > \frac{n-1}{2}$ and on $L^2(\mathbb{R}^n)$ for $\delta > 0$. By Marcinkiewicz Theorem (see Section 2.3) with $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ and $0 < \theta < 1$, we obtain the L^p-bound of S^δ for $1 \leq p \leq 2$ and $\delta > \frac{n-1}{2}$.

For $2 < p < \infty$, we will prove the duality between L^p and L^q, $\frac{1}{p} + \frac{1}{q} = 1$, and the fact that the theorem is proved for $L^q, 1 < q < 2$. Observe the following ; if a function ψ is locally integrable and if $\sup \left| \int \psi \phi \, dx \right| = A < \infty$, where the sup is taken over all continuous ϕ with compact support which verify $\| \phi \|_q \leq 1$, then $\psi \in L^p$ and $\| \psi \|_q = A$. We take $f \in L^1 \cap L^p$, $(2 < p < \infty)$, and ϕ of the type described above. Since $H^\delta \in L^2$, and because of our choice of f and ϕ, the double integral

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H^\delta(x-y) f(y) \phi(x) \, dx \, dy$$

converges absolutely ; its value is therefore

$$I = \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} H^\delta(x-y) \phi(x) \, dx \right) dy.$$

But the theorem is valid for $1 < q < 2$ (with the kernel $H^\delta(-x)$ instead of $H^\delta(x)$, but with the same constant A_q). Therefore $\int_{\mathbb{R}^n} H^\delta(x-y) \phi(x) \, dx$ belongs to L^q, and its L^q norm is majorized by $A_q \| \phi \|_q = A_q$. Hölder's inequality then shows that $\left| \int_{\mathbb{R}^n} (S^\delta f) \phi \, dx \right| = |I| \leq A_q \| f \|_q$, and taking the supremum of all the ϕ's indicated above gives the result that

$$\| S^{\delta} f \|_{L^p(\mathbb{R}^n)} \leq A_q \| f \|_{L^p(\mathbb{R}^n)}, \qquad 2 < p < \infty. \qquad \square$$

We consider for $\delta > 0$

$$\widehat{T^{\delta} f}(\xi) = \left(1 - \frac{|\xi'|^2}{|\xi_{n+1}|^2}\right)_+^{\delta} \hat{f}(\xi), \quad \xi = (\xi', \xi_{n+1}).$$

Using inverse Fourier transform, we denote by $T^{\delta} f(x) = K^{\delta} * f(x)$ where

$$K^{\delta}(x) = F^{-1}\left[\left(1 - \frac{|\xi'|^2}{|\xi_{n+1}|^2}\right)_+^{\delta}\right](x).$$

Let $\psi \in C_0^{\infty}(\mathbb{R})$ be supported in $(\frac{1}{2}, 2)$ such that $\sum_{l=-\infty}^{\infty} \psi(2^{-l} t) = 1$ for $t > 0$.

If we write the kernel $K_l^{\delta}(x) = F^{-1}\left[\left(1 - \frac{|\xi'|^2}{|\xi_{n+1}|^2}\right)_+^{\delta} \psi(2^{-l}\xi_{n+1})\right](x)$, we notice that

$$K^{\delta}(x) = \sum_{l=-\infty}^{\infty} 2^{l(n+1)} K_0^{\delta}(2^l x).$$

From the kernel estimates in [SH], we have for any $N > 0$

$$\begin{aligned} &\left|K_0^{\delta}(x)\right| + \left|\nabla K_0^{\delta}(x)\right| \\ &\quad \leq C \frac{1}{(1+|x'|)^{(n+1)/2}} \frac{1}{(1+|x_{n+1}|)^{\delta+1}} \chi_{\{|x'| \leq |x_{n+1}|\}}(x) \\ &\quad + C \frac{1}{(1+|x'|)^{\delta+(n+1)/2}} \frac{1}{(1+||x_{n+1}| - |x'||)^N} \chi_{\{|x'| \geq |x_{n+1}|\}}(x). \end{aligned}$$

Remark. The key tool in the proof of the weak type (1,1) estimate is the Calderón-Zygmund decomposition of L^1 functions.

Proposition 4.3 If $\delta > \dfrac{n-1}{2}$, then for $\alpha > 0$

$$\left|\{x : |T^{\delta} f(x)| > \alpha\}\right| \le C\alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})},$$

where $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^{n+1}$.

Proof. From the Calderón-Zygmund decomposition we assume that $f = g + b$, where $b = \sum_{k=1}^{\infty} b_k$. We now have

$$\begin{aligned}\{x : |T^{\delta} f(x)| > \alpha\} &\subset \left\{x : |T^{\delta} g(x)| > \frac{\alpha}{2}\right\} \cup \left\{x : |T^{\delta} b(x)| > \frac{\alpha}{2}\right\} \\ &\doteq I \cup II.\end{aligned}$$

Since $|g(x)| < 2^{n+1}\alpha$ a.e., we use the L^2 boundedness of T^{δ} and Chebyshev's inequality to get

$$|I| \le C\alpha^{-2} \| g \|^2_{L^2(\mathbb{R}^{n+1})} \le C\alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})}.$$

Let Q_k be certain non-overlapping cubes and Q_k^* be the cubes with the same center as Q_k but twice the sidelength. If $\Omega^* = \cup Q_k^*$, then $|\Omega^*| \le C 2^{n+1} \alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})}$.

So, $\left|\left\{x \in \Omega^* : |T^{\delta} b(x)| > \dfrac{\alpha}{2}\right\}\right| \le C\alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})}$.

It remains to show that

$$\left|\left\{x \notin \Omega^* : |T^{\delta} b(x)| > \frac{\alpha}{2}\right\}\right| \le C\alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})}.$$

Since T^{α} is translation invariant, we may assume that

$$Q_k = \{x : \max |x_j| \le R\}.$$

We now consider

$$\int_{x \notin Q_k^*} \left|2^{l(n+1)} (K_0^{\delta}(2^l) * b_k)(x)\right| dx = \int_{y \in Q_k^*} \int_{x \notin Q_k^*} \left|2^{l(n+1)} K_0^{\delta}(2^l (x-y)) b_k(y)\right| dx\, dy$$

$$\leq \ \| b_k \|_{L^1(\mathbb{R}^{n+1})} \int_{\{x\,:\,\max|x_j| > 2^l R\}} \left| K_0^\delta(x) \right| dx.$$

By using the kernel estimates and $\delta > \dfrac{n-1}{2}$, we have

$$\int_{\{x\,:\,\max|x_j| > 2^l R\}} \left| K_0^\delta(x) \right| dx$$

$$\leq \int\int_{\{|x'| \leq 2^l R,\, |x_{d+1}| > 2^l R\}} \frac{1}{|x'|^{(n+1)/2}} \frac{1}{|x_{n+1}|^{\delta+1}} dx'\, dx_{n+1}$$

$$\leq \int\int_{\{|x'| > 2^l R,\, |x_{n+1}| > 2^l R,\, \| x_{n+1}| - |x'\| \leq 1\}} \frac{1}{|x'|^{\delta+(n+1)/2}} dx'\, dx_{n+1}$$

$$+ \int\int_{\{|x'| > 2^l R,\, |x_{d+1}| > 2^l R,\, \| x_{n+1}| - |x'\| > 1\}} \frac{1}{|x'|^{\delta+(n+1)/2}} \frac{1}{\| x_{n+1}| - |x'\|^N} dx'\, dx_{n+1}$$

$$\leq C(2^l R)^{-\left\{\delta - \frac{n-1}{2}\right\}}.$$

On the other hand, since $\int b_k = 0$, it follows that

$$2^{l(n+1)}(K_0^\delta(2^l) * b_k)(x) = \int_{\mathbb{R}^{d+1}} 2^{l(n+1)} \left\{ K_0^\delta(2^l(x-y)) - K_0^\delta(2^l x) \right\} b_k(y)\, dy.$$

The mean value theorem, and $\delta > \dfrac{n-1}{2}$ to have

$$\int_{x \notin Q_k^*} \left| 2^{l(n+1)}(K_0^\delta(2^l) * b_k)(x) \right| dx$$

$$= \int_{y \in Q_k^*} \int_{x \notin Q_k^*} 2^{l(n+1)} \left| K_0^\delta(2^l(x-y)) - K_0^\delta(2^l x) \right| |b_k(y)|\, dx\, dy$$

$$\leq \int_{y \in Q_k^*} \int_{x \notin Q_k^*} 2^{l(n+1)} \left| \nabla K_0^\delta(2^l x) \right| |2^l y| |b_k(y)|\, dx\, dy$$

$$\leq \ C(2^l R) \ \| b_k \|_{L^1(\mathbb{R}^{n+1})} \int_{\mathbb{R}^{n+1}} 2^{l(n+1)} \left| \nabla K_0^\delta(2^l x) \right| dx$$

$$\leq \ C(2^l R) \ \| b_k \|_{L^1(\mathbb{R}^{n+1})}.$$

Putting (4.5) and (4.6) together and applying the triangle inequality gives

$$\int_{x \notin Q_k^*} \left| (K^\delta * b_k)(x) \right| dx \le C \| b_k \|_{L^1(\mathbb{R}^{n+1})} \left(\sum_{2^l R \ge 1} (2^l R)^{-\left\{\delta - \frac{n-1}{2}\right\}} + \sum_{2^l R < 1} 2^l R \right)$$

$$\le C \| b_k \|_{L^1(\mathbb{R}^{n+1})}.$$

From $b = \sum_{k=1}^{\infty} b_k$, it follows that

$$\left| \left\{ x \notin \Omega^* : |T^\delta b(x)| > \frac{\alpha}{2} \right\} \right| \le C\alpha^{-1} \sum_{k=1}^{\infty} \int_{x \notin Q_k^*} \left| (K^\delta * b_k)(x) \right| dx$$

$$\le C\alpha^{-1} \sum_{k=1}^{\infty} \| b_k \|_{L^1(\mathbb{R}^{n+1})}$$

$$\le C\alpha^{-1} \| f \|_{L^1(\mathbb{R}^{n+1})}.$$

From (4.4) and (4.7), we have

$$|II| \le C \| f \|_{L^1(\mathbb{R}^{n+1})},$$

and thus (4.3) and (4.8) give (4.2). □

Lemma 4.4 For $\delta > 0$, we have

$$\| T^\delta f \|_{L^2(\mathbb{R}^{n+1})} \le C \| f \|_{L^2(\mathbb{R}^{n+1})}.$$

Proof. By Plancherel Theorem (see Section 2), we note that

$$\| T^\delta f \|_{L^2(\mathbb{R}^{n+1})} = \| \widehat{T^\delta f} \|_{L^2(\mathbb{R}^{n+1})}.$$

Since the multiplier $m^\delta(\xi', \xi_{n+1}) = \left(1 - \frac{|\xi'|^2}{|\xi_{n+1}|^2} \right)_+^\delta$ is bounded by 1, we have

$$\| \widehat{T^\delta f} \|_{L^2(\mathbb{R}^{n+1})} = \| m^\delta \hat{f} \|_{L^2(\mathbb{R}^{n+1})}$$

$$\le \| \hat{f} \|_{L^2(\mathbb{R}^{n+1})} = \| f \|_{L^2(\mathbb{R}^{n+1})}. \qquad \square$$

We turn to prove Theorem 2.

Proof of Theorem 2. Applying Lemmas 1 and 2, we interpolate with real method between the results on $L^2(\mathbb{R}^{n+1})$ for $\delta > 0$ and on $L^{1,\infty}(\mathbb{R}^{n+1})$ for $\delta > \frac{n-1}{2}$. By Marcinkiewicz interpolation Theorem (see Section 2.3) with $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ and $0 < \theta < 1$, we obtain the L^p bound of T^δ for $1 < p \leq 2$ and $\delta > \frac{n-1}{2}$. Likewise Theorem 1 we use duality to have the L^p-bound for $2 < p < \infty$. Therefore, we have the desired bound of T^δ for $\delta > \frac{n-1}{2}$ and $1 < p < \infty$. $\square$

Remark. When $p = \infty$, T^δ is unbounded on $L^\infty(\mathbb{R}^{n+1})$, since the kernel of T^δ is not integrable, when $\delta > \frac{n-1}{2}$.

5. Appendix

In this section we study definition and properties of Bessel functions.

Definition A.1 We shall only consider Bessel functions J_k of real order $k > -\frac{1}{2}$(although some of the results can be extended easily to complex numbers k with real part bigger than $-\frac{1}{2}$).

We will define the Bessel function J_k of order k by its Poisson representation formula

$$J_k(z) = \frac{(\frac{z}{2})^2}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} e^{izs}(1-s^2)^k \frac{ds}{\sqrt{1-s^2}},$$

where $k > -\frac{1}{2}$ and $z \in \mathbb{C}$. Among all equipment definitions of Bessel functions, the preceding definition will be the most useful to us. Observe that for t real, $J_k(t)$ is also a real number.

Proposition A.2 Let us summarize a few properties of Bessel functions.

(1) We have the following recurrence formula:

$$\frac{d}{dz}(z^{-k}J_k(z)) = -z^{-k}J_{k+1}(z), \quad k > -\frac{1}{2}.$$

(2) We also have the companion recurrence formula:

$$\frac{d}{dz}(z^k J_k(z)) = z^k J_{k-1}(z), \quad k > -\frac{1}{2}.$$

(3) J_k satisfies the differential equation

$$z^2 f''(z) + z f'(z) + (z^2 - k^2) f(z) = 0.$$

(4) If $k \in \mathbb{Z}^+$, then J_k can be written in the form

$$J_k(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta} e^{-ik\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\sin\theta - k\theta) d\theta.$$

This was taken by Bessel as the definition of these functions for k integer.

(5) For $k > -\frac{1}{2}$ and t real we have the following identity:

$$J_k(t) = \frac{1}{\Gamma(\frac{1}{2})}(\frac{t}{2})^k \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!}.$$

Proof. We first verify property (1). We have

$$\frac{d}{dz}(z^{-k}J_k(z)) = \frac{i}{2^k\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} s e^{izs}(1-s^2)^{k-\frac{1}{2}} ds$$

$$= \frac{i}{2^k\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} \frac{iz}{2} e^{izs} \frac{(1-s^2)^{k+\frac{1}{2}}}{k+\frac{1}{2}} ds$$

$$= -z^{-k}J_{k+1}(z),$$

where we integrated by parts and we used the fact that $\Gamma(x+1) = x\Gamma(x)$. Property (2) can be proved similarly.

Property (3) follows from a direct calculation. A calculation using the definition of the Bessel function gives that the left-hand side of (3) is equal to

$$\frac{2^{-k}z^{k+1}}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{izs}\left((1-s^2)z + 2is(k+\frac{1}{2})\right)(1-s^2)^{k-\frac{1}{2}} ds,$$

which in turn is equal to

$$-i\int_{-1}^{+1} \frac{d}{ds}(e^{isz}(1-s^2)^{k+\frac{1}{2}}) ds = 0.$$

Property (4) can be derived directly from (1). Let

$$G_k(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta}e^{-ik\theta}d\theta,$$

for $k > -1/2$ and $z \in C$. We can show easily that $G_0 = J_0$. If we had

$$\frac{d}{dz}(z^{-k}G_k(z)) = -z^{-k}G_{k+1}(z), \quad z \in C$$

for $k \geq 0$ we would immediately conclude that $G_k = J_k$ for $k \in \mathbb{Z}^+$. We have

$$\frac{d}{dz}(z^{-k}G_k(z)) = -z^{-k}\left(\frac{k}{z}G_k(z) - \frac{dG_k}{dz}(z)\right)$$

$$= -z^{-k}\int_0^{2\pi}\frac{k}{2\pi z}e^{iz\sin\theta}e^{-ikz} - \frac{1}{2\pi}\left(\frac{d}{dz}e^{iz\sin\theta}\right)e^{-ik\theta}d\theta$$

$$= -\frac{z^{-k}}{2\pi}\int_0^{2\pi} i\frac{d}{d\theta}(\frac{e^{iz\sin\theta - ik\theta}}{z}) + (\cos\theta - i\sin\theta)e^{iz\sin\theta}e^{-ik\theta}d\theta$$

$$= -z^{-k}\int_0^{2\pi} e^{iz\sin\theta}e^{-i(k+1)\theta}d\theta = -z^{-k}G_{k+1}(z).$$

Finally, the identity in (5) can be derived by inserting the expression

$$\sum_{j=0}^{\infty}(-j)^j\frac{(ts)^{2j}}{(2j)!} + i\sin(ts)$$

for e^{its} in the definition of the Bessel function $J_k(t)$. Carrying out the algebra gives

$$J_k(t) = \frac{\left(\frac{t}{2}\right)^k}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}(-1)^j\frac{1}{\Gamma(k+\frac{1}{2})}\frac{t^{2j}}{(2j)!}2\int_0^1 s^{2j-1}(1-s^2)^{k-\frac{1}{2}}s\,ds$$

$$= \frac{\left(\frac{t}{2}\right)^k}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}(-1)^j\frac{1}{\Gamma(k+\frac{1}{2})}\frac{t^{2j}}{(2j)!}\frac{\Gamma(j+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(j+k+1)}$$

$$= \frac{\left(\frac{t}{2}\right)^k}{\Gamma(\frac{1}{2})} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!}. \qquad \square$$

Proposition A.3 Let $\mu > -\frac{1}{2}$, $\nu > -1$ and $t > 0$. Then the following identity is valid:

$$\int_0^1 J_u(ts) s^{u+1} (1-s^2)^{\nu} ds = \frac{\Gamma(\nu+1) 2^{\nu}}{t^{\nu+1}} J_{\mu+\nu+1}(t).$$

To prove this identity we use formula (5) in proposition 1. We have

$$\int_0^1 J_u(ts) s^{u+1} (1-s^2)^{\nu} ds$$

$$= \frac{(\frac{t}{2})^{\mu}}{\Gamma(\frac{1}{2})} \int_0^1 \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j+\frac{1}{2}) t^{2j}}{\Gamma(j+\mu+1)(2j)!} s^{2j+\mu+\mu} (1-s^2)^{\nu} s\, ds$$

$$= \frac{1}{2} \frac{(\frac{t}{2})^{\mu}}{\Gamma(\frac{1}{2})} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j+\frac{1}{2}) t^{2j}}{\Gamma(j+\mu+1)(2j)!} \int_0^1 u^{j+\mu} (1-u)^{\nu}\, du$$

$$= \frac{1}{2} \frac{(\frac{t}{2})^{\mu}}{\Gamma(\frac{1}{2})} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j+\frac{1}{2}) t^{2j}}{\Gamma(j+\mu+1)(2j)!} \frac{\Gamma(\mu+j+1)\Gamma(\nu+1)}{\Gamma(j+\mu+\nu+2)(2j)!}$$

$$= \frac{\Gamma(\nu+1) 2^{\nu}}{t^{\nu+1}} J_{\mu+\nu+1}(t). \qquad \square$$

Theorem A.4 Let $d\sigma$ denote surface measure on S^{n-1} for $n \geq 2$. Then the following is true:

$$\widehat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i \xi \cdot \theta} d\theta = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|).$$

We have

$$\widehat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i\xi \cdot \theta} d\theta$$

$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{+1} e^{-2\pi i|\xi| \cdot s}(1-s^2)^{\frac{n-2}{2}} \frac{ds}{\sqrt{1-s^2}}$$

$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{n-2}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{(\pi|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|)$$

$$= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi|\xi|). \qquad \square$$

Theorem A.5 The Fourier Transform of a Radial Function is radial on $\mathbb{R}^n$

Let $f(x) = f_0(|x|)$ be a radial function defined on $\mathbb{R}^n$, where f_0 is defined on $[0,\infty)$. Then the Fourier transform of f is given by the formula

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n}{2}-1}(2\pi r|\xi|) r^{\frac{n}{2}} dr.$$

To obtain this formula, use polar coordinates to write

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

$$= \int_0^\infty \int_{S^{n-1}} f_0(r) e^{-2\pi i\xi \cdot r\theta} d\theta\, r^{n-1} dr$$

$$= \int_0^\infty f_0(r) \widehat{d\sigma}(r\xi) r^{n-1} dr$$

$$= \int_0^\infty \frac{2\pi}{(r|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{n-1} dr$$

$$= \frac{2\pi}{(|\xi|)^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{\frac{n}{2}} dr.$$

As an application we take $f(x) = \chi_{B(0,1)}$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$. we obtain

$$\widehat{\chi_{B(0,1)}}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|\xi| r) r^{\frac{n}{2}} dr = \frac{J_{\frac{n}{2}}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}}},$$

in view of the result in proposition 2. More generally, for $\delta > -1$, let

$$m_\delta(x) = \begin{cases} (1-|x|^2)^\delta & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Then

$$\widehat{m_\delta}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|\xi| r) r^{\frac{n}{2}} (1-r^2)^\delta dr = \frac{\Gamma(\delta+1)}{\pi^\delta} \frac{J_{\frac{n}{2}+\delta}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}+\delta}},$$

also in view of the identity in proposition 2.

Proposition A.5 Here we take $z = r$ a positive real number and we seek the asymptotic behavior $J_k(r)$ as $r \to 0$ and as $r \to \infty$. let us fix $k > -1/2$. The following is true:

$$J_k(r) = \begin{cases} \dfrac{r^k}{2^k \Gamma(k+1)} + O(r^{k+1}) & \text{as } r \to 0 \\ \sqrt{\dfrac{2}{\pi r}} \cos\left(r - \dfrac{\pi k}{2} - \dfrac{\pi}{4}\right) + O(r^{-3/2}) & \text{as } r \to \infty \end{cases}$$

The asymptotic behavior of $J_k(r)$ as $r \to 0$ is rather trivial. We simply need to note that

$$\int_{-1}^{+1} e^{irt} (1-t^2)^{k-\frac{1}{2}} dt = \int_{-1}^{+1} (1-t^2)^{k-\frac{1}{2}} + O(r)$$

$$= \int_0^{\pi} (\sin^2\phi)^{k-\frac{1}{2}} (\sin\phi) d\phi + O(r)$$

$$= \frac{\Gamma(k+1/2)\Gamma(1/2)}{\Gamma(k+1)} + O(r)$$

as $r \to 0$.

The asymptotic behavior of $J_k(r)$ as $r \to \infty$ is more delicate. Consider the region in the complex plane obtained by excluding the rays $(-\infty, -1)$ and $(1, \infty)$. We choose an analytic branch of $(1-z^2)^{k-\frac{1}{2}}$ in this region that is real valued and nonnegative on the interval $[-1,1]$. We integrate the analytic function

$$(1-z^2)^{k-\frac{1}{2}} e^{irz}$$

over the boundary of the rectangle whose lower side is $[-1,1]$ and whose height is $R > 0$.

We obtain

$$i\int_0^R e^{ir(1+)} (t^2-2)^{k-\frac{1}{2}} dt + \int_{-1}^{+1} e^{irt} (1-t^2)^{k-\frac{1}{2}} dt$$

$$+ \int_R^0 e^{ir(-1+it)} (t^2+2it)^{k-\frac{1}{2}} dt + \varepsilon(R) = 0,$$

where $\varepsilon(R) \to 0$ as $R \to \infty$. It follows that

$$I = \int_{-1}^{+1} e^{irt} (1-t^2)^{k-\frac{1}{2}} dt = I_+ + I_-,$$

where

$$I_+ = +ie^{ir} \int_0^{\infty} e^{-rt} (t^2+2it)^{k-\frac{1}{2}} dt, \qquad \text{and}$$

$$I_{-} = -ie^{ir}\int_0^{\infty} e^{-rt}(t^2+2it)^{k-\frac{1}{2}}dt$$

Next we observe that

$$(t^2+2it)^{k-\frac{1}{2}} = -i(2t)^{k-\frac{1}{2}}e^{i(\frac{k\pi}{2}+\frac{\pi}{4})}+\phi_{+}(t)$$

$$(t^2-2it)^{k-\frac{1}{2}} = +i(2t)^{k-\frac{1}{2}}e^{i(\frac{k\pi}{2}+\frac{\pi}{4})}+\phi_{-}(t),$$

where $|\phi_{+}(t)|+|\phi_{-}(t)| \le Ct^{k+\frac{1}{2}}$. Note that the Laplace transform

$$\mathcal{L}(f)(r) = \int_0^{\infty} f(t)e^{-tr}dt$$

of the function t^b is $r^{-b-1}\Gamma(b+1)$ when $b>-\frac{1}{2}$, and that the functions ϕ_{+} and ϕ_{-} have Laplace transforms bounded by a constant multiple of $r^{-k-3/2}$. Therefore, we obtain

$$I_{+} = (-i)(+i)e^{-ir}e^{i(\frac{k\pi}{2}+\frac{\pi}{4})}r^{-r-\frac{1}{2}}2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})+O(r^{-k-3/2})$$

$$I_{-} = (+i)(-i)e^{ir}e^{-i(\frac{k\pi}{2}+\frac{\pi}{4})}r^{-r-\frac{1}{2}}2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2})+O(r^{-k-3/2})$$

as $r\to\infty$. Adding these two last inequalities and multiplying by the missing factor $\dfrac{(\frac{r}{2})^k}{\Gamma(k+\frac{1}{2})}\Gamma(\frac{1}{2})$, we obtain the equality

$$J_k(r) = \sqrt{\frac{2}{\pi r}}\cos(r-\frac{\pi k}{2}-\frac{\pi}{4})+O(r^{-3/2})$$

as $r\to\infty$.

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