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敎育學碩士(數學敎育)學位論文

리만 흰곱多樣體 위의 흰函數의 非存在性

朝鮮大學校 敎育大學院

數學敎育專攻

梁 允 華

리만 휜곱多樣體 위의 휜函數의 非存在性

The Nonexistence of warping functions on Riemannian warped product manifolds.

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指導敎授 鄭 潤 泰

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리만 흰곱多樣體 위의 흰函數의 非存在性

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미분기하학에서 기본적인 문제 중의 하나는 미분다양체가 가지고 있는 곡률 함수에 관한 연구이다.

연구방법으로는 종종 해석적인 방법을 적용하여 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

Kazdan and Warner([11, 12, 13])의 결과에 의하면 N위의 함수 f가 N위의 Riemannian metric의 scalar curvature가 되는 세 가지 경우의 타입이 있는데 먼저

(A) N위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 적당한 점에서 $f(x_0)<0$일 때이다. 즉 N위에 negative constant scalar curvature를 갖는 metric이 존재하는 경우이다.

(B) N위의 함수 f가 Riemannian metric의 scalar curvature이면 그 함수 f가 항등적으로 $f\equiv 0$이거나 모든 점에서 $f(x)<0$인 경우이다. 즉, N위에서 zero scalar curvature를 갖는 Riemannian metric이 존재하는 경우이다.

(C) N위의 어떤 f라도 positive constant curvature를 갖는 Riemannian metric이 존재하는 경우이다.

본 논문에서는 엽다양체 N이 (C)에 속하는 compact Riemannian manifold

일 때, Riemannian warped product manifold인 $M=[a,\infty)\times_f N$위에 함수$R(t,x)$가 어떤 조건을 만족하면 $R(t,x)$가 Riemannian warped product metric의 scalar curvature가 될 수 있는 warping function $f(x)$가 존재할 수 없음에 관하여 연구하고자 한다.

1. Introduction

One of the basic problems in the differential geometry is to study the set of curvature function over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwise conformal to some metric with constant scalar curvature or not.

In a recent study ([10]), Jung and Kim have studied the problem of scalar curvature functions on Lorentzian warped product manifolds and obtaind partial results about the existence and nonexistence of Lorentzian warped metric with some prescribed scalar curvature function. In this paper, we study also the existence and nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds.

By the results of Kazdan and Warner ([11], [12], [13]), if N is a compact Riemannian $n-$manifold without boundary $n \geq 3$, then N belongs to one of the following three catagories :

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian

metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [11], [12] and [13] Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([9]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([9], [16, p.322]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative

scalar curvature ([6]). It follows from the results of Aviles and McOwen ([1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In [14] and [15], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric.

In this paper, when N is a compact Riemannian manifold, we consider the nonexistence of warping functions on a warped product manifold $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. That is, it is shown that if the fiber manifold N belongs to class (C) then M does not admit a Riemannian metric with some positive scalar curvature near the end outside a compact set.

2. Preliminaries

First of all, in order to induce a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1. *Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\Im(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that

(D1) $\nabla_V W$ *is* $\Im$ *-linear in* V,

(D2) $\nabla_V W$ *is* R*-linear in* W,

(D3) $\nabla_V(fW) = (Vf)W + f\nabla_v W \quad$ *for* $f \in \Im(M)$,

(D4) $[V, W] = \nabla_V W - \nabla_W V$, *and*

(D5) $X < V, W >=< \nabla_X V, W > + < V, \nabla_X W >$ *for all* $X, V, W \in \mathfrak{X}(M)$.

If ∇ satisfies axioms (D1)$\sim$(D3), then $\nabla_V W$ is called the *covariant derivative* of W with respect to V for the connection ∇. If ∇ satisfies axioms

(D4)$\sim$ (D5), then ∇ is called the *Levi - Civita connection* of M, which is characterized by the *Koszul formula* ([17]).

A geodesic $c : (a, b) \to M$ is a smooth curve of M such that the tangent vector $c^{'}$ moves by parallel translation along c. In order words, c is a geodesic if

$$\nabla_{c^{'}} c^{'} = 0 \quad \text{(geodesic equation)}.$$

A pregeodesic is a smooth curve c which may be reparametrized to be a geodesic. Any parameter for which c is a geodesic is called an affine parameter. If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be complete if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c^{'}} c^{'} = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U, (x^1, x^2, ..., x^n))$ be local coordinates on M and let $\dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^2}, ..., \dfrac{\partial}{\partial x^n}$ denote the natural basis with respect to these coordinates.

The connection coefficients Γ_{ij}^k of ∇ with respect to $(x^1, x^2, ..., x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{(connection coefficients)}.$$

Using these coefficients, we may write equation as the system

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \quad \text{(geodesic equations in coordinates).}$$

Definition 2.2. *The curvature tensor of the connection ∇ is a linear transformation valued tensor R in $Hom(\mathfrak{X}(M), \mathfrak{X}(M))$ defined by :*

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathfrak{X}(M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

It is well-known that $R(X,Y)Z$ at p depends only upon the values of X, Y and Z at p ([17]).

If $w \in T_p^*(M)$ is a cotangent vector at p and $x, y, z \in T_p(M)$ are tangent vectors at p, then one defines

$$R(\omega, X, Y, Z) = (\omega, R(X,Y)Z) = \omega(R(X,Y)Z)$$

for X, Y and Z smooth vector fields extending x, y and z, respectively.

The curvature tensor R is a (1,3) tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R_{jkm}^i \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m,$$

where the curvature components R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

Notice that $R(X,Y)Z = -R(Y,X)Z$, $R(\omega, X, Y, Z) = -R(\omega, Y, X, Z)$ and $R^i_{jkm} = -R^i_{jmk}$.

Furthermore, if $X = \sum \frac{x^i \partial}{\partial x^i}$, $Y = \sum \frac{y^i \partial}{\partial x^i}$, $Z = \sum \frac{z^i \partial}{\partial x^i}$ and $\omega = \sum \omega_i dx^i$ then

$$R(X,Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} z^j x^k y^m \frac{\partial}{\partial x^i}$$

and

$$R(w, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} w_i z^j x^k y^m.$$

Consequently, one has $R\left(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}\right) = R^i_{jkm}$.

Definition 2.3. *From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the Ricci tensor. Its components are $R_{ij} = \sum_{k=1}^{n} R^k_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{ij=1}^{n} R_{ij} g^{ij}$ is called the scalar curvature* ([2], [3], [4]).

Definition 2.4. *Suppose Ω is a smooth, bounded domain in R^n, and let $g : \Omega \times R \to R$ be a Caratheodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation*

$$\Delta u = g(x,u) \quad in \quad \Omega$$

$$u = u_0 \quad on \quad \partial\Omega$$

$u \in H^{1,2}(\Omega)$ is a (weak) sub-solution if $u \leq u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega g(x,u)\varphi dx \leq 0 \quad for\ all \quad \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0.$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) super-solution if in the above the reverse inequalities hold.

We briefly recall some results on warped product manifolds. Complete details may be found in [3] or [17]. On a semi-Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.5. *The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^* g_F$$

where g_B and g_F are metric tensors of B and F, respectively. In order words, if v is tangent to M at (p,q), then

$$g(v,v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber ([17]).

We denote the metric g by $\langle \quad , \quad \rangle$. In view of Remark 2.13 (1) and Lemma 2.14, we may also denote the metric g_B by $\langle \quad , \quad \rangle$. The metric g_F will be denoted by $(\quad , \quad)$.

Remark 2.6. *Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows* :

(1) For each $q \in F$, the map $\pi \mid_{\sigma^{-1}(q)=B\times q}$ is an isometry onto B.

(2) For each $p \in B$, the map $\sigma \mid_{\pi^{-1}(p)=p\times F}$ is a positive homothetic map onto F with homothetic factor $\frac{1}{f(p)}$.

(3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q).

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.

(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p,q) by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is π-related to X and σ-related to the zero vector field on F. Similarly, if Y is a vector field of F, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

Lemma 2.7. *If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B.*

Proof. We must show that $grad(h \circ \pi)$ is horizonal and π-related to $grad(h)$ on B. If v is vertical tangent vector to M, then

$$\langle grad(h \circ \pi), v\rangle = v(h \circ \pi) = d\pi(v)h = 0, \quad \text{since} \quad d\pi(v) = 0.$$

Thus $grad(h \circ \pi)$ is horizonal. If x is horizonal,

$$\langle d\pi(grad(h \circ \pi), d\pi(x)\rangle = \langle grad(h \circ \pi), x\rangle = x(h \circ \pi) = d\pi(x)h$$

$$= \langle grad(h), d\pi(x)\rangle.$$

Hence at each point, $d\pi(grad(h \circ \pi)) = grad(h)$. ■

In view of Lemma 2.14, we simplify the notations by writing h for $h \circ \pi$ and $grad(h)$ for $grad(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to

M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a $(1,s)$-tensor, and if $v_1, v_2, ..., v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, ..., v_s) = A(d\pi(v_1), ..., d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in [5].

Now we recall the formula for the Ricci curvature tensor Ric on the warped product manifold $M = B \times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.8. *On a warped product manifold $M = B \times_f F$ with $n = dim F > 1$, let X, Y be horizontal and V, W vertical.*

Then

(1) $Ric(X,Y) = Ric^B(X,Y) - \frac{n}{f}H^f(X,Y)$,

(2) $Ric(X,V) = 0$,

(3) $Ric(V,W) = Ric^F(V,W) - \langle V,W\rangle f^\sharp$,

where $f^\sharp = \dfrac{\Delta f}{f} + (n-1)\dfrac{\langle grad(f), grad(f)\rangle}{f^2}$ *and* $\Delta f = trace(H^f)$ *is the Laplacian on B.*

Proof. See Corollary 7.43 in [17]. ■

On the given warped product manifold $M = B \times_f F$, we also write S^B for the pullback by π of the scalar curvature S_B of B and similarly for S^F. From now on, we denote $grad(f)$ by ∇f.

Lemma 2.9. *If S is the scalar curvature of $M = B \times_f F$ with $n = dim F > 1$, then*

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2} \tag{2.1}$$

where Δ is the Laplacian on B.

Proof. For each $(p, q) \in M = B \times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)}M$. We can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 = \langle \overline{d_j}, \overline{d_j} \rangle = f(p)^2(d_j, d_j) = (f(p)d_j, f(p)d_j)$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF. By Lemma 2.8 (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i}),$$

and

$$Ric(\overline{d_j},\overline{d_j}) = Ric^F(\overline{d_j},\overline{d_j}) - f^2(p)g_F(d_j,d_j)\left(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}\right).$$

Hence, for $\epsilon_\alpha = g(\epsilon_\alpha, \epsilon_\alpha)$

$$\begin{aligned} S(p,q) &= \sum_\alpha \epsilon_\alpha R_{\alpha\alpha} \\ &= \sum_i \epsilon_i Ric(\overline{e_i},\overline{e_i}) + \sum_j \epsilon_j Ric(\overline{d_j},\overline{d_j}) \\ &= S^B(p,q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2} \end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$.

■

3. Main Results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f defined to be the product manifold $([a, \infty) \times_f N, g^{'})$ with

$$g^{'} = dt^2 + f^2(t)g. \tag{3.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of $g^{'}$ is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}[R(g)(x) - 2nf(t)f^{''}(t) - n(n-1)|f^{'}(t)|^2] \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$ (For details, [7] or [9]).

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u^{''}(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary.

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^\alpha$ in (3.3), where $\alpha > 1$ is a constant. We have

$$R(t,x) \leq \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > \alpha.$$

Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [8], we have the following theorem.

Theorem 3.1. *For $n \geq 3$, let $M = [a,\infty) \times_f N$ be the Riemannian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a geodesically complete Riemannian metric of negative scalar curvature outside a compact set.*

Here the following lemma plays an important role in this paper, whose proof is similar to that of Lemma 1.8 in [15].

Lemma 3.2. *On $[a,\infty)$, there does not exist a positive solution $u(t)$ such that*

$$t^2 u''(t) + \frac{c}{4}u(t) \leq 0 \quad \textit{for} \quad t \geq t_0, \tag{3.4}$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that $u(t)$ satisfies

$$t^2 u^{''}(t) + \frac{c}{4} u(t) \leq 0 \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Let

$$u(t) = t^{\alpha} v(t), \quad t \geq t_0$$

where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u^{''}(t) = \alpha(\alpha - 1) t^{\alpha - 2} v(t) + 2\alpha t^{\alpha - 1} v^{'}(t) + t^{\alpha} v^{''}(t).$$

And we obtain

$$t^{\alpha} v(t) \left[\alpha(\alpha - 1) + \frac{c}{4} \right] + 2\alpha t^{\alpha + 1} v^{'}(t) + t^{\alpha + 2} v^{''}(t) \leq 0. \tag{3.5}$$

Let δ be a positive constant such that $\delta^2 = \dfrac{c-1}{4}$. Then we have

$$\alpha(\alpha - 1) + \frac{c}{4} = \left(\alpha - \frac{1}{2} \right)^2 + \frac{c-1}{4} \geq \delta^2.$$

Then δ is a constant independent on α. Equation (3.5) gives

$$2\alpha t v^{'}(t) + t^2 v^{''}(t) \leq -\delta^2 v(t). \tag{3.6}$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta < 1$, that is, $\alpha < \frac{1}{2}$. Then equation (3.6) becomes

$$\left(t^{\beta} v^{'}(t) \right)^{'} \leq -\frac{\delta^2 v(t)}{t^{2-\beta}}.$$

Upon integration we have

$$t^{\beta}v^{'}(t) - \tau^{\beta}v^{'}(\tau) \leq -\int_{\tau}^{t} \frac{\delta^2 v(s)}{s^{2-\beta}} ds, \quad t > \tau > t_0. \tag{3.7}$$

Here we have two following cases :

[case1] If $v^{'}(\tau) \leq 0$ for some $\tau > t_0$, then (3.7) implies that

$$t^{\beta}v^{'}(t) \leq -C$$

for some positive constant C. We have

$$v(t) \leq v(\tau) - \int_{\tau}^{t} \frac{C}{s^{\beta}} ds = v(\tau) - C\frac{s^{1-\beta}}{1-\beta} \mid_{\tau}^{t} \to -\infty,$$

as $\beta < 1$. Hence $v(t) < 0$ for some t, contradicting that $v(t) > 0$ for all $t \geq t_0$.

[case2] We have $v^{'}(\tau) > 0$ for all $\tau > t_0$. Equation (3.7) implies that

$$\tau^{\beta}v^{'}(\tau) - \int_{\tau}^{t} \frac{\delta^2 v(s)}{s^{2-\beta}} ds \geq 0$$

for all $t > \tau > t_0$. As $v^{'}(t) > 0$ for all $t > t_0$, we have

$$\tau^{\beta}v^{'}(\tau) \geq v(\tau) \int_{\tau}^{t} \frac{\delta^2}{s^{2-\beta}} ds = v(\tau)[\frac{1}{s^{1-\beta}}[-\frac{\delta^2}{1-\beta}]] \mid_{\tau}^{t} .$$

Let $t \to \infty$ we have

$$\tau^{\beta}v^{'}(\tau) \geq \frac{v(\tau)}{\tau^{1-\beta}} \frac{\delta^2}{1-\beta}.$$

Or after changing the parameter we have

$$\frac{v^{'}(t)}{v(t)} \geq \frac{1}{t}\frac{\delta^2}{1-\beta}, \quad t > t_0.$$

Choose $\alpha < \frac{1}{2}$ close to $\frac{1}{2}$ so that $\beta < 1$ is close to 1. Using the fact that δ is independent on α or β, we have

$$\frac{v^{'}(t)}{v(t)} \geq \frac{N}{t}$$

for a big integer $N > 2$. This gives

$$v(t) \geq Ct^N, \quad t > t_0$$

where C is a positive constant. The inequality (3.7) implies that

$$t^{\beta}v^{'}(t) \leq \tau^{\beta}v^{'}(\tau) - \int_{\tau}^{t} \frac{C\delta^2 s^N}{s^{2-\beta}} ds \to -\infty \quad \text{as} \quad t \to \infty.$$

Thus $v^{'}(t) < 0$ for t large, which is also a contradiction. Hence there is no solution to equation (3.4). ■

From now on, we assume that $R(t, x)$ is the function of only t-variable. Then we have the following theorems.

Theorem 3.3. *If N belongs to class (A) or (B), that is, $R(g) \leq 0$, then there is no positive solution to equation* (3.3) *with*

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \textit{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Equation (3.3) gives

$$t^2 u''(t) + \frac{c}{4}u(t) \leq 0.$$

By Lemma 3.2, we complete the proof. ■

If N belongs to (A), then a negative constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = -\dfrac{4n}{n+1}k^2$, where k is a positive constant. Then equation (3.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2 u(t)^{1-\frac{4}{n+1}} + R(t,x)u(t) = 0. \tag{3.8}$$

In order to prove the nonexistence of some warped product metric, we need the following lemma.

Lemma 3.4. *Let $u(t)$ be a positive smooth function on $[a, \infty)$. If $u(t)$ satisfies*

$$\frac{u^{''}(t)}{u(t)} \leq \frac{C}{t^2}$$

for some constant $C > 0$, then there exists $t_0 > a$ such that for all $t > t_0$.

$$u(t) \leq C_0 t^{\epsilon}$$

for some positive constant C_0 and $\epsilon > 1$.

Proof. Since $C > 0$, we can choose $\epsilon > 1$ such that $\epsilon(\epsilon - 1) = C$. Then from the hypothesis, we have

$$t^{\epsilon} u^{''}(t) \leq \epsilon(\epsilon - 1) t^{\epsilon - 2} u(t).$$

Upon integration from $t_1 (\geq a)$ to $t (> t_1 \geq a)$, and using twice integration by parts, we obtain

$$t^{\epsilon} u^{'}(t) - \epsilon t^{\epsilon - 1} u(t) - t_1^{\epsilon} u^{'}(t_1) + \epsilon {t_1}^{\epsilon - 1} u(t_1) + \epsilon(\epsilon - 1) \int_{t_1}^{t} s^{\epsilon - 2} u(s) ds$$

$$\leq C \int_{t_1}^{t} s^{\epsilon - 2} u(s) ds.$$

Therefore we have

$$t^{\epsilon} u^{'}(t) - \epsilon t^{\epsilon - 1} u(t) \leq t_1^{\epsilon} u^{'}(t_1) - \epsilon t_1^{\epsilon - 1} u(t_1). \tag{3.9}$$

We consider two following cases :

[Case 1] There exists $t_1 \geq a$ such that $u'(t_1) \leq 0$. If there is a number $t_1 \geq a$ such that $u'(t_1) \leq 0$, then we have

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq 0.$$

This gives

$$(\ln u(t))' \leq \epsilon(\ln t)'.$$

Hence

$$u(t) \leq c_1 t^{\epsilon}$$

for all $t > t_1$, where c_1 is a positive constant.

[Case 2] There does not exist $t_1 \geq a$ such that $u'(t_1) \leq 0$.

In other words, if $u'(t) > 0$ for all $t \geq a$, then $u(t) \geq c'$ for some positive constant c'. Let c_2 be a positive constant such that

$$t_1^{\epsilon}u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1) \leq c_2,$$

then equation (3.9) gives

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq c_2$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \leq \frac{\epsilon}{t} + \frac{c_2}{u(t)t^{\epsilon}} \leq \frac{\epsilon}{t} + \frac{c_2}{c't^{\epsilon}}.$$

Integrating from t_1 to t we have

$$\ln\frac{u(t)}{u(t_1)} \leq \epsilon \ln\left(\frac{t}{t_1}\right) + \frac{c_2}{(\epsilon-1)c't_1^{\epsilon-1}} \leq \epsilon \ln\left(\frac{c_3 t}{t_1}\right),$$

as $\epsilon > 1$. Here c_3 is a positive constant such that $\ln c_3 \geq \frac{c_2}{\epsilon(\epsilon-1)c't_1^{\epsilon-1}}$.

Hence we again obtain the inequality

$$u(t) \leq bt^{\epsilon}$$

for some positive constant b and for all $t \geq t_1$.

Thus from two cases we always find $t_0 > a$ and a constant $C_0 > 0$ such that

$$u(t) \leq C_0 t^{\epsilon}$$

for all $t \geq t_0$.

■

Using the above lemma, we can prove the following theorem about the nonexistence of warping function, whose proof is similar to that of Lemma 3.3 in [15].

Theorem 3.5. *Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n(\geq 3)$. We may assume that $R(g) = -\frac{4n}{n+1}k^2$, where k is a positive constant. On $[a, \infty) \times N$, there does not exist a Riemannian warped product metric*

$$g^{'} = dt^2 + f^2(t)g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

Proof. Assume that we can find a Riemannian warped product metric on $[a, \infty) \times N$ with

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$. In equation (3.3), we have

$$\frac{4n}{n+1}\left[\frac{u^{''}(t)}{u(t)} + \frac{k^2}{u(t)^{\frac{4}{n+1}}}\right] = -R(t) \leq \frac{n(n-1)}{t^2} \tag{3.10}$$

and

$$\frac{u^{''}(t)}{u(t)} \leq \frac{\frac{(n-1)(n+1)}{4}}{t^2}. \tag{3.11}$$

In equation (3.11), we can apply Lemma 3.4 and take $\epsilon = \frac{n+1}{2}$. Hence we have $t_0 > a$ such that

$$u(t) \leq c_0 t^{\frac{n+1}{2}}$$

for some positive constants c_0 and all $t > t_0$. Then

$$\frac{k^2}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{'}}{t^2}$$

where $0 < c^{'} \leq \frac{k^2}{c_0^{\frac{4}{n+1}}}$ is a positive constant. Hence equation (3.10) gives

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n+1)(n-1)-\delta}{4t^2},$$

where $4c^{'} \geq \delta \geq 0$ is a constant. We can choose $\delta^{'} > 0$ such that

$$\frac{(n+1)(n-1)-\delta}{4} = \left(\frac{n+1}{2} - \delta^{'}\right)\left(\frac{n-1}{2} - \delta^{'}\right)$$

for small positive δ. Applying the Lemma 3.4 again, we have $t_1 > a$ such that

$$u(t) \leq c_1 t^{\frac{n+1}{2}-\delta^{'}}$$

for some $c_1 > 0$ and all $t > t_1$. And

$$\frac{k^2}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{''}}{t^{2-\epsilon}}, \tag{3.12}$$

where $\epsilon = \dfrac{4}{n+1}\delta^{'}$ and $0 < c^{''} \leq \dfrac{k^2}{c_1^{\frac{4}{n+1}}}$. Thus equation (3.11) and (3.12) give

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n-1)(n+1)}{4t^2} - \frac{c^{''}}{t^{2-\epsilon}},$$

which implies that

$$u^{''}(t) \leq 0$$

for t large. Hence $u(t) \leq c_2 t$ for some constant $c_2 > 0$ and large t. From equation (3.10) we have

$$\frac{u^{''}(t)}{u(t)} \leq \frac{-c_3}{t^{\frac{4}{n+1}}} + \frac{(n+1)(n-1)}{4t^2} \leq -\frac{c_3}{t}$$

for t large enough, as $n \geq 3$. Here c_3 is a positive constant. Multiplying $u(t)$ and integrating from $t^{'}$ to t, we have

$$u^{'}(t) - u^{'}(t^{'}) \leq -c_3 \int_{t^{'}}^{t} \frac{u(s)}{s} ds, \quad t > t^{'}.$$

We consider two following cases :

[Case 1] There exists $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$.

If $u^{'}(t^{'}) \leq 0$ for some $t^{'}$, then $u^{'}(t) \leq -c_4$ for some positive constant c_4. Hence $u(t) \leq 0$ for t large enough, contradicting the fact that u is positive.

[Case 2] There does not exist $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$.

In order words, if $u^{'}(t) > 0$ for all t large, then $u(t)$ is increasing, hence

$$\int_{t^{'}}^{t} \frac{u(s)}{s} ds \geq u(t^{'}) \int_{t^{'}}^{t} \frac{1}{s} ds \to \infty.$$

Thus $u^{'}(t)$ has to be negative for some t large, which is a contradiction to the hypothesis.

Therefore there does not exist such warped product metric. ■

If N belongs to class (C), then by the results of Kazdan and Warner ([11], [12], [13]), some positive constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = \frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (3.3) becomes

$$\frac{4n}{n+1}u''(t) - \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} + R(t,x)u(t) = 0. \tag{3.13}$$

If $R(t,x) = R(t)$ is the bounded function of only t-variable, our first main theorem is as follows :

Theorem 3.6. *Suppose that* $R(g) = \frac{4n}{n+1}k^2$ *for* $n \geq 3$ *and* $R(t,x) = R(t) \in C^\infty([a,\infty))$. *Assume that for* $t > t_0$, *there exists a positive solution* $u(t)$ *of equation* (3.13) *with* $0 > R(t) > -M$ *for some positive constant* M. *Then* $u(t) \geq t^\alpha$ *for large* t *and all* $\alpha > 0$.

Proof. Suppose $u(t) > 0$ satisfies equation (3.13), i.e.,

$$\frac{4n}{n+1}u''(t) = -R(t)u(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}}. \tag{3.14}$$

Since $R(t) \leq 0$ and $\dfrac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} > 0$, integrating equation (3.14) from $\tau(\geq a)$ to t, we have

$$u^{'}(t) - u^{'}(\tau) > 0$$

for all $t(> \tau)$.

Here we have two following cases :

[Case 1] There exists $\tau(\geq a)$ such that $u^{'}(\tau) \geq 0$. Then $u^{'}(t) \geq 0$, so $u(t)$ is an increasing function. Thus $u(t) \geq u(\tau) > 0$. Therefore from equation (3.14) we have

$$u^{''}(t) \geq k^2u(t)^{1-\frac{4}{n+1}} \geq c_0$$

for large t and some positive constant c_0. Hence we have

$$u(t) \geq \frac{c_0}{2}t^2 + c_1t + c_2 \tag{3.15}$$

for some constants c_1, c_2. Again substituting equation (3.15) to equation (3.14), we have

$$u^{''}(t) > k^2\left(\frac{c_0}{2}t^2 + c_1t + c_2\right)^{1-\frac{4}{n+1}}$$

$$u(t) > c_3t^{2+2\left(1-\frac{4}{n+1}\right)}$$

for some positive constant c_3. Reiterating this method, we complete the theorem.

[Case 2] There does not exist $\tau(\geq a)$ such that $u^{'}(\tau) \geq 0$. In other words, we have $u^{'}(t) < 0$ for all $t(\geq a)$. Then $u(t)$ is a decreasing function. If $u(t) \geq c_0$ for some positive constant c_0, then by [Case 1] our theorem holds. Otherwise, since $u(t)$ is a decreasing function, $u(t) \to +0$ as $t \to \infty$. From equation (3.14) $R(t)$ is bounded and $1 - \frac{4}{n+1} > 0$, so $u^{''}(t) \to +0$ as $t \to \infty$ and $u^{'}(t)$ is an increasing function.

Put $u(t) = e^{-g(t)}$. Then $u^{'}(t) = -e^{-g(t)}g^{'}(t) < 0$, so $g^{'}(t) > 0$ and $g(t)$ is an increasing function. Thus we have

$$u^{''}(t) = e^{-g(t)}((g^{'}(t))^2 - g^{''}(t)) \to +0$$

as $t \to \infty$, so $g^{''}(t) > 0$ because $\frac{(g^{'}(t))^2}{e^{g(t)}} \approx \frac{2g^{'}(t)g^{''}(t)}{e^{g(t)}g^{'}(t)} = \frac{2g^{''}(t)}{e^{g(t)}}$ by L'Hospital's Theorem.

Therefore from equation (3.14) we have

$$(g^{'}(t))^2 - g^{''}(t) = -\frac{n+1}{4n}R(t) + k^2 e^{\frac{4}{n+1}g(t)}.$$

Since $g^{''}(t) > 0$ and $R(t) \leq 0$, we have

$$(g^{'}(t))^2 \geq k^2 e^{\frac{4}{n+1}g(t)}.$$

And since $g^{'}(t) > 0$, we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}g(t)}. \tag{3.16}$$

Since $g(t)$ is increasing, from equation (3.16) $g^{'}(t) \geq c_0$ for some positive constant c_0. Hence we have

$$g(t) \geq c_0 t + c_1 \tag{3.17}$$

for some constant c_1. Again, substituting equation (3.17) to equation (3.16), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}(c_0 t + c_1)}$$

and, integrating the above equation, we have

$$g(t) \geq a_1 e^{\frac{2}{n+1} c_0 t} \tag{3.18}$$

for some positive constants a_1 and c_0. Again plugging equation (3.18) into equation (3.16), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1} a_1 e^{\frac{2}{n+1} c_0 t}}$$

and, integrating the above equation, we have

$$g(t) \geq a_2 e^{\frac{2}{n+1} a_1 e^{\frac{2}{n+1} c_0 t}}$$

for some positive constant a_2. And again, iterating this way, we have

$$g(t) \geq a_n e^{\frac{2}{n+1} a_{n-1} e^{\frac{2}{n+1} \cdot^{\cdot^{\cdot} e^{a_1 e^{\frac{2}{n+1} c_0 t}}}}},$$

which is impossible.

From [Case 1] and [Case 2], we complete the theorem. ■

If $R(t,x)$ is also the function of only t-variable, our second main theorem is as follows :

Theorem 3.7. *Suppose that* $R(g) = \frac{4n}{n+1}k^2$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a function such that*

$$-\frac{n(n-1)}{t^2} \leq R(t) < 0 \quad \textit{for} \quad t > t_0,$$

where $t_0 > a$ *and* $1 \leq C$ *is a constant. Then equation* (3.13) *has no positive solution on* $[a,\infty)$.

Proof. Assume that for $t > t_0$, there exists a solution $u(t)$ of equation (3.13) with $0 > R(t) > -M$. Then by Theorem 3.6 u(t) is an increasing function such that $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$. From equation (3.13), we have

$$\frac{4n}{n+1}\frac{u''(t)}{u(t)} = -R(t) + \frac{4n}{n+1}k^2 u(t)^{-\frac{4}{n+1}} \leq \frac{C}{t^2}$$

for some constant $C \geq 1$ and large t. Hence the Lemma 3.4 implies that for all large t

$$u(t) \leq C_0 t^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$, which is a contradiction to the fact that $u(t) \geq t^{\alpha}$ for large t and all $\alpha > 0$. Therefore equation (3.13) has no positive solution on $[a, \infty)$. ■

References

[1] P. Aviles and R. McOwen, *Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds*, DiR. Geom. 27 (1998), 225-239.

[2] T. Aubin *Nonlinear analysis on manifolds. Monge - Ampere equations*, Springer - verlag New York Heidelberg Berlin.

[3] J.K. Beem and P.E. Ehrlich, *"Global Lorentzian Geometry"*, Pure and Applied Mathematics, Vol. 67, Dekker, New York, 1981

[4] J.K. Beem , P.E. Ehrlich and K.L. Easley, *"Global Lorentzian Geometry"*, Pure and Applied Mathematics, Vol. 202, Dekker, New York, 1996

[5] J.K. Beem , P.E. Ehrlich and Th.G. Powell, *Warped product manifolds in relativity*, Selected Studies (Th.M. Rassias, G.M. Rassias, eds.), North-Holland, 1982, 41-56.

[6] J. Bland and M. Kalka, *Negative scalar curvature metrics on noncompact manifolds*, Trans. A.M.S.316(1989), 433-446.

[7] F. Dobarro and E. Lami Dozo, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. I.H.E.S. 58 (1893), 295-408.

[8] P. E. Ehrlich, Yoon-Tae Jung and Seon-Bu Kim, *Constant scalar*

curvatures on warped product manifolds, Tsukuba J. Math. Vol. 20 no.1 (1996), 239-256.

[9] M. Gromov and H.B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Math. I.H.E.S. 58(1983), 295-408.

[10] Y.T. Jung and I.H. Kim, *The Nonexistence of Warping Functions On Space-Times*, Honam Mathematical J. 30(2008), No.4, 693-701.

[11] J.L. Kazdan and F.W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. DiR.Geo. 10(1975), 113-134.

[12] J.L. Kazdan and F.W. Warner, *Existence and conformal deformation of metrics with prescribed Guassian and scalar curvature*, Ann. of Math. 101(1975), 317-331.

[13] J.L. Kazdan and F.W. Warner, *Curvature functions for compact* 2-*manifolds*, Ann. of Math. 99(1974), 14-74.

[14] M.C. Leung, *Conformal scalar curvature equations on complete manifolds*, Comm. in P.D.E. 20(1995), 367-417.

[15] M.C. Leung, *Conformal deformation of warped products and scalar curvature functions on open manifolds*, preprint.

[16] H.B. Lawson and M.Michelsohn, *"Spin Geometry"*, Princeton University Press, Princeton, 1989.

[17] B. O'Neill. *Semi-Riemannian geometry with applocations to relativity*, Academic Press, New York, (1983).

저작물 이용 허락서

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논문제목	한글 : 리만 휜곱多樣體 위의 휜函數의 非存在性 영어 : The Nonexistence of warping functions on Riemannian warped product manifolds.				

본인이 저작한 위의 저작물에 대하여 다음과 같은 조건아래 조선대학교가 저작물을 이용할 수 있도록 허락하고 동의합니다.

- 다 음 -

1. 저작물의 DB구축 및 인터넷을 포함한 정보통신망에의 공개를 위한 저작물의 복제, 기억장치에의 저장, 전송 등을 허락함
2. 위의 목적을 위하여 필요한 범위 내에서의 편집·형식상의 변경을 허락함. 다만, 저작물의 내용변경은 금지함.
3. 배포·전송된 저작물의 영리적 목적을 위한 복제, 저장, 전송 등은 금지함.
4. 저작물에 대한 이용기간은 5년으로 하고, 기간종료 3개월 이내에 별도의 의사표시가 없을 경우에는 저작물의 이용기간을 계속 연장함.
5. 해당 저작물의 저작권을 타인에게 양도하거나 또는 출판을 허락을 하였을 경우에는 1개월 이내에 대학에 이를 통보함.
6. 조선대학교는 저작물의 이용허락 이후 해당 저작물로 인하여 발생하는 타인에 의한 권리 침해에 대하여 일체의 법적 책임을 지지 않음
7. 소속대학의 협정기관에 저작물의 제공 및 인터넷 등 정보통신망을 이용한 저작물의 전송·출력을 허락함.

동의여부 : 동의(ㅇ) 반대( )

2011년 2월

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