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The completeness of some metrics on Lorentzian warped product manifolds

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CONTENTS

Ⅰ. INTRODUCTION ··1

Ⅱ. PRELIMINARIES ON A WARPED PRODUCT MANIFOLD··6

Ⅲ. FIBER MANIFOLDS IN CLASS (A) or (B)····················16

Ⅳ. FIBER MANIFOLDS IN CLASS (C)························24

【REFERENCE】 ··29

國 文 抄 錄

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미분기하학의 기본적인 문제 중의 하나는 미분다양체상의 있는 곡률 함수(curvature function)를 연구하는 것이다. 종종 해석적인 방법을 택해 연구방법으로 다양체 위에서의 편미분방정식을 유도하여 해의 존재성을 보인다.

본 논문에서는 다양체 N이 n (≥3) 차원 콤팩트 리만(compact Riemannian) 다양체일 때, 휜곱다양체(warped product manifold) $M=[a,\infty)\times_f N$ 위에서 미래방향의 시간류 측지적 완비로렌쯔 거리 (future timelike geodesically complete Lorentzian metrics)의 존재성을 보이고자 한다.

따라서, 본 논문에서는 다양체 위에서 비선형 편미분방정식을 유도하고, 속 다양체(fiber manifold)가 상수 스칼라 곡률(constant scalar curvature)을 갖고 어떤 함수 R(t,x)가 t만의 함수일 때, 상해(upper solution) 와 하해(lower solution)의 방법을 이용하여 방정식

$$\frac{4n}{n+1}u''(t)-R(t)u(t)+cu(t)^{1-\frac{4}{n+1}}=0\cdots\cdots\cdots\cdots\cdots\cdots\text{(*)}$$

을 만족하는 양의 해(positive solution)의 존재성을 밝힌다.

좀 더 구체적인 연구내용은 다음과 같다.

제 2장에서 휜곱다양체(warped product manifold)에 관한 기본적인 개념과 몇 가지 결과를 설명하였다.

제 3장에서 휜곱다양체(warped product manifold)상에서 n (≥ 3)차원 콤팩트 리만(compact Riemannian)다양체 N이 class (A) 혹은 (B)인 경우 식 (*)의 양의 해(positive soiution)의 존재성을 보였다.

제 4장에서 휜곱다양체(warped product manifold) 상에서 n (≥ 3)차원 콤팩트 리만(compact Riemannian)다양체 N이 class (C) 인 경우일 때, 식 (*)을 만족하는 양의 해(positive solution)의 존재성을 연구하였다.

I. INTRODUCTION

One of the basic problems in the differential geometry is studying the set of curvature functions which a given manifold possesses.

The well-known problem in differential geometry is whether there exists a warping function of warped metric with some prescribed scalar curvature function. One of the main methods of studying differential geometry is by the existence and the nonexistence of a Lorentzian warped metric with a prescribed scalar curvature function on some Lorentzian warped product manifold. In order to study these kinds of problems, we need some analytic methods in differential geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [B.K], [B.O], [D.D], [D.D.V], [D.G], [Eb], [Ej], [G.L], [K.K.P], [L.M], [M.M]), and also in studying L_2-cohomology (cf. [Z]).

For Lorentzian manifolds, warped products have been used widely in studying the space-times (cf. [Al], [B.E.P], [E.J.K], [G], [J], [J.L], [J.K.L.S 1, 2, 3], [P]). Since warped product have been proven important in global Riemannian geometry, it is thus not surprising that the equivalent Lorentzian concept is also quite useful.

Perhaps even more interestingly on physical grounds than purely Riemannian constructions employing warped products, many of known exact solutions of the Einstein field equations of General Relativity are warped product metrics of the form $B \times_f F$, where (B, g_B) is a Lorentzian manifold and (F, g_F) is a Riemannian manifold. The most notable class of examples is the Robertson-Walker space-times of cosmology theory as well as the Schwarzschild space-time. So, in Lorentzian geometry, the warped product is also widely used for studying space-times with various applications (cf. [Al], [D.D.V], [D.V.V], [G], [M], etc.).

A space-time (M, g) is said to be geodesically complete if all geodesics of (M, g) are complete. Also, (M, g) is said to be nonspacelike (resp. null, timelike) geodesically complete if all nonspacelike (resp. null, timelike) geodesics are complete.

In recent work, we have considered the problem of scalar curvature functions on semi-Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of semi-Riemannian warped metric with some prescribed scalar curvature function (cf. [J], [J.K.L.S 1, 2, 3], [J.L]).

In [J.L], when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future complete Lorentzian metrics on $M = [a, b) \times_f N$ with specific scalar curvatures, where a and b are positive constants.

In this paper, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f F$ in terms of its warping function f and the scalar curvatures of B and F.

In a recent study [L 1, 2], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. In this paper, we study the existence and nonexistence of Lorentzian warped metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner ([K.W 1, 2, 3]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [K.W 1, 2, 3], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question of how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([G.L]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds. Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([L.M], p.

322).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature ([B.K]). It follows from the resutls of Aviles and McOwen ([A.M]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In [B.K] and [L 1, 2], authors considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [E.J.K], the authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [E.J.K], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [E.J.K] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In [J], the author considered the existence of a warping function on a Lorentzian warped product manifold $M = [a, \infty) \times_f N$. Similarly in [J.K], authors also considered the existence of a warping function on a Lorentzian warped product manifold $M = (-\infty, \infty) \times_f N$.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future completeness of Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. It is shown that if the fiber manifold N

belongs to class (A), (B) or (C), then M admits a Lorentzian metric with some prescribed scalar curvature outside a compact set.

This thesis is constituted as follows:

In Chapter II, we introduce the basic concepts and some results about warped product manifolds.

In Chapter III, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures. It is shown that if the fiber manifold N belongs to class (A) or (B), then M admits a Lorentzian metric with negative scalar curvature approaching zero near the end outside a compact set.

In Chapter IV, it is shown that if the fiber manifold N of $M = [a, \infty) \times_f N$ belongs to class (C), then M admits a Riemannian metric of positive scalar curvature.

II. PRELIMINARIES ON A WARPED PRODUCT MANIFOLD

First of all, in order to derive a partial differential equation, we need some definitions of connections, curvatures and some results about warped product manifolds.

Definition 2.1 A Lorentzian manifold (M, g) is a connected smooth manifold of dimension ≥ 2 with a countable basis together with a smooth Lorentzian metric g of signature $(-, +, +, \cdots, +)$. The Lorentzian manifold (M, g) is said to be a space-time if ,in addition, (M, g) may also be given a time orientation ([O]).

Definition 2.2 A tangent vector $v \in T_pM$ is classified as *timelike*, *nonspacelike*, *null* or *spacelike* if $g(v, v)$ is negative, nonpositive, zero, or positive, respectively:
(1) $g(v, v) < 0$, (timelike)
(2) $g(v, v) \leq 0$, (nonspacelike or causal)
(3) $g(v, v) = 0$, (null or lightlike)
(4) $g(v, v) > 0$, (spacelike)

The set of all null vectors in $T_p(M)$ is called the *nullcone* at $p \in M$ and null vectors are also said to be *lightlike*.

Definition 2.3 A vector field X on M is timelike if $g(X(p), X(p)) < 0$ at all points of $p \in M$. A Lorentzian manifold with a given timelike vector field X is said to be *time-oriented* by X. A *space-time* is a *time-oriented* Lorentzian manifold.

Not all Lorentzian manifolds may be time oriented, but a Lorentzian manifold which is not time orientable always admits a two-fold covering which is time orientable.

Definition 2.4 Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields defined on M, and let $\mathfrak{F}(M)$ denote the ring of all smooth real-valued functions on M. A connection ∇ on a smooth manifold M is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

such that
(D1) $\nabla_V W$ is $\mathfrak{F}-$linear in V,
(D2) $\nabla_V W$ is $\mathbb{R}-$linear in W,
(D3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \mathfrak{F}(M)$.
(D4) $[V, W] = \nabla_V W - \nabla_W V$, and
(D5) $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$
for all $X, V, W \in \mathfrak{X}(M)$.

If ∇ satisfies axioms (D1) $\sim$ (D3), then $\nabla_V W$ is called the *covariant derivative* of W with respect to V for the connection ∇. If, in addition, ∇ satisfies axioms (D4) $\sim$ (D5), then ∇ is called the *Levi-Civita connection* of M, which is characterized by the *Koszul formula* ([O]).

A *geodesic* $c : (a, b) \to M$ is a smooth curve of M such that the tangent vector c' moves by parallel translation along c. In other words, c is a geodesic if

$$\nabla_{c'} c' = 0. \quad \text{(geodesic equation)}$$

A *pregeodesic* is a smooth curve c which may be reparametr ized to be a geodesic. Any parameter for which c is a geodesic is called an *affine parameter.* If s and t are two affine parameters for the same pregeodesic, then $s = at + b$ for some constants $a, b \in \mathbb{R}$. A pregeodesic is said to be *complete* if for some affine parameterizion (hence for all affine parameterizations) the domain of the parametrization is all of $\mathbb{R}$.

The equation $\nabla_{c'} c' = 0$ may be expressed as a system of linear differential equations. To this end, we let $(U, (x^1, x^2, \cdots, x^n))$ be local coordinates on M and let $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\}$ denote the *natural basis* with respect to these coordinates.

The *connection coefficients* Γ^k_{ij} of ∇ with respect to $(x^1, \cdots, x^n)$ are defined by

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x^k}. \quad \text{(connection coefficients)}$$

Using these coefficients we may write the equation as the system

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \quad \text{(geodesic equations)}$$

Definition 2.5 The curvature tensor of the connection ∇ is a linear transformation valued tensor R in $Hom(\mathfrak{X}(M), \mathfrak{X}(M))$ defined by:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus, for $Z \in \mathfrak{X}(M)$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well-known that $R(X, Y)Z$ at p depends only upon the values of X, Y, and Z at p ([O]). If $\omega \in T_p^* M$ is a cotangent vector at p and $x, y, z \in T_p M$ are tangent vectors at P, then one defines

$$R(\omega, x, y, z) = (\omega, R(X, Y)Z) = \omega(R(X, Y)Z)$$

for X, Y, and Z smooth vector fields extending x, y, and z, respectively.

The *curvature tensor* R is a $(1,3)$ tensor field which is given in local coordinates by

$$R = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^m$$

where the *curvature components* R^i_{jkm} are given by

$$R^i_{jkm} = \frac{\partial \Gamma^i_{mj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^m} + \sum_{a=1}^{n} (\Gamma^a_{mj}\Gamma^i_{ka} - \Gamma^a_{kj}\Gamma^i_{ma}).$$

Notice that $R(X,Y)Z = -R(Y,X)Z$, $R(\omega, X, Y, Z) = -R(\omega, Y, X, Z)$, and $R^i_{jkm} = -R^i_{jmk}$. Furthermore, if $X = \sum X^i \frac{\partial}{\partial x^i}$, $Y = \sum Y^i \frac{\partial}{\partial x^i}$, $Z = \sum Z^i \frac{\partial}{\partial x^i}$, and $\omega = \sum \omega_i dx^i$, then

$$R(X,Y)Z = \sum_{i,j,k,m=1}^{n} R^i_{jkm} Z^j X^k Y^m \frac{\partial}{\partial x^i}$$

and

$$R(\omega, X, Y, Z) = \sum_{i,j,k,m=1}^{n} R^i_{jkm} \omega_i Z^j X^k Y^m.$$

Consequently, one has $R(dx^i, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^j}) = R^i_{jkm}$.

Definition 2.6 From the curvature tensor R, one nonzero tensor (or its negative) is obtained by contraction. It is called the *Ricci tensor*. Its components are $R_{ij} = \sum_{k=1}^{n} R^k_{ikj}$. The Ricci tensor is symmetric and its contraction $S = \sum_{i,j=1}^{n} R_{ij} g^{ij}$ is called the *scalar curvature* ([Au],[B.E], [B.E.E]).

Definition 2.7 Let $\phi : M \longrightarrow N$ be a smooth mapping. If $A \in \mathfrak{F}^0_s(N)$ with $s \geq 1$, that is, an $(0, s)$ tensor over $T_{\phi(p)}(N)$, let

$$(\phi^* A)(v_1, v_2, \cdots, v_s) = A(d\phi(v_1), \cdots, d\phi(v_s))$$

for all $v_i \in T_p(M)$, $p \in M$. Then $\phi^*(A)$ is called the *pullback* of A by ϕ ([O]).

At each point p in M, $\phi^*(A)$ gives an $\mathbb{R}$-multilinear function from $T_p(M)^s$ to $\mathbb{R}$, that is, an $(0, s)$ tensor over $T_p(M)$. In the special case if a $(0, 0)$ tensor $f \in \mathfrak{F}(N)$ is given, the pullback to M is defined to be $\phi^*(f) = f \circ \phi \in \mathfrak{F}(M)$. Note that $\phi^*(df) = d(\phi^* f)$.

Remark 2.8 We are ready at last to define the *Lorentzian distance function*

$$d = d(g) : M \times M \longrightarrow [0, \infty]$$

of an arbitrary space-time. If $c : [0, 1] \longrightarrow M$ is a piecewise smooth nonspacelike curve differentiable except at $0 = t_1 < t_2 < \cdots < t_k = 1$, then the length $L(c) = L_g(c)$ of c is given by the formula

$$L(c) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{-g(c'(t), c'(t))}dt.$$

If there is a smooth future directed timelike curves from p to q, then there are timelike curves from p to q (very close to piecewise null curves) of arbitrarily small length.

Proposition 2.9 A spacelike or timelike pregeodesic $\alpha : [0, b) \to M$ is *complete* (to the right) if and only if it has infinite length.
Proof. See p.154 in [O]. $\square$

This is clear since the unit speed reparametrization of α is a geodesic defined on the interval $[0, L(\alpha))$. In this way it is easy to check that the Poincaré half-plane $\mathbb{H}^2$ is complete.

Example 2.10. A typical semicircular pregeodesic in $\mathbb{H}^2$ $\alpha(s) = (\sin\, s,\, \cos\, s)$, $0 \leq s < \frac{\pi}{2}$, has $< \alpha^{'}, \alpha^{'} >= \sec^2 s$, hence

$$L(\alpha) = \int_0^{\frac{\pi}{2}} \sec\, s\, ds = \infty.$$

For null geodesics there is no such simple criterion, but null geodesics are often easier to compute.

Definition 2.11 Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation

$$\triangle\, u = g(x, u) \quad \text{in} \quad \Omega,$$

$$u = u_0 \qquad \text{on} \quad \partial\Omega.$$

$u \in H^{1,2}(\Omega)$ is a (weak) *sub-solution* if $u \leq u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi\, dx + \int_\Omega g(x, u)\varphi\, dx \leq 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega), \varphi \geq 0.$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) *super-solution* if in the above the reverse inequalities hold.

We briefly recall some results on warped product manifolds. Complete details may be found in [B.E], or [O]. On a semi-Riemannian product manifold $B \times F$, let π and σ be the projections of $B \times F$ onto B and F, respectively, and let $f > 0$ be a smooth function on B.

Definition 2.12 The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$$

where g_B and g_F are metric tensors of B and F, respectively. In other words, if v is tangent to M at (p,q), then

$$g(v,v) = g_B(d\pi(v), d\pi(v)) + f^2(p) g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber([O]).

We denote the metric g by $\langle \ , \ \rangle$. In view of Remark 2.13 (1) and Lemma 2.14, we may also denote the metric g_B by $\langle \ , \ \rangle$. The metric g_F will be denoted by $(\ , \ )$.

Remark 2.13 Some well known elementary properties of the warped product manifold $M = B \times_f F$ are as follows:

(1) For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B\times q}$ is an isometry onto B.
(2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(p)=p\times F}$ is a positive homothetic map onto F with homothetic factor $\frac{1}{f(p)}$.
(3) For each $(p,q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p,q).
(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M.
(5) If ϕ is an isometry of F, then $1 \times \phi$ is an isometry of M, and if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M.

Recall that vectors tangent to leaves are called horizontal and vector tangent to fibers are called vertical. From now on, we will often use a natural identification

$$T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_pB \times T_qF.$$

The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If X is a vector field on B, we define $\overline{X}$ at (p,q) by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is $\pi-$related to X and $\sigma-$related to the zero vector field on F. Similarly, if Y is a vector field on F, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

Lemma 2.14 If h is a smooth function on B, then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B.

Proof. We must show that grad$(h \circ \pi)$ is horizontal and π-related to grad h on B.

If v is a vertical tangent vector to M, then $< \text{grad}(h \circ \pi), v >= v(h \circ \pi) = d\pi(v)h = 0$, since $d\pi(v) = 0$.
Thus grad$(h \circ \pi)$ is horizontal. If x is horizontal,

$$\begin{aligned} &< d\pi(\text{grad}(h \circ \pi)), d\pi(x) >=< \text{grad}(h \circ \pi), x > \\ &= x(h \circ \pi) = d\pi(x)h =< \text{grad}\, h, d\pi(x) > . \end{aligned}$$

Hence at each point, $d\pi(\text{grad}(h \circ \pi)) = \text{grad}\, h$. □

In view of Lemma 2.14, we simplify the notations by writing h for $h \circ \pi$ and grad(h) for grad$(h \circ \pi)$. For a covariant tensor A on B, its lift $\overline{A}$ to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \to B$. That is, if A is a $(1,s)-$tensor, and if $v_1, \cdots, v_s \in T_{(p,q)}M$, then $\overline{A}(v_1, \cdots, v_s) = A(d\pi(v_1), \cdots, d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\overline{A} = 0$ on B. For example, if f is a smooth function on B, the lift to M of the Hessian of f is also denoted by H^f. This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detailed computations, see Lemma 5.1 in [B.E.P].

Now we recall the formula for the Ricci curvature tensor Ric if the warped product manifold $M = B\times_f F$. We write Ric^B for the pullback by π of the Ricci curvature of B and similarly for Ric^F.

Lemma 2.15 On a warped product manifold $M = B\times_f F$ with $n=\dim F > 1$, let X, Y be horizontal and V, W vertical.
Then

(1) $Ric(X,Y) = Ric^B(X,Y) - \frac{n}{f}H^f(X,Y)$

(2) $Ric(X,V) = 0$

(3) $Ric(V,W) = Ric^F(V,W) - \langle V,W\rangle f^{\#}$,

where $f^{\#} = \frac{\Delta f}{f} + (n-1)\frac{\langle \mathrm{grad}(f), \mathrm{grad}(f)\rangle}{f^2}$, and $\Delta f = \mathrm{trace}(H^f)$ is the Laplacian on B.

Proof. See Corollary 7.43 in [O]. □

On the given warped product manifold $M = B\times_f F$, we also write S^B for the pullback by π of the scalar curvature S_B of B and similarly for S^F. From now on, we denote grad(f) by ∇f.

Corollary 2.16 If S is the scalar curvature of $M = B\times_f F$ with $n=\dim F > 1$, then

$$S = S^B + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2} \tag{2.1}$$

where Δ is the Laplacian on B.

Proof. For each $(p,q) \in M = B\times_f F$, let $\{e_i\}$ be an orthonormal basis for T_pB. Then by the natural isomorphism $\{\overline{e_i} = (e_i, 0)\}$ is an orthonormal set in $T_{(p,q)}M$. We can choose $\{d_j\}$ on T_qF such that $\{\overline{e_i}, \overline{d_j}\}$ forms an orthonormal basis for $T_{(p,q)}M$. Then

$$1 = \langle \overline{d_j}, \overline{d_j} \rangle = f(p)^2(d_j, d_j) = (f(p)d_j, f(p)d_j)$$

which implies that $\{f(p)d_j\}$ forms an orthonormal basis for T_qF.

By Lemma 2.15 (1) and (3), for each i and j

$$Ric(\overline{e_i}, \overline{e_i}) = Ric^B(\overline{e_i}, \overline{e_i}) - \sum_i \frac{n}{f} H^f(\overline{e_i}, \overline{e_i}),$$

and

$$Ric(\overline{d_j}, \overline{d_j}) = Ric^F(\overline{d_j}, \overline{d_j}) - f^2(d_j, d_j)(\frac{\Delta f}{f} + (n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}).$$

Hence, for $\varepsilon_i = g(\overline{e_i}, \overline{e_i})$, $i = 1, ..., m$ and $\varepsilon_j = g(\overline{d_j}, \overline{d_j})$, $j = m+1, ..., m+n$

$$\begin{aligned} S(p,q) &= \sum_\alpha \varepsilon_\alpha R_{\alpha\alpha} \\ &= \sum_i \varepsilon_i Ric(\overline{e_i}, \overline{e_i}) + \sum_j \varepsilon_j Ric(\overline{d_j}, \overline{d_j}) \\ &= S^B(p,q) + \frac{S^F}{f^2} - 2n\frac{\Delta f}{f} - n(n-1)\frac{\langle \nabla f, \nabla f \rangle}{f^2}, \end{aligned}$$

which is a nonlinear partial differential equation on $B \times q$ for each $q \in F$. $\square$

III. FIBER MANIFOLDS IN CLASS (A) OR (B)

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to \mathbb{R}^+$ be a smooth function, where a is a positive number. The Lorentzian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \tag{3.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{3.2}$$

for $t \in [a, \infty)$ and $x \in N$. (For details, cf. [D.D] or [E.J.K])

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (3.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{3.3}$$

In this paper, we assume that the fiber manifold N is a nonempty, connected and compact Riemannian n-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [E.J.K], we have the following proposition.

Proposition 3.1. If the scalar curvature of the fiber manifold N is a constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a,\infty)\times_f N$ produces positive constant scalar curvature.

Proposition 3.1 implies that in Lorentzian warped product there is no obstruction of the existence of metric with positive scalar curvature. However, the results of [K.W.1] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 3.2. Theorem 5.5 in [P] implies that all timelike geodesics are future (resp. past) complete on $(-\infty,+\infty)\times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}dt = +\infty$ (resp. $\int_{-\infty}^{t_0}\left(\frac{v}{1+v}\right)^{\frac{1}{2}}dt = +\infty$) for some t_0 and Remark 2.58 in [B.E] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}}dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}}dt = +\infty$) for some t_0 (cf. Theorem 4.1 and Remark 4.2 in [B.E.P]. In this reference, the warped product metric is $g' = -dt^2 + v(t)g$).

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^{\alpha}$ in equation (3.3), where $\alpha \in (0,1)$ is a constant, and we have

$$R(t,x) \leq -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > a.$$

Therefore, from the above fact, Remark 3.2 implies the following:

Theorem 3.3. For $n \geq 3$, let $M = [a,\infty)\times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B). Then on

M there is a future geodesically complete Lorentzian metric of negative scalar curvature outside a compact set.

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = t^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = -\frac{4n}{n+1} \cdot \frac{1}{4} \cdot \frac{1}{t^2}, \quad t > a.$$

If $R(t,x)$ is the function of only t-variable, then we have the following proposition.

Proposition 3.4. If $R(g) = 0$, then there is no positive solution to equation (3.3) with

$$R(t) \leq -\frac{4n}{n+1} \cdot \frac{c}{4} \cdot \frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. See Proposition 2.4 in [J]. □

In particular, if $R(g) = 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

Proposition 3.5. Suppose that $R(g) = 0$ and $R(t,x) = R(t) \in C^{\infty}([a,\infty))$. Assume that for $t > t_0$, there exist an (weak) upper solution $u_+(t)$ and a (weak)lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a (weak) solution $u(t)$ of equation (3.3) such that for $t > t_0$, $\quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. See Theorem 2.5 in [J]. $\square$

Theorem 3.6. Suppose that $R(g) = 0$. Assume that $R(t, x) = R(t) \in C^{\infty}([a, \infty))$ is a function such that

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq \frac{4n}{n+1}bt^s \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a$, $b > 0$, $0 < c < 1$ and $s(> 0)$ are constants. Then equation (3.3) has a positive solution on $[a, \infty)$.

Proof. Since $R(g) = 0$, put $u_+(t) = t^{\frac{1}{2}}$. Then $u_+''(t) = \frac{-1}{4}t^{\frac{1}{2}-2}$. Hence

$$\begin{aligned} \frac{4n}{n+1}u_+''(t) - R(t)u_+(t) &= \frac{4n}{n+1}\frac{-1}{4}t^{\frac{1}{2}-2} - R(t)t^{\frac{1}{2}} \\ &= \frac{4n}{n+1}t^{\frac{1}{2}}[\frac{-1}{4}t^{-2} - \frac{n+1}{4n}R(t)] \\ &\leq \frac{4n}{n+1}\frac{1}{4}t^{\frac{1}{2}-2}[-1+c] \leq 0 \end{aligned}$$

Therefore $u_+(t)$ is our (weak) upper solution. And put $u_-(t) = e^{-t^{\alpha}}$, where $\alpha > \frac{s+2}{2}$ is a positive constant. Then $u_-''(t) = e^{-t^{\alpha}}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}]$. Hence

$$\begin{aligned} &\frac{4n}{n+1}u_-''(t) - R(t)u_-(t) \\ = &\frac{4n}{n+1}e^{-t^{\alpha}}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}] - R(t)e^{-t^{\alpha}} \\ \geq &\frac{4n}{n+1}e^{-t^{\alpha}}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2} - bt^s] \geq 0 \end{aligned}$$

for large t and α such that $\alpha > \frac{s+2}{2}$. Thus, for large t, $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$. So, by Proposition

3.5, equation (3.3) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$ for large t. $\square$

Corollary 3.7. Suppose that $R(g) = 0$. Assume that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a function such that

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq \frac{4n}{n+1}b\frac{1}{t^2} \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a$, $0 < c < 1$ and $0 < b < \frac{(n+1)(n+3)}{4}$ are constants. Then equation (3.3) has a positive solution on $[a, \infty)$ and on M the resulting Lorentzian warped product metric is a future geodesically complete metric.

Proof. We take the same (weak) upper solution $u_+(t) = t^{\frac{1}{2}}$ as in Theorem 3.6. Since $R(g) = 0$ and $R(t) \leq \frac{4n}{n+1}b\frac{1}{t^2}$, we take the lower solution $u_-(t) = t^{-\beta}$ where the constant $\beta(0 < \beta < \frac{n+1}{2})$ will be determined later. Then $u_-''(t) = \beta(\beta+1)t^{-\beta-2}$. Hence

$$\begin{aligned} \frac{4n}{n+1}u_-''(t) - R(t)u_-(t) &\geq \frac{4n}{n+1}\beta(\beta+1)t^{-\beta-2} - \frac{4n}{n+1}b\frac{1}{t^2}t^{-\beta} \\ &= \frac{4n}{n+1}t^{-\beta-2}[\beta(\beta+1) - b] \geq 0 \end{aligned}$$

if β is sufficiently close to $\frac{n+1}{2}$. Thus $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$ for large t. Hence Proposition 3.5 implies that equation (3.3) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$ for large t. And since β is sufficiently close to $\frac{n+1}{2}$, $-\frac{2\beta}{n+1} + 1 > 0$. Therefore

$$\begin{aligned}
\int_{t_0}^{+\infty}\left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}}dt &= \int_{t_0}^{+\infty}\frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \\
&\geq \int_{t_0}^{+\infty}\frac{u_-(t)^{\frac{2}{n+1}}}{\sqrt{1+u_-(t)^{\frac{4}{n+1}}}}dt \\
&= \int_{t_0}^{+\infty}\frac{t^{-\frac{2\beta}{n+1}}}{\sqrt{1+t^{-\frac{4\beta}{n+1}}}}dt \\
&\geq \frac{1}{\sqrt{2}}\int_{t_0}^{+\infty}t^{-\frac{2\beta}{n+1}}dt = +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{+\infty}f(t)dt = \int_{t_0}^{+\infty}u(t)^{\frac{2}{n+1}}dt \\
&\geq \int_{t_0}^{+\infty}u_-(t)^{\frac{2}{n+1}}dt = \int_{t_0}^{+\infty}t^{-\frac{2\beta}{n+1}}dt = +\infty,
\end{aligned}$$

which, by Remark 3.2, implies that the resulting warped product metric is a future geodesically complete one. □

Theorem 3.8. Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that

$$\frac{4n}{n+1}bt^{-2} < R(t) \leq \frac{4n}{n+1}dt^s,$$

where b, d, and s are positive constants. If $b > \frac{(n+1)(n+3)}{4}$, then equation (3.3) has a positive solution on $[a,\infty)$ and on M the resulting Lorentzian warped product metric is not a future geodesically complete metric.

Proof. Put $u_-(t) = e^{-t^\alpha}$, where $\alpha > \frac{s+2}{2}$ is a positive constants. Then $u_-''(t) = e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}]$. Hence

$$\begin{aligned}
& \frac{4n}{n+1}u_-''(t) - R(t)u_-(t) \\
= \;& \frac{4n}{n+1}e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}] - R(t)e^{-t^\alpha} \\
= \;& \frac{4n}{n+1}e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2} - dt^s] \geq 0
\end{aligned}$$

for large t and α such that $\alpha > \frac{s+2}{2}$. thus for large t , $u_-(t)$ is a (weak) lower solution.

Since $R(g) = 0$ and $R(t) \geq \frac{4n}{n+1}b\frac{1}{t^2}$, we take the upper solution $u_+(t) = t^{-\delta}$ where the constant $\delta(\delta > \frac{n+1}{2})$ will be determined later. Then $u_+''(t) = \delta(\delta+1)t^{-\delta-2}$. Hence

$$\begin{aligned}
\frac{4n}{n+1}u_+''(t) - R(t)u_+(t) &\leq \frac{4n}{n+1}\delta(\delta+1)t^{-\delta-2} - \frac{4n}{n+1}b\frac{1}{t^2}t^{-\delta} \\
&= \frac{4n}{n+1}t^{-\delta-2}[\delta(\delta+1) - b] \leq 0
\end{aligned}$$

if δ is sufficiently close to $\frac{n+1}{2}$. Thus $u_+(t)$ is a (weak)upper solution and $0 < u_-(t) < u_+(t)$ for large t. Hence Proposition 3.5 implies that equation (3.3) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$ for large t. And since δ is sufficiently close to $\frac{n+1}{2}$, $-\frac{2\delta}{n+1} + 1 < 0$. Therefore

$$
\begin{aligned}
\int_{t_0}^{+\infty}\left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}}dt &= \int_{t_0}^{+\infty}\frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \\
&\le \int_{t_0}^{+\infty} u_+(t)^{\frac{2}{n+1}}dt \\
&= \int_{t_0}^{+\infty} t^{-\frac{2\delta}{n+1}}dt < +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{t_0}^{+\infty} f(t)dt = \int_{t_0}^{+\infty} u(t)^{\frac{2}{n+1}}dt \\
&\le \int_{t_0}^{+\infty} u_+(t)^{\frac{2}{n+1}}dt = \int_{t_0}^{+\infty} t^{-\frac{2\delta}{n+1}}dt < +\infty,
\end{aligned}
$$

which, by Remark 3.2, implies that the resulting warped product metric is not a future geodesically complete one. □

Remark 3.9. In case that $R(g) = 0$, and we see that the function $R(t) = \frac{4n}{n+1}\cdot\frac{(n+1)(n+3)}{4}\cdot\frac{1}{t^2}$ is a fiducial point whether the resulting warped product metric is geodesically complete or not. Note that $u(t) = t^{-\frac{n+1}{2}}$ is a solution of equation (3.3) when $R(t) = \frac{4n}{n+1}\cdot\frac{(n+1)(n+3)}{4}\cdot\frac{1}{t^2}$. In case that $u(t) = t^{-\frac{n+1}{2}}$ we know that the resulting warped product metric is a geodesically complete one.

IV. FIBER MANIFOLDS IN CLASS (C)

In this section, we assume that the fiber manifold N of $M = [a, \infty) \times_f N$ belongs to class (C), where a is a positive number. In this case, N admits a Riemannian metric of positive scalar curvature. If we let $u(t) = t^\alpha$, where $\alpha \in (0, 1)$ is a constant, then we have

$$R(t, x) > -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} \geq -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

By the similar proof like as proposition 3.4, we have the following:

Proposition 4.1. If $R(g)$ is positive and $R(t, x)$ is the function of only t-variable, then there is no positive solution to equation (3.3) with

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. See Theorem 3.1 in [J]. □

Proposition 4.1 implies that if $R(g)$ is positive, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset.

If N belongs to (C), then any smooth function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metic g on N with scalar curvature $R(g) = \frac{4n}{n+1}k$, where k is a positive constant. Then equation (3.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}ku(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0. \tag{4.1}$$

Proposition 4.2. Suppose that $R(g) = \frac{4n}{n+1}k$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exist an (weak) upper solution $u_+(t)$ and a (weak) lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a (weak) solution $u(t)$ of equation (4.1) such that for $t > t_0, 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. See Theorem 3.2 in [J]. □

If $R(t,x)$ is the function of only t-variable, then we have the following theorem about the existence of some warped product metric.

Theorem 4.3. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1}bt^s \geq R(t) \geq \frac{4n}{n+1}\frac{C}{t^\alpha} \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_0,$$

where $t_0 > a$, $\alpha < 2$,and C, s, b are positive constants. Then equation (4.1) has a positive solution on $[a,\infty)$.

Proof. We let $u_+(t) = t^m$, where m is some positive number. If we take m large enough so that $m\frac{4}{n+1} > 2$, then we have

$$
\begin{aligned}
& \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
\leq\ & \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
=\ & \frac{4n}{n+1}t^m[\frac{m(m-1)}{t^2} + \frac{k}{t^{m\frac{4}{n+1}}} - \frac{C}{t^\alpha}] \\
\leq\ & 0, \quad t \geq t_0 \quad \text{for} \quad \text{some} \quad \text{large} \quad \mathrm{t}_0,
\end{aligned}
$$

which is possible for large fixed m since $\alpha < 2$. Hence $u_+(t)$ is an (weak) upper solution. And we take the (weak) lower solution $u_-(t) = t^{-\beta}$ where $\beta > 0$ will be determined later. Then $u_-''(t) = \beta(\beta+1)t^{-\beta-2}$. Hence

$$
\begin{aligned}
& \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \\
\geq\ & \frac{4n}{n+1}\beta(\beta+1)t^{-\beta-2} + \frac{4n}{n+1}k(t^{-\beta})^{1-\frac{4}{n+1}} - \frac{4n}{n+1}bt^s t^{-\beta} \\
=\ & \frac{4n}{n+1}t^{-\beta}[\beta(\beta+1)t^{-2} + kt^{\frac{4}{n+1}\beta} - bt^s] \geq 0
\end{aligned}
$$

for large t and large β such that $\frac{4}{n+1}\beta > s$. And it is also clear that $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$. By Proposition 4.2, we can obtain a positive solution. □

Remark 4.4. In case that $R(g) = \frac{4n}{n+1}k$ and $-\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2} < R(t) < \frac{4n}{n+1}\frac{d}{t^2}$, where k, d are positive constants, we do not know the existence of solutions of equation (4.1)

Corollary 4.5. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that

$$\frac{4n}{n+1}bt^s \geq R(t) \geq \frac{4n}{n+1}\frac{C}{t^\alpha} \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_0,$$

where $t_0 > a$, $\alpha < 2$, and C, s, b are positive constants. If $0 < s < 2$, then equation (4.1) has a positive solution on $[a, \infty)$ and on M the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

Proof. We take the same (weak) upper solution $u_+(t) = t^m$, where m is some positive number, as in the proof of Theorem 4.3. And , as in the proof of Theorem 4.3, we also take the (weak) lower solution $u_-(t) = t^{-\beta}$ where $\beta > 0$ will be determined later. Then $u_-''(t) = \beta(\beta+1)t^{-\beta-2}$. Since $0 < s < 2$, if β is very close to $\frac{n+1}{2}$, then $2 > \frac{4}{n+1}\beta > s$. Hence

$$\begin{aligned} & \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \\ \geq\ & \frac{4n}{n+1}\beta(\beta+1)t^{-\beta-2} + \frac{4n}{n+1}k(t^{-\beta})^{1-\frac{4}{n+1}} - \frac{4n}{n+1}bt^s t^{-\beta} \\ =\ & \frac{4n}{n+1}t^{-\beta}[\beta(\beta+1)t^{-2} + kt^{\frac{4}{n+1}\beta} - bt^s] \geq 0 \end{aligned}$$

for large t. And it is also clear that $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$. Proposition 4.2 implies that we can obtain a positive solution of equation (4.1). And since $-\frac{2\beta}{n+1} + 1 > 0$, we get

$$\begin{aligned}\int_{t_0}^{+\infty}\left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}}dt &= \int_{t_0}^{+\infty}\frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \\ &\geq \int_{t_0}^{+\infty}\frac{(t^{-\beta})^{\frac{2}{n+1}}}{\sqrt{1+(t^{-\beta})^{\frac{4}{n+1}}}}dt \\ &\geq \frac{1}{\sqrt{2}}\int_{t_0}^{+\infty}t^{-\frac{2\beta}{n+1}}dt = +\infty\end{aligned}$$

and

$$\int_{t_0}^{+\infty}f(t)dt = \int_{t_0}^{+\infty}u(t)^{\frac{2}{n+1}}dt \geq \int_{t_0}^{+\infty}t^{\frac{-2\beta}{n+1}}dt = +\infty,$$

which, by Remark 3.2, implies that the resulting warped product metric is a future geodesically complete one. □

Remark 4.6. In case that $R(g) = \frac{4n}{n+1}k$ and $\frac{4n}{n+1}dt^2 \leq R(t)$ where k, d are positive constants, we do not know whether or not our resulting Lorentizian warped metric is a future geodesically complete one.

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